Discount of Illiquid Asset Value under Utility Indifference Pricing

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Abstract: This paper applies the stochastic dynamic programming theory in Markov Decision Process (MDP) to examine the discounts of illiquid assets under short sale constraints using utility indifference pricing measure. The simulation results show that the illiquid assets only discount under short sale constraints. The discount ratios are related with the percentage of illiquid wealth, the restricted terms of illiquid assets and the risk aversion of investors.

Keywords: Illiquid asset, pricing, short sales constraints.

1 INTRODUCTION

Since the propose of portfolio theory by Markowitz (1952), how to allocate the assets efficiently has been of particular interest for academic and industries. It is now gradually extended to include illiquid assets, such as real estate, human capital and restricted stocks. These assets have two important characteristic. Firstly, they are illiquid and subject to strong limitations on trading. Secondly, their initial endowments are difficult to adjust due to the trading limitation. How to price such illiquid assets is of key importance for general asset allocation, and there is a considerable literature that looks at the impact of illiquidity on asset value.

Longstaff (2001) studies the impact of illiquidity on the portfolio choice in a framework of two assets under continuous time. One asset is risk free asset, the other is risky and not allowed to trade for some period. The trading restriction influences the portfolios of the investors significantly, and results in the decrease of value of the risky asset. In a more general model with three assets, Kahl, Liu and Longstaff (2003) study the impact of selling restrictions on the asset allocations of the investors. In the three assets, one is riskfree asset, one is market portfolio which is traded in market, and another is the restricted share for which there is also common unrestricted share in the same company that can be traded in market. Under the assumptions that: (i) there are no short sale constraints; (ii) the optimal weight of restricted asset is zero if there is no restriction, they also demonstrate that the selling restriction has a big impact on the asset value. However, as will shown later in this paper, they fail to consider the ability of self-adjustment when the investors construct their market portfolios, which is important in studying the impact of illiquidity.

In a recent work, Longstaff (2005) studies the general equilibrium pricing for the two assets (one is liquid, the other is restricted) with two heterogeneous investors. The pricing results show that even if the dividends of the two assets are the same, the price of liquid asset could be 25% higher than that of illiquid asset. In a latest paper, Lin and Zheng (2008) introduce the short sale constraints into Kahl, Liu and Longstaff (2003) model and show that short sale constraints have significant impacts on discounts of illiquid asset value. The discounts with short sale constraints are higher than those without short sale constraints. Other relevant researches include Mayers (1973, 1976), Longstaff (1995), Schwartz and Tebaldi (2004), Cuoco (1997), Miller (1974), and Chamberlain and Wilson (2000).

This paper is an extension of Kahl, Liu and Longstaff (2003) and Lin and Zheng (2008). Kahl, Liu and Longstaff (2003), Lin and Zheng (2008) both assume that one risky asset is market portfolio and the optimal weight on the other risky asset is always zero if there are no liquidity restrictions. One of the main drawbacks of this assumption is that it rules out the possibility that the investors could construct the market portfolio by themselves, which will affect the discount value. This paper relaxes this assumption by assuming that two assets are both risky assets that the investors could use to construct market portfolio. The results show that the discount of illiquid asset is zero if short sale is permitted, which is different from Kahl, Liu and Longstaff (2003) and Lin and Zheng (2008). The discount of illiquid asset is positive only if short sale is not permitted. Moreover, the discount ratios are related with many factors, such as illiquid wealth level, the restricted terms and risk aversion of investors. The remainder of this paper is organized as follows. Section II introduces the model. Section III formalizes the model as MDP. Section IV introduces the algorithm and simulation results. Section V provides the utility indifference pricing results. Finally, Section VI concludes the paper.

2 MODEL

In this section, we model the consumption-portfolio choice of a representative investor where some portion of his wealth is in illiquid assets that he cannot sell for a given period of time. We describe the model in details.

(1) There is a representative investor with initial wealth W (liquid wealth W_1, illiquid wealth W_2) and power utility (CARA):

$$U(C)=\begin{cases}\dfrac{C^{1-\gamma}}{1-\gamma}, C\ge 0\\ -\infty, C<0\end{cases} \qquad (1),$$

Where γ is the degree of relative risk aversion

(2) There are three assets. One is risk-free asset B_t with constant return r,

$$B_{t+\Delta t}=B_t\exp(r\Delta t), \qquad (2)$$

one is totally liquid asset S_t with price dynamics following

$$S_{t+\Delta t}=S_t\exp(y_{1t}(\Delta t)) \qquad (3)$$

$$y_{1t}(\Delta t)=\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}\varepsilon_1 \qquad (4)$$

where μ is the risk premium, σ is the volatility, ε_1 is the standard normal distribution random variable. Another asset has liquid issue and illiquid issue. The price of liquid issue Pt follows

$$P_{t+\Delta t}=P_t\exp(y_{2t}(\Delta t)) \qquad (5)$$

$$y_{2t}(\Delta t)=\left(r+\lambda-\frac{v^2}{2}\right)\Delta t+v\sqrt{\Delta t}\varepsilon_2 \qquad (6)$$

where λ is the risk premium, v is the volatility, ε_2 is the standard normal distribution random variable. The correlation between ε_1 and ε_2 is ρ.

(3) The investor is to maximize the expected utility in a finite horizon T by choosing the consumption C, the investment in two risky asset X_1, X_2 (investment in risk-free asset is then W_1-C-X_1-X_2):

$$\begin{aligned}&V(W_1,W_2)\\&=\max_{\{C,X_1,X_2\}}E\left(\sum_{t=0}^{T-1}\exp(-\beta t)U(C_t)+\exp(-\beta T)U(W_{1T},W_{2T})\right)\end{aligned} \qquad (7)$$

where β is the time discount factor.

(4) The restricted term τ is no less than T, i.e.,

$$U(W_{1T},W_{2T})=U(W_{1T}+W_{2T})=U(W_T) \qquad (8)$$

(5)

$$U(W_T)=\begin{cases}\dfrac{W_T^{\,1-\gamma}}{1-\gamma}, W_T\ge 0\\ -\infty, W_T<0\end{cases} \qquad (9)$$

3 MARKOV DECISION PROCESS (MDP)

It is clear that the objective in the economic model could be stated as the discounted reward maximization under a finite horizon. We could then formulate the model as a Markov Decision Process.

(1) Decision Epoch:

$$t=\{0,\Delta t,\ldots,n\Delta t\} \qquad (10)$$

where $n\Delta t=T-\Delta t$.

(2) State:

$$S_t=(W_{1t},W_{2t}) \qquad (11)$$

with dynamics

$$\begin{aligned}W_{1,t+\Delta t}&=X_t\exp(y_{1t}(\Delta t))\\&\quad+(W_{1,t}-C_t-X_{1t}-X_{2t})\exp(r\Delta t)\\&\quad+X_{2t}\exp(y_{2t}(\Delta t))\end{aligned} \qquad (12)$$

$$W_{2,t+\Delta t}=W_{2,t}\exp(y_{2t}(\Delta t)) \qquad (13)$$

(3) Action:

$$A_t=(C_t,X_t) \qquad (14)$$

Case 1: if short sale is permitted then

$$X_{1t}\in[-\infty,\infty], X_{2t}\in[-\infty,\infty],\ C_t\in(0,\infty), \qquad (15)$$

Case 2: if short sale is not permitted, then

$$\begin{aligned}&X_{1t}\in[0,W_{1,t}], X_{2t}\in[0,W_{1,t}]\\&C_t\in(0,W_{1,t})\\&C_t+X_{1t}+X_{2t}\le W_{1,t}\end{aligned} \qquad (16)$$

(4) Reward function:

$$\begin{aligned}&R_t(s_t,a_t)=R_t((W_{1,t},W_{2,t}),(C_t,X_t))=U(C_t),\\&t=0,\Delta t,2\Delta t,\ldots,n\Delta t,\\&R_T(W_{1,T},W_{2T})=U(W_{1,T}+W_{2,T})\end{aligned} \qquad (17)$$

(5) Transition Probability Matrix:

$$\begin{aligned}&f((W_{1t+\Delta t},W_{2t+\Delta t})=(w_1,w_2)\mid s_t,a_t)\\&=f((W_{1,t+\Delta t},W_{2,t+\Delta t})=(x,y)\mid(W_{1,t},W_{2,t}),(C_t,X_{1t},X_{2t}))\\&=f\left(\varepsilon_1=\frac{\ln\left(w_1-D_t-\frac{X_{2t}w_2}{W_{2t}}\right)-E_t}{\sigma\sqrt{\Delta t}},\varepsilon_2=\frac{\ln(w_2)-\ln(W_{2,t})-F_t}{v\sqrt{\Delta t}}\right)\end{aligned} \qquad (18)$$

where $f(\varepsilon_1=z_1,\varepsilon_2=z_2)$ is the density function of bivariate normal distribution with correlation ρ, and

$$D_t=(W_{1,t}-C_t-X_t-X_{2t})\exp(r\Delta t),$$

$$E_t=\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t,\ F_t=\left(r+\lambda-\frac{v^2}{2}\right)\Delta t.$$

4 ALGORITHM AND SIMULATION

We may use the backward induction algorithm to do with the above MDP problem. In order to simulate the dynamics of asset price, we follow binominal method proposed by Kahl, Liu and Longstaff (2003), i.e.,

$$M_{t+\Delta t} = M_t \exp\left(\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t \pm \sigma\sqrt{\Delta t}\right) \quad (19)$$

$$S_{t+\Delta t} = S_t \exp\left((r+\lambda-\frac{v^2}{2})\Delta t \pm v\sqrt{\Delta t}\right) \quad (20)$$

and

$$P\left(M_{t+\Delta t} = M_t \exp\left(D\Delta t+\sigma\sqrt{\Delta t}\right), S_{t+\Delta t} = S_t \exp\left(E\Delta t + v\sqrt{\Delta t}\right)\right)$$
$$= P\left(M_{t+\Delta t} = M_t \exp\left(D\Delta t-\sigma\sqrt{\Delta t}\right), S_{t+\Delta t} = S_t \exp\left(E\Delta t - v\sqrt{\Delta t}\right)\right)$$
$$=(1+\rho)/4 \quad (21)$$

$$P\left(M_{t+\Delta t} = M_t \exp\left(D\Delta t+\sigma\sqrt{\Delta t}\right), S_{t+\Delta t} = S_t \exp\left(E\Delta t - v\sqrt{\Delta t}\right)\right)$$
$$= P\left(M_{t+\Delta t} = M_t \exp\left(D\Delta t-\sigma\sqrt{\Delta t}\right), S_{t+\Delta t} = S_t \exp\left(E\Delta t + v\sqrt{\Delta t}\right)\right)$$
$$=(1-\rho)/4 \quad (22)$$

where $D = r+\mu-\frac{\sigma^2}{2}$, $E = r+\lambda-\frac{v^2}{2}$. Thus it can ensure the correlation of two dynamics to be ρ.

The parameters are: $W=1$, $r=5\%$, $\mu=8\%$, $\sigma=25\%$, v = 30%, ρ = 0.9, T = 3. In order to examine the impact of illiquid wealth level, restricted term and risk aversion on the discount, we let $W_2 = 0, 0.3, 0.5, 0.7$, the restricted term $\tau = 1,2,3$, and γ = 2, 4. $\Delta t = 1$.

Table I reports the optimal consumption-portfolio choice under different conditions. It is surprisingly to find that the restriction on liquidity has no impact on the optimal consumption-portfolio choice if short sale is permitted. The investors could adjust their position in risky asset 2 according to his illiquid wealth. The consumption and investment in risky asset 1 keeps unchanged. However, if short sale is not permitted, the investors fail to adjust the position in risky asset 2 by short sale if the illiquid wealth exceeds his optimal weight. Moreover, $C + X_1 + X_2 < W_1$, which means the investor has positive position in risk free assets, which is different from the conclusion in Lin and Zheng (2008) using log utility. Finally, when the risk aversion degree increases, the investment in risky assets will decrease, while the optimal consumption keeps unchanged. As a result, the impact of risk aversion on investor's choice is mainly on portfolio rather than the consumption.

5 UTILITY INDIFFERENCE DISCOUNT

The idea of Utility Indifference Pricing (UIP) comes form the concept of certainty equivalent amount, which is the certain amount of money that makes the agents indifferent between the return from the gamble and this amount. The UIP is economically intuitive in the sense that it measures the amount the investor is willing to pay today for some claim or right such that she is no worse off in expected utility terms than she would have been without them. Let the claim or right be k and assume the initial wealth of the investor be x, then the UIP for the claim or right is the amount p that satisfies

$$V(x-p,k) = V(x,0) \quad (23)$$

if the action is buying or

$$V(x,k) = V(x+p,0) \quad (24)$$

if the action is selling.

Here we apply the UIP framework to study the discount of illiquid asset value. Let H_t be UIP for the illiquid asset W_{2t}, then

$$V_t(W_{1t};W_{2t}) = V_t(W_{1t}+H_t;0) \quad (25)$$

it is obvious that

$$W_{1t} + H_t \le W_{1t} + W_{2t} = 1 \quad (26)$$

Then, the Utility Indifference Discount (UID), d_t is defined as the percentage

$$d_t = \frac{W_{2t} - H_t}{W_{2t}} \quad (27)$$

Table I: *Impact of illiquidity on portfolio choice*[a]

		Illiquid Wealth (W2)			
		(a) Short sale is permitted			
γ	τ	0	0.3	0.5	0.7
2	1		(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)
	2	(0.14,0.35,0.33)	(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)
	3		(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)
		(b) Short sale is not permitted			
2	1		(0.14,0.35,0.03)	(0.14,0.25,0.00)	(0.14,0.14,0.00)
	2	(0.14,0.35,0.33)	(0.14,0.34,0.03)	(0.14,0.25,0.00)	(0.13,0.16,0.00)
	3		(0.14,0.34,0.04)	(0.14,0.24,0.00)	(0.10,0.09,0.05)
		(a) Short sale is permitted			
4	1		(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)
	2	(0.14,0.17,0.16)	(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)
	3		(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)
		(b) Short sale is not permitted			
4	1		(0.14,0.09,0.00)	(0.14,0.01,0.00)	(0.13,0.00,0.00)
	2	(0.14,0.17,0.16)	(0.14,0.10,0.00)	(0.13,0.00,0.00)	(0.13,0.00,0.00)
	3		(0.14,0.11,0.00)	(0.13,0.03,0.00)	(0.10,0.05,0.02)

[a] The values are the C, X1 and X2, i.e., the optimal consumption, investment in risky asset 1 and risky asset 2.

Table II: *Impact of illiquidity on discount*

		Illiquid wealth(W_2)					
γ	τ	(a) Short sale is permitted (%)			(b) Short sale is not permitted (%)		
		30%	50%	70%	30%	50%	70%
2	1	0.00	0.00	0.00	0.00	0.45	1.45
	2	0.00	0.00	0.00	0.07	1.31	3.64
	3	0.00	0.00	0.00	0.41	2.80	11.76
4	1	0.00	0.00	0.00	0.99	3.37	5.89
	2	0.00	0.00	0.00	2.59	7.36	12.29
	3	0.00	0.00	0.00	3.39	10.97	25.34

Table II reports the UID results of different conditions. The UIDs are zero if short sale is permitted, which is consistent with the findings in Table I. As a results, the illiquidity only matters under short sale constraints. If short sale is not permitted, the UID increases with the illiquid wealth level, the restricted terms and risk aversion.

6 CONCLUSION

This paper applies the stochastic dynamic programming theory of MDP to examine the impact of illiquidity on optimal consumption-portfolio choice and asset pricing. The simulation results show that illiquidity has no impact on optimal consumption-portfolio choice and asset value if short sale is permitted. However, the illiquidity has significant impact on asset value under short sale constraints. The discounts of illiquid assets are related with illiquid wealth level, illiquid terms and risk aversion of investors. These findings have good implications for understanding the discount of illiquid assets.

On the other hand, this paper just considers two extreme conditions: short sale is permitted or not permitted. In developed countries, short sale is generally permitted but with transaction costs, such as fixed transaction costs (Lo, Mamaysky and Wang, 2004) and proportion transaction costs (Vayanos, 1998).The discounts of illiquid assets under transaction costs are for future research.

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