When does resale restriction make sense to the value of stocks?[1]

Hai Lin

Department of Finance and Wangyanan Institute for Studies in Economics, Xiamen University, Xiamen, China

Zhenlong Zheng

Department of Finance and Wangyanan Institute for Studies in Economics, Xiamen University, Xiamen, China

1. Introduction

Many stocks are subject to some kinds of resale restrictions in the financial markets.One example is the restricted stocks issued in USA, which are issued by a company but not registered with the Securities and Exchange Commision (SEC) and can be sold via private placement to sophicticated investors but cannot resold in the open market for one-year holding period[2], and limited amounts can be sold after that. Another example is Chinese markets, where most listed companies have two classes of shares: Restritee Institutional Shares (RIS) which are almost completely illiquid[3] and common shares that are traded on stock exchanges. Most of the RIS are held by the state with a purpose to control the economy. More broadly, sale restrictions are usually included in executive stock or stock-option based compensation contracts. The average waiting period before the options can be exercised is 23.6 months (Kole (1997)). Similar lockups are also usually part of the contract between the issuer and the underwriter in the vast majority of IPOs. Most lockups do not allow the company insiders (officers, directors, employers, their friends and family, and venture capitalists) to sell their shares for a period of 180

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[2] The holding period was two year prior to 1997.

[3] RIS have to be transacted through private placement. Starting in 2000, the Chinese government has allowed for auctions of these shares. Now, in order to further the market reform, the government is considering to give up the resale restriction and make the RIS publicly traded so that she could sell them and quit some competitive areas. Since it will change the value of the restricted stocks immediately and deliver a large impact on the market, how to price the restircted shares with correspondence to the common shares a key problem with a lot of concerns.

days. Ibbotson and Ritter (1995) reported that Morgan Stanley agreed to a two year lockup period in its IPO. The holders of these restricted stocks, therefore, bear some costs of holding an illiquid portfolio for some periods.

Since these restricted stocks have the common stocks alongside, the discount of the value of restricted stock with the value of an otherwise identical class of common stock measures the impact of restriction on the stock value. Some empirical studies looked at the magnitude of this discount. Maher (1976) examined restricted stock purchase made by four mutual funds in the period 1979-1973 and concluded that they traded an average discount of 35.43% on publicly traded stock in the same companies. Silber (1991) examined 310 restricted stock issues from 1981 to 1988 and found the price discount was average 33.75%. Johnson (1999) found a smaller discount of 20%. However, just as Damadaran (2005) pointed out, there are reasons to be skeptical about the findings. These studies are based on small sample sizes, spread out over long time periods and the standard errors in the estimates are substantial. Moreover, the investors with whom equity is privately placed may be providing other services to the firm, for which the discount is compenstion.

Other studies compared unregistered private placement that represent the restricted stock issues to registered private placement and found much smaller illiquidity discounts. Wruck (1989) estimated a difference of 17.6% in average discounts and only 10.4% in the mean discount. Hertzel and Smith (1993) concluded that discount of restricted stock is 13.5% higher than that of registered stock. Bajaj el. (2001) attribute only 7.23% to be the illiquidity discount. In other markets, Chen and Xiong (2001) compared the market prices of traded common stock in 258 listed companies with the auction and private placement price of the RIS shares and conclude that the discount was 78% for the auctions and almost 86% for private placements.

In the meantime, a lot of researchers tried to propose some models to study the impact of resale restriction on the asset value during the last decades. Longstaff (1995) regarded the liquidity as a look-back option and presented an upper bound for the options by considering an investor with perfect market timing abitilites who owns an asset on which she is not allowed to trade for a period. In the absence of trading restrictions, this investor would sell at the maximum price that an asset reaches during the time period and the value of the look-back option estimated using this maximum price should be the upper bound for the value of illiquidity. However, since it is the upper bounds of illiquidity, the value of illiquidity in practice will be lower if the investors are unsure about when an asset has reached its maximum price.

Longstaff (2001) studied the impact of illiquidity on the portfolio choice in a framework of two assets under continuous time. One asset is risk free asset, the other is risky and not allowed to trade for some period. The resale restriction influenced the portfolios of the investors significantly, and resulted in the decrease of value of the risky asset.In a more general model with three assets, Kahl, Liu & Longstaff (2003) studied the impact of resale restriction on the asset allocation of the investors. In the three assets, one is riskfree asset, one is market portfolio which is traded in market, another is the restricted asset with its common unrestricted asset alongside. Under the assumptions that: (i) there is no short sales constraint; (ii) the optimal weight of restricted asset is zero, they also demonstrated that the resale restriction made a big impact on the asset value. However, they fail to consider the case that the investor constructs the portfolio himself rather than just choose the market portfolio, which is important in studying the impact of restriction.

In a recent work, Longstaff (2005) studied the general equalibrium pricing for the two assets (one is liquid, the other is restricted) with two heterogeneous investors. The pricing results showed that even if the dividends of the two assets are the same, the price of liquid asset could be 25% higher than that of illiquid asset. Other relevant works include Mayers (1972,1973, 1976), Stapleton and Subrahmanyam (1979), Vayanos (1998), Hong and Wang (2000), Subramanian and Jarrow (2001),Lo, Mamaysky and Wang (2004), Shwartz and Tebaldi (2004),etc.

Both the empirical evidence and theoretical analysis suggest that the resale restriction matters and the restricted asset should be discounted to otherwise similar liquid asset. However, the magnitude of the discounts remain unknown. While most researches show discount of restricted asset value, it is widely held practitioner view that restricted stock has only a minor cost to the recipient[4]. In 2003 court case between McCord and Commisioner, the taxpayer's expert argued for a discount of 35% based on the restricted stock studies. The Internal Revenue Service (IRS), however, argued for only 7%, on the basis tha a big portion of the observed discount in restricted stock and IPO studies reflects factors other than liquidity.

The factors that affect the discounts is another interesting question that one would ask. Most of the researches on the discount in restricted stock focus on the firm specific characteristics. Silber (1991) noted that the discount varies directly with the amount of restricted stock relative to the publicly traded stock and inversely with the credit-worthiness of the issuing company. In Chinese stock market, Chen and Xiong (2001) found that the discount in RIS vary across firms with smaller discounts at larger, less volatile firms. Longstaff (1995) applied the general lessons of option pricing and ar-

[4] Wall Street Journal, April 12, 2001.

gued that the cost of illiquidity, stated as a percent of firm value, will be greater for more volatile assets and will increase with the length of the period for which trading is restricted.

One of the main costs of restriction for the investor is the failure to adjust the portfolio if necessary. Therefore, when considering the discount in restricted stock with common stock alongside, it is very intuitive to argue that the discount make sense only under the short sale constraints, since the holders can always adjust his portfolio by short selling the common stocks if the weight of restricted stock is too high. In this sense, short sale constraints give the economic sense of discount in restricted stock.

However, there has been no research that related the discounted in restricted stock to the short sales constraint, which is one major market friction in practice. Although the magnitude of short sale constraints are different for the different markets, their existence of is quite well known. For example, the short sale is prohibited in China. In USA, although short sale is permitted, it may induce higher transaction costs. The existence of short sale constraint thus make the investors reluctant to hold the restricted stock with respect to otherwise similar common stock, which result in the discount of restricted stock.

This paper would like to introduce the short sale constraints into a three asset portfolio choice model that is similar to Kahl, Liu & Longstaff (2003). In order to consider the case that the investor could construct the portfolio himself, we relax the assumption in Kahl, Liu & Longstaff (2003) that one risky asset is market portfolio so the best weight of restricted stock is zero to be that two risky assets are both stocks. One is totally common stock and the other is restricted stock with its common stock alongside. Therefore, the best weight of restricted stock is decided by the investor and is not zero any more. The short sale constraints limit the ability of the investor in adjusting the weight in restricted stock by short selling its common stock alongside.

This analysis has some important implications for us to understand the economic sense of short selling constraints in determining the discount of restricted stock. The result indicates that if short sale is permitted, the resale constraint has no impact on the portfolio selection of the investor and thus has no impact on the restricted stock value. Moreover, the conclusion still holds even if we take the transaction cost that is not related to short sale constraints[5] in to account. Therefore, the resale restriction only makes sense under the short sale constraints if we are interested in the consumption-portfolio decision of the investor. Our simulated results also suggested that the discount

[5]for example, fixed transaction cost (Lo, Mamaysky (2004))and proportional transaction cost (Vayanos (1998))

ratio is affected by the correlation of the risky assets and the risk aversion attitude of the investor, which are both very intuitive in finance theory.

Our other differences from Kahl, Liu & Longstaff (2003) are: (i) Kahl, Liu & Longstaff (2003) assume the short sale is permitted, while we would like to introduce the short sale constraints and consider different case; (ii) they applied the continuous time framework, while we would like to use discrete Markov Decision Process (MDP) theory. One reason for the preference of discrete time is that we could add the constraints more easily to the action sets to consider different case; (iii) This research can also constributed to study the impact of short sales constraints on asset allocations and pricing.

The remainder of this paper is organized as follows. Section 2 describles the economic model we consider. Section 3 presents the economic model into Markov Decision Process (MDP). Section 4 uses the backward induction algorithm to do with the stochastic dynamic programming problem. Section 5 examines the impact of resale restriction by simulation. Section 6 makes the concluding remark.

2. Economic Model

In this section, we model the consumption-portfolio choice of an representative investor ~~where some portion of his~~ wealth is in restricted stocks that he cannot sell for a given period of time. We describle the model in details.

(1) There is a representative investor with initial wealth W, where the liquid part is W_1, and restricted part is W_2.

(2) The utility function for the investor is represented by Costant Relative Risk Aversion (CRRA) function:

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma}, & C \geq 0 \\ -\infty, & C < 0 \end{cases}$$

where γ is the relative risk aversion parameter.

(3) There are three assets in the market. one is riskfree asset B_t with constant return r

$$B_{t+\Delta t} = B_t \exp(r\Delta t)$$

The other two assets are risky stocks. stock 1 is totally common stock S_t with price dyanmics following

$$S_{t+\Delta t} = S_t \exp(y_{1t}(\Delta t))$$

where $y_{1t}(\Delta t)$ is the log return during the Δt period and can be written by:

$$y_{1t}(\Delta t) = \left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_1$$

where μ is the risk premium, σ is the volatility, ε_1 is the standard normal distribution random variable. This assumption is consistent with the log-normal distribution of stock price and the Geometric Brownian Motion assumption of stock price in continuous time.

Stock 2 denoted by P_t, is restricted with common stock alongside. The price dynamics similarly follows:

$$P_{t+\Delta t} = P_t \exp\left(y_{2t}(\Delta t)\right)$$

where $y_{2t}(\Delta t)$ is the log return during the Δt period and can be written by:

$$y_{2t}(\Delta t) = \left(r + \lambda - \frac{v^2}{2}\right)\Delta t + v\sqrt{\Delta t}\varepsilon_1$$

where λ, v are the risk premium and the volatility, ε_2 is the standard normal distribution random variable. ρ is the correlation between $\varepsilon_1, \varepsilon_2$.

(4) the investor is to maximize the expected utility in a finite horizon T by choosing the consumption C, the investment in the totally common stock (X_1) and restricted stock (X_2)[6].

$$V(W_1, W_2) = \max_{\{C, X_1, X_2\}} E\left(\sum_{t=0}^{T-1} \exp(-\beta t)\, U(C_t) + \exp(-\beta T)\, U(W_{1T}, W_{2T})\right)$$

where β is the rate of time preference.

(5) The period of resale restriction τ is less than the investment horizon.

$$\tau \leq T$$

this implies

$$U(W_{1T}, W_{2T}) = U(W_{1T} + W_{2T}) = U(W_T)$$

(6)

$$U(W_T) = \begin{cases} \frac{W_T^{1-\gamma}}{1-\gamma}, & W_T \geq 0 \\ -\infty, & W_T < 0 \end{cases}$$

3. Markov Decision Process (MDP)

[6] thus the investment in riskfree asset is $W_1 - C - X_1 - X_2$ if there is no transaction cost.

It is clear that the objective in the economic model could be stated as the discounted reward maximization under a finite horizon. We could then formulate the model as a Markov Decision Process.

(1) Decision Epoch (t):

$$t = \{0, \Delta t, ..., n\Delta t\}$$

where $n\Delta t = T - \Delta t$.

(2) State (S_t):

(i) if there is no transaction cost, the state is just the value of liquid asset and restricted asset:

$$S_t = (W_{1t}, W_{2t})$$

with dynamics

$$\begin{aligned} W_{1t+\Delta t} &= X_{1t} \exp(y_{1t}(\Delta t)) + (W_{1t} - C_t - X_{1t} - X_{2t}) \exp(r\Delta t) \\ &\quad + X_{2t} \exp(y_{2t}(\Delta t)) \end{aligned}$$

$$W_{2t+\Delta t} = W_{2t} \exp(y_{2t}(\Delta t))$$

(ii) If there is transaction cost, the wealth dynamics are related with the portfolio structure, the state vector S_t was the portfolio rather than the wealth level, i.e.,

$$S_t = (Z_{0t}, Z_{1t}, Z_{2t}; W_{2t})$$

where Z_{0t}, Z_{1t}, Z_{2t} are the market value of risk free asset and two risky stocks respectively, $W_{1t} = \sum_{j=1}^{3} Z_{jt}$.The state dynamics are:

$$Z_{0t+\Delta t} = X_{0t} \exp(r)$$

$$\begin{aligned} Z_{1t+\Delta t} &= X_{1t} \exp(y_{1t}(\Delta t)) \\ Z_{2t+\Delta t} &= X_{2t} \exp(y_{2t}(\Delta t)) \end{aligned}$$

$$W_{2t+\Delta t} = W_{2t} \exp(y_{2t}(\Delta t))$$

and

$$X_{0t} = W_{1t} - C_t - X_{1t} - X_{2t} - c_1(X_{1t}, Z_{1t}) - c_2(X_{2t}, Z_{2t})$$

where $c_j(X_{jt}, Z_{jt}), j = 1, 2$ is the transaction cost function for the two stocks respectively. It can be fixed transaction cost

$$c_j(X_{jt}, Z_{jt}) = k_j{}^{7}$$

or the proportional transaction cost:

$$c(X_{jt}, Z_{jt}) = k_j |X_{jt} - Z_{jt}|^{8}$$

(iii) We can also consider the transaction cost related to the short sales, one simple example is the generalization of fixed cost to consider the short sale constraints:

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j, & \text{if } X_{jt} - Z_{jt} > 0 \\ k_j, & \text{if } X_{jt} \geq 0, X_{jt} - Z_{jt} < 0 \\ k_j + h_j, & \text{if } X_{jt} < 0, X_{jt} - Z_{jt} < 0 \\ 0, & \text{if } X_{jt} - Z_{jt} = 0 \end{cases}$$

where $h_j > 0$ is the transaction cost induced by short selling.

(3) Action set (A_t):

$$A_t = (C_t, X_{1t}, X_{2t})$$

(i) if short sale is permitted then

$$C_t \in (0, \infty), X_{1t} \in (-\infty, \infty), X_{2t} \in (-\infty, \infty)$$

(ii) if short sale is not permitted, then

$$C_t \in (0, \infty), X_t \in [0, \infty), X_{2t} \in [0, \infty), C_t + X_{1t} + X_{2t} \leq W_{1t}$$

[7] or more generally

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j^+, \text{if } X_{jt} - Z_{jt} > 0 \\ k_j^{-1}, \text{if } X_{jt} - Z_{jt} < 0 \\ 0, \text{ if } X_{jt} - X_{jt-\Delta t} = 0 \end{cases}$$

[8] or more generally,

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j^+(X_{jt} - X_{jt-\Delta t}), \text{if } X_{jt} - Z_{jt} \geq 0 \\ -k_j^{-1}(X_{jt} - X_{jt-\Delta t}), \text{if } X_{jt} - Z_{jt} < 0 \end{cases}$$

(4) Reward function $(r(s_t, a_t))$

$$r(s_t, a_t) = U(C_t)$$

$$r_T(s_T) = U(W_{1T} + W_{2T})$$

where

$$U(X) = \begin{cases} \frac{X^{1-\gamma}}{1-\gamma}, & X \geq 0 \\ -\infty, & X < 0 \end{cases}$$

(5) Transition Probability function $(f(s_{t+\Delta t}|(s_t, a_t)))$: If there is no transaction cost, the state transition probability function $f(s_{t+\Delta t}|(s_t, a_t))$ can be written by[9]:

$$\begin{aligned} & f\left((W_{1t+\Delta t}, W_{2t+\Delta t}) = (w_1, w_2) \,|\, (W_{1t}, W_{2t}), (C_t, X_{1t}, X_{2t})\right) \\ = \;& f\left(\exp(y_{1t}(\Delta t)) = w_1 - D_t - \frac{X_{2t} w_2}{W_{2t}}, \exp(y_{2t}(\Delta t)) = \frac{w_2}{W_{2t}}\right) \\ = \;& f\left(\varepsilon_1 = \frac{\ln\left(w_1 - A_t - \frac{X_{2t} w_2}{W_{2t}}\right) - E_t}{\sigma\sqrt{\Delta t}}, \varepsilon_2 = \frac{\ln w_2 - \ln W_{2t} - F_t}{v\sqrt{\Delta t}}\right) \end{aligned}$$

where

$f(\varepsilon_1 = z_1, \varepsilon_2 = z_2)$ is the density function of bivariate normal distribution with correlation ρ, and

$$D_t = (W_{1t} - C_t - X_{1t} - X_{2t}) \exp(r\Delta t)$$

$$E_t = \left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t;$$

$$F_t = \left(r + \lambda - \frac{v^2}{2}\right)\Delta t$$

.

Remark: if $W_{2t} = 0$, the above model return to the case with no restricted assets, and the similar MDP model can be set up.

4. Backward Induction Algorithm

[9] the transition probability function under transaction cost can be computed in similar way.

In this section, we introduce the backward induction algorithm to do with the above MDP problem. In order to make the introduction simple, we assume (i) there is no transaction cost, (ii) $\tau = T$,$\Delta t = 1$.

The basic ideas of backward induction algorithm are as follows:

(1) $t = T$, the resale restriction disappears. Then

$$V_T\left(W_{1T}, W_{2T}\right) = U\left(W_{1T} + W_{2T}\right)$$

(2) $t = t - 1$, compute

$$V_t\left(W_{1t}, W_{2t}\right) = \max_{a_t \in A_t}\left(r\left(s_t, a_t\right) + \exp\left(-\beta\right)\int\int f\left(j_1, j_2 | s_t, a_t\right) V_{t+1}\left(j_1, j_2\right) dj_1 dj_2\right)$$

Let

$$A^*_{s_t,t} = \arg\max_{a_t \in A_t}\left(r\left(s_t, a_t\right) + \exp\left(-\beta\right)\int\int f\left(j_1, j_2 | s_t, a_t\right) V_{t+1}\left(j_1, j_2\right) dj_1 dj_2\right)$$

(3) if $t = 0$,stop the induction. Otherwise return to (2).

It should be noted that this backward induction methodology is a general way to do with the discount reward maximization. We could consider the transaction cost by adjusting the transition probability function $f\left(j_1, j_2 | s_t, a_t\right)$ .We could also consider the impact of short sale constraints by imposing the restriction on action set A_t.

If $\tau < T$, we could modify the backward induction a little to fit the case. The backward induction with no restricted stock is used between τ and T to get $V_\tau\left(W_{1\tau}, W_{2\tau}\right) = V_\tau\left(W_{1\tau} + W_{2\tau}\right)$ .Then the above algorithm applies to the period from 0 to τ.

If there is transaction cost, we may modify the state vector to be the portfolio and apply the similar backward induction to do with the dynamic programming problem.

Based on the backward induction algorithm, we could propose some propositions on the discount in restricted stock.

Propsition 1. Suppose $\widetilde{V}_t\left(W_{1t}, W_{2t}\right)$ is the value function for the investor with liquid asset W_{1t} and restricted asset W_{2t}, and $V_t\left(W_t\right)$ is the value function for the investor with liquid asset W_t and no restricted asset. The resale restriction period is τ. If short sale is permitted and there is no transaction cost,

$$\widetilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_{1t} + W_{2t}\right), \quad \forall t < \tau$$

i.e., the restriction has no impact on the maximal expected utility level that the investor could achieve.

Proof.

(1) $t=\tau$,

$$\widetilde{V}_\tau\left(W_{1\tau},W_{2\tau}\right)=V_\tau\left(W_{1\tau}+W_{2\tau}\right)$$

the proposition holds at time τ.

(2) $t=\tau-1$,suppose the optimal strategy for investor with no restricted wealth $W_t=W_{1t}+W_{2t}$ is

$$a_t^*=\left(C_t^*,X_{1t}^*,X_{2t}^*\right)$$

where X_{it}^* denotes the value of stock i the investor holds optimally at time t. By budget constraint, the investment in riskfree asset X_{0t}^* is

$$X_{0t}^*=W_t-C_t^*-\sum_{j=1}^{2}X_{1t}^*$$

then,

$$V_t\left(W_t\right)=U\left(C_t^*\right)+\exp\left(-\beta\right)\int f\left(j|s_t,a_t^*\right)V_{t+1}\left(j\right)dj$$

where

$$\begin{aligned} j &= X_{0t}^*\exp\left(r\right)+\sum_{i=1}^{2}X_{it}^*\exp\left(y_{it}\right) \\ &= \left(W_t-C_t^*\right)\exp\left(r\right)+\sum_{i=1}^{2}X_{it}^*\left(\exp\left(y_{it}\right)-\exp\left(r\right)\right) \end{aligned}$$

and

$$s_t=W_t$$

Then we consider the restricted case.

It is obvious that

$$\widetilde{V}_t\left(W_{1t},W_{2t}\right)\leq V_t\left(W_t\right) \qquad (1)$$

consider the strategy

$$\widetilde{a}_t^*=\left(\widetilde{C}_t^*,\widetilde{X}_{1t}^*,\widetilde{X}_{2t}^*\right)$$

where

$$\widetilde{C}_t^*=C_t^*,\widetilde{X}_{1t}^*=X_{1t}^*$$

$$\widetilde{X}_{2t}^{*} = X_{2t}^{*} - W_{2t}$$

By budget constraint,

$$\begin{aligned}
\widetilde{X}_{0t}^{*} &= W_{1t} - \widetilde{C}_{t}^{*} - \widetilde{X}_{1t}^{*} - \widetilde{X}_{2t}^{*} \\
&= W_{1t} - C_{t}^{*} - X_{1t}^{*} - X_{2t}^{*} + W_{2t} \\
&= W_{t} - C_{t}^{*} - X_{1t}^{*} - X_{2t}^{*} \\
&= X_{0t}^{*}
\end{aligned}$$

The expected utility of this strategy is

$$\widetilde{v}_{t}\left(\widetilde{a}_{t}^{*}\right) = U\left(C_{t}^{*}\right) + \exp\left(-\beta\right) \int\int f'\left(\left(j_{1}, j_{2}\right)|\widetilde{s}_{t}, \widetilde{a}_{t}^{*}\right) \widetilde{V}_{t+1}\left(j_{1}, j_{2}\right) dj_{1} dj_{2}$$

where

$$\begin{aligned}
j_{1} &= X_{0t}^{*} \exp\left(r\right) + X_{1t}^{*} \exp\left(y_{1t}\right) + \left(X_{2t}^{*} - W_{2t}\right) \exp\left(y_{2t}\right) \\
&= j - W_{2t} \exp\left(y_{2t}\right)
\end{aligned}$$

$$j_{2} = W_{2t} \exp\left(y_{2t}\right)$$

and

$$\widetilde{s}_{t} = \left(W_{1t}, W_{2t}\right)$$

therefore,

$$\begin{aligned}
& f'\left(\left(j_{1}, j_{2}\right)|\widetilde{s}_{t-1}, \widetilde{a}_{t-1}^{*}\right) \\
= \;& f\left(j = j_{1} + j_{2}|s_{t-1}, a_{t-1}^{*}\right) g\left(j_{1}, j - j_{1}|j\right)
\end{aligned}$$

and

$$
\begin{aligned}
& \widetilde{v}_t\left(\widetilde{a}_t^*\right) \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int\int f'\left(\left(j_1,j_2\right)|\widetilde{s}_t,\widetilde{a}_t^*\right)\widetilde{V}_{t+1}\left(j_1,j_2\right)dj_1dj_2 \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int f\left(j=j_1+j_2|s_t,a_t^*\right)g\left(j_1,j-j_1|j\right)\widetilde{V}_{t+1}\left(j_1,j-j_1\right)dj_1dj \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int f\left(j=j_1+j_2|s_{T-1},a_{T-1}^*\right)\int g\left(j_1,j-j_1|j\right)V_{t+1}(j)\,dj_1dj \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int f\left(j|s_t,a_t^*\right)V_{t+1}(j)\,dj \\
= \;& V_t\left(W_t\right) \qquad\qquad\qquad\qquad (2)
\end{aligned}
$$

Combine (1) and (2), we can easily get

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right)=\widetilde{V}_t\left(W_{1t},W_{2t}\right)=V_t\left(W_t\right)$$

(3) if $t=0$,stops. Otherwise $t=t-1$ and return to (2).

Q.E.D.

Collary 1. Suppose $a_t^*=(C_t^*,X_{1t}^*,X_{2t}^*)$ is the optimal consumption-portfolio choice with no restricted assets at time t, W_{2t} is the value of restricted asset at time t, then if short sales is permitted and there is no transaction cost, the investor with restricted asset will choose the optimal strategy $\widetilde{a}_t^*=(\widetilde{C}_t^*,\widetilde{X}_{1t}^*,\widetilde{X}_{2t}^*)$ according to:

$$\widetilde{C}_t^*=C_t^*,\widetilde{X}_{1t}^*=X_{1t}^*$$

$$\widetilde{X}_{2t}^*=X_{2t}^*-W_{2t}$$

i.e., the investor will adjust his portfolio to keep the same optimal portfolio as with no restricted assets.

We then consider the transaction cost that is not related to short sales, i.e., fixed transaction cost and proportional transaction cost. Under these conditions, the transaction cost function $c_j\left(X_{jt},Z_{jt}\right)$ can be written as

$$c_j\left(X_{jt},Z_{jt}\right)=c_j\left(X_{jt}-Z_{jt}\right)$$

Proposition 2. Suppose $\widetilde{V}_t\left(Z_{0t},Z_{1t},Z_{2t};W_{2t}\right)$ is the value function for the investor with liquid asset Z_{0t},Z_{1t},Z_{2t} and restricted stock W_{2t}, and $V_t\left(Z_{0t},Z_{1t},Z_{2t}\right)$ is the value function for the investor with liquid asset Z_{0t},Z_{1t},Z_{2t} and no restricted asset. The resale restriction period is τ.If short sale is permitted, then under the fixed transaction cost or proportional transaction cost,

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) = V_t\left(Z_{0t}, Z_{1t}, Z_{2t} + W_{2t}\right) \quad \forall t < \tau$$

i.e., the restriction has no impact on the maximal expected utility level that the investor could achieve.

Proof.

(1) $t = \tau$,

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) = V_t\left(Z_{0t}, Z_{1t}, Z_{2t} + W_{2t}\right)$$

the proposition holds at time τ.

(2) $t = t-1$,suppose the optimal strategy for investor with portfolio $\left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right)$ where $Z'_{0t} = Z_{0t}, Z'_{1t} = Z_{1t}, Z'_{2t} = Z_{2t} + W_{2t}$ and no restricted stock is

$$a_t^* = \left(C_t^*, X_{1t}^*, X_{2t}^*\right)$$

By budget constraint,

$$\begin{aligned} X_{0t}^* &= \sum_{j=0}^{2} Z'_{jt} - C_t^* - \sum_{j=1}^{2} X_{jt}^* - \sum_{j=1}^{2} c_j\left(X_{jt}^* - Z'_{jt}\right) \\ &= Z'_{0t} - C_{T-1}^* + \sum_{j=1}^{2}\left(Z'_{jt} - X_{jt}^* - c_j\left(X_{jt}^* - Z'_{jt}\right)\right) \end{aligned}$$

then,

$$\begin{aligned} & V_t\left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right) \\ = \; & U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d=3} f\left(j|s_t, a_t^*\right) V_{t+1}\left(j\right) dj \end{aligned}$$

where

$$j = \begin{pmatrix} X_{0t}^* \exp\left(r\right) \\ X_{1t}^* \exp\left(y_{1t}\right) \\ X_{2t}^* \exp\left(y_{2t}\right) \end{pmatrix}$$

$$s_t = \left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right)$$

Then we consider the restricted case.

Similarly, it is obvious that

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) \leq V_t\left(Z_{0t}, Z_{1t}, Z'_{2t}\right) \tag{3}$$

consider the strategy

$$\widetilde{a}_t^* = \left(\widetilde{C}_t^*, \widetilde{X}_{1t}^*, \widetilde{X}_{2t}^*\right)$$

where

$$\widetilde{C}_t^* = C_t^*, \widetilde{X}_{1t}^* = X_{1t}^*$$

$$\widetilde{X}_{2t}^* = X_{2t}^* - W_{2t}$$

Based on the budget constraint

$$\begin{aligned}
\widetilde{X}_{0t}^* &= \sum_{j=1}^{3} Z_{jt} - \widetilde{C}_t^* - \sum_{j=1}^{2}\left(\widetilde{X}_{jt}^* + c_j\left(\widetilde{X}_{jt}^* - Z_{jt}\right)\right) \\
&= Z_{0t} - C_t^* + \left(Z_{1t} - X_{1t}^*\right) + \left(Z_{2t} - X_{2t}^* + W_{2t}\right) \\
&\quad - c_1\left(X_{1t}^* - Z_{1t}\right) - c_2\left(X_{2t}^* - Z_{2t} - W_{2t}\right) \\
&= Z'_{0t} - C_t^* + \sum_{j=1}^{2}\left(Z'_{jt} - X_{jt}^* - c_j\left(X_{jt}^* - Z'_{jt}\right)\right) \\
&= X_{0t}^*
\end{aligned}$$

The expected utility of this strategy is

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = U\left(C_t^*\right) + \exp\left(-\beta\right)\int_{d_1=3}\int_{d_2=3} f'\left(j_1, j_2 | \widetilde{s}_t, \widetilde{a}_t^*\right)\widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2$$

where

$$j_1 = \begin{pmatrix} X_{0t}^* \exp\left(r\right) \\ X_{1t}^* \exp\left(y_{1t}\right) \\ \left(X_{2t}^* - W_{2t}\right)\exp\left(y_{2t}\right) \end{pmatrix}$$

$$j_2 = \begin{pmatrix} 0 \\ 0 \\ W_{2t} \exp(y_{2t}) \end{pmatrix} = j - j_1$$

$$\widetilde{s}_t = (Z_{0t}, Z_{1t}, Z_{2t}; W_{2t})$$

therefore,

$$\begin{aligned} & f'\left((j_1, j_2) \,|\widetilde{s}_{T-1}, \widetilde{a}^*_{T-1}\right) \\ = \; & f\left(j = j_1 + j_2 | s_{T-1}, a^*_{T-1}\right) g\left(j_1, j_2 | j\right) \end{aligned}$$

and

$$\begin{aligned} & \widetilde{v}_t\left(\widetilde{a}^*_t\right) \\ = \; & U\left(C^*_t\right) + \exp\left(-\beta\right) \int\int f'\left((j_1, j_2)\,|\widetilde{s}_t, \widetilde{a}^*_t\right) \widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2 \\ = \; & U\left(C^*_t\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j = j_1 + j_2 | s_t, a^*_t\right) \int_{d_2=3} g\left(j_1, j - j_1 | j\right) \widetilde{V}_{t+1}\left(j_1, j - j_1\right) dj_1 dj \\ = \; & U\left(C^*_t\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j = j_1 + j_2 | s_t, a^*_t\right) \int_{d_2=3} g\left(j_1, j - j_1 | j\right) V_{t+1}\left(j\right) dj_1 dj \\ = \; & U\left(C^*_t\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j | s_t, a^*_t\right) V_{t+1}\left(j\right) dj \\ = \; & V_t\left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right) \qquad (4) \end{aligned}$$

Combine (3) and (4), we can easily get

$$\widetilde{v}_t\left(\widetilde{a}^*_t\right) = \widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) = V_t\left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right)$$

(3) if $t = 0$, stop. Otherwise $t = t - 1$ and return to (2).

Q.E.D.

Collary 2. Suppose $a^*_t = (C^*_t, X^*_{1t}, X^*_{2t})$ is the optimal consumption-portfolio choice with no restricted assets at time t, and W_{2t} is the value of restricted asset at time t, then if short sales is permitted and the transaction cost is fixed or proportional, the investor with restricted asset will choose the optimal strategy $\widetilde{a}^*_t = (\widetilde{C}^*_t, \widetilde{X}^*_{1t}, \widetilde{X}^*_{2t})$ according to:

$$\widetilde{C}^*_t = C^*_t, \widetilde{X}^*_{1t} = X^*_{1t}$$

$$\widetilde{X}_{2t}^{*} = X_{2t}^{*} - W_{2t}$$

i.e., the investor will adjust his portfolio to keep the same optimal portfolio as with no restricted stock.

The propositions and collaries indicate that if short sale is permitted, the restriction has no effect on the consumption-portfolio behavior of the investor even if there is transaction cost such as fixed transaction cost and proportional transaction cost. Assuming (a_t^*, X_{0t}^*) is optimal if there is no restriction, then $\left(\widetilde{a_t^*}, X_{0t}^*\right)$ is always feasbile under the restricted case and the investors could achieve the same expected level by simply choosing $\left(\widetilde{a_t^*}, X_{0t}^*\right)$. Thus the investor automatically adjusts his portfolio to consider his weight in restricted stock and trades as if the restriction dose not exist. In this sense, the resale restriction has minor effect on the asset value. It is also worthwhile to note that the propositions could be generalized to consider N restricted stocks with their common stocks alongside[10].

However, if the short sale is not permitted or is permitted but with higher transaction cost, the restriction does make sense. The main reason is that $\left(\widetilde{a_t^*}, X_{0t}^*\right)$ will not be feasible sometimes so that the investor with restricted stock fail to achieve the same expected utility level as with no restricted stock. Therefore, the restricted stock should be discounted with respect to its common stock to consider such failure. In other words,

$$\widetilde{V}_t\left(W_{1t}; W_{2t}\right) \leq V_t\left(W_t\right)$$

if there is no transaction cost and

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) \leq V_t\left(Z_{0t}, Z_{1t}, Z_{2t}'\right)$$

if there is transaction cost.

In order the study the discount of restricted stock under the short sale constraint, we may compute the value function $\widetilde{V}_t\left(W_{1t}; W_{2t}\right)$ (if there is no transaction cost) and choose W_t' to make[11]

$$\widetilde{V}_t\left(W_{1t}; W_{2t}\right) = V_t\left(W_t'\right)$$

[10] the proof is given in Appendix

[11] The other equivalent method is to assume the return of restricted stock to be $y_{2t} + l\left(\tau - t\right)$ and choose the illiquidity premium $l\left(\tau - t\right)$ to make

$$\widetilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_t\right)$$

the impact of restriction is represented by the illiquidity premium $l\left(\tau - t\right)$.

it is obvious that

$$W_t' \leq W_{1t} + W_{2t}$$

then the percentage

$$d_t = \frac{W_t - W_t'}{W_{2t}}$$

can be represented as the discount ratio of the restricted stock. It is clear that $d_t = 0$ if there is no short sale constraint.

5. Simulated Results

In this section, we study the effect of restriction by some simulated results. We first examine how the restriction affect the optimal consumption-portfolio decision of the investor, then estimate the discount of restricted stock with respect to its common stock.

In order to simulate the dynamics of asset price, we follow Longstaff (2005) binominal method, i.e.

$$\begin{aligned} S_{t+\Delta t} &= S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t \pm \sigma\sqrt{\Delta t}\right) \\ P_{t+\Delta t} &= P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t \pm v\sqrt{\Delta t}\right) \end{aligned}$$

and

$$\begin{aligned} & p\left(\begin{array}{c} S_{t+\Delta t} = S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\right) \\ P_{t+\Delta t} = P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t + v\sqrt{\Delta t}\right) \end{array}\right) \\ = \; & p\left(\begin{array}{c} S_{t+\Delta t} = S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}\right) \\ P_{t+\Delta t} = P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t - v\sqrt{\Delta t}\right) \end{array}\right) \\ = \; & (1 + \rho)/4 \end{aligned}$$

$$
\begin{aligned}
& p\begin{pmatrix} M_{t+\Delta t} = M_t \exp\left(\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\right) \\ S_{t+\Delta t} = S_t \exp\left(\left(r+\lambda-\frac{v^2}{2}\right)\Delta t - v\sqrt{\Delta t}\right) \end{pmatrix} \\
= \; & p\begin{pmatrix} M_{t+\Delta t} = M_t \exp\left(\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}\right) \\ S_{t+\Delta t} = S_t \exp\left(\left(r+\lambda-\frac{v^2}{2}\right)\Delta t + v\sqrt{\Delta t}\right) \end{pmatrix} \\
= \; & (1-\rho)/4
\end{aligned}
$$

Thus it can ensure the correlation of two asset price dynamics to be ρ. $\Delta t = 1$.

The parameters are: $r = 5\%$, $\mu = 8\%$,$\lambda = 10\%$, $\sigma = 25\%$, $v = 30\%$, $T = 10$. we let $\rho = -0.5$, 0, 0.5; $W_1 = 10\%$,30%,50%,70%, 90% ; the restriction period $\tau = 1, 2, 5$; $\gamma = 2, 4$. Three market conditons are considered in simulation: (i) No transaction cost ($k_j = 0$); (ii) fixed transaction cost 0.05 (h_j) is induced by short sale; (iii) short sale is not permitted.

[TO BE COMPLETED]

6. Concluding Remark:

[TO BE COMPLETED]

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APPENDIX

Generalization of Proposition 2.

Proposition. Suppose $\widetilde{V}_t(Z_{0t}, Z_{1t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt})$ is the value function for the investor with liquid asset $Z_{1t}, Z_{2t}, ..., Z_{Nt}$ (asset 0 is riskfree) and restricted asset $K_{1t}, ..., K_{Nt}$ and $V_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt})$ is the value function for the investor liquid asset $Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}$ and no restricted asset. The restriction period is τ.The short sale is permitted.Then under fixed or proportional transaction cost,

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}) = V_t(Z_{0t}, Z_{1t} + K_{1t}, ..., Z_{Nt} + K_{Nt}), \ \forall t < \tau$$

i.e., the restriction has no impact on the expected utility that the investor could achieve.

Proof.

(1) $t = \tau$,

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}) = V_t(Z_{0t}, Z_{1t} + K_{1t}, ..., Z_{Nt} + K_{Nt})$$

the proposition holds at time τ.

(2) $t = \tau - 1$,suppose the optimal strategy for investor with with portfolio $(Z'_{0t}, Z'_{1t}, ..., Z'_{Nt})$ where $Z'_{0t} = Z_{0t}, Z'_{jt} = Z_{jt} + K_{jt}, \ j = 1, ..., N$ and no restricted stock is

$$a^*_t = (C^*_t, X^*_{1t}, ..., X_{Nt})$$

where X^*_{it} means the value of asset i the investor holds optimally at time t. By budget constraint,

$$\begin{aligned} X^*_{0t} &= \sum_{j=0}^{N} Z'_{jt} - C^*_t - \sum_{j=1}^{N} X^*_{jt} - \sum_{j=1}^{N} c_j\left(X^*_{jt} - Z'_{jt}\right) \\ &= Z'_{0t} - C^*_t + \sum_{j=1}^{N} \left(Z'_{jt} - X^*_{jt} - c_j\left(X^*_{jt} - Z'_{jt}\right)\right) \end{aligned}$$

where $c_j\left(X^*_{jt} - Z_{jt}\right)$ is the transaction cost function.

then,

$$\begin{aligned}
& V_t\left(Z_{0t}', Z_{1t}', Z_{2t}', ...Z_{Nt}'\right) \\
= \quad & U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d=N+1} f\left(j|s_t, a_t^*\right) V_{t+1}\left(j\right) dj
\end{aligned}$$

where

$$j = \begin{pmatrix} X_{0t}^* \exp\left(r\right) \\ X_{1t}^* \exp\left(y_{1t}\right) \\ . \\ X_{Nt}^* \exp\left(y_{Nt}\right) \end{pmatrix}$$

$$s_t = \left(Z_{1t}, ..., Z_{Nt}\right)$$

Then we consider the restricted case.

It is obvious that

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}\right) \leq V_t\left(Z_{0t}', Z_{1t}', ..., Z_{Nt}'\right) \qquad (5)$$

consider the strategy

$$\widetilde{a}_t^* = \left(\widetilde{C}_t^*, \widetilde{X}_{1t}^*, \widetilde{X}_{2t}^*, ..., \widetilde{X}_{Nt}^*\right)$$

where

$$\widetilde{C}_t^* = C_t^*$$

$$\widetilde{X}_{jt}^* = X_{jt}^* - K_{jt}, \quad j = 1, ..., N$$

and the riskfree asset

$$\widetilde{X}_{0t}^* = X_{0t}^*$$

Since

$$
\begin{aligned}
& \widetilde{C}_t^* + \sum_{j=0}^{N} \widetilde{X}_{jt}^* + \sum_{j=1}^{N} c_j \left(\widetilde{X}_{jt}^* - Z_{jt} \right) \\
= \quad & C_t^* + \sum_{j=0}^{N} \left(X_{jt}^* - K_{jt} \right) + \sum_{j=1}^{N} c_j \left(X_{jt}^* - K_{jt} - Z_{jt} \right) \\
= \quad & C_t^* + \sum_{j=0}^{N} X_{jt}^* + \sum_{j=1}^{N} c_j \left(X_{j,T-1}^* - Z'_{jt} \right) - \sum_{j=1}^{N} K_{jt} \\
= \quad & \sum_{j=0}^{N} Z'_{jt} - \sum_{j=1}^{N} K_{jt} \\
= \quad & \sum_{j=1}^{N} Z_{jt}
\end{aligned}
$$

therefore,then it is admissible under the restricted case.

The expected utility of this strategy is

$$
\widetilde{v}_t \left(\widetilde{a}_t^* \right) = U \left(C_t^* \right) + \exp \left(-\beta \right) \int \int f' \left(\left(j_1, j_2 \right) | \widetilde{s}_t, \widetilde{a}_t^* \right) \widetilde{V}_{t+1} \left(j_1, j_2 \right) dj_1 dj_2
$$

where

$$
j_1 = \begin{pmatrix} X_{0t}^* \exp \left(r \right) \\ \left(X_{1t}^* - K_{1t} \right) \exp \left(y_{1t} \right) \\ . \\ \left(X_{Nt}^* - K_{Nt} \right) \exp \left(y_{Nt} \right) \end{pmatrix}
$$

$$
j_2 = \begin{pmatrix} 0 \\ K_{1t} \exp \left(y_{1t} \right) \\ . \\ K_{Nt} \exp \left(y_{Nt} \right) \end{pmatrix} = j - j_1
$$

$$
\widetilde{s}_t = \left(Z_{0t}, Z_{1t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt} \right)
$$

therefore,

$$
\begin{aligned}
& f' \left(\left(j_1, j_2 \right) | \widetilde{s}_t, \widetilde{a}_t^* \right) \\
= \quad & f \left(j = j_1 + j_2 | s_t, a_t^* \right) g \left(j_1, j_2 | j \right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \widetilde{v}_t\left(\widetilde{a}_t^*\right) \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int_{d_1=N+1}\int_{d_2=N+1} f'\left(\left(j_1,j_2\right)|\widetilde{s}_t,\widetilde{a}_t^*\right)\widetilde{V}_{t+1}\left(j_1,j_2\right)dj_1dj_2 \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int_{d_1=N+1} f\left(j=j_1+j_2|s_t,a_t^*\right)\int_{d_2=N+1} g\left(j_1,j-j_1|j\right)\widetilde{V}_{t+1}\left(j_1,j-j_1\right)dj_1dj \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int_{d_1=N+1} f\left(j=j_1+j_2|s_t,a_t^*\right)\int_{d_2=N+1} g\left(j_1,j-j_1|j\right)V_{t+1}\left(j\right)dj_1dj \\
= \;& U\left(C_t^*\right)+\exp(-\beta)\int_{d_1=N+1} f\left(j|s_t,a_t^*\right)V_{t+1}\left(j\right)dj \\
= \;& V_t\left(W_t\right) \qquad (6)
\end{aligned}
$$

Combine (5) and (6), we can easily get

$$
\begin{aligned}
\widetilde{v}_t\left(\widetilde{a}_t^*\right) &= \widetilde{V}_t\left(Z_{0t},Z_{1t},Z_{2t},...,Z_{Nt};K_{1t},...,K_{Nt}\right) \\
&= V_t\left(Z'_{0t},Z'_{1t},...,Z'_{Nt}\right)
\end{aligned}
$$

(3) if $t=0$, stop. Otherwise $t=t-1$ and return to (2).

Q.E.D.