Utility Indifference Discount of Restricted Stock Value[1]

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Abstract

Many stocks are subject to some kinds of resale restrictions in the financial markets.In this paper, we introduces the short sale constraints into a three asset consumption-portfolio choice model and propose the Utility Indifference Discount (UID) to measure the discount of the restricted stocks. We consider different short sale constraints and study the their impact on the UID of restricted stock. The results could help us understand the economic sense of short sales constraints on the value of restricted stocks and help resolve the argument on the magnitude of discount due to resale restriction.

JEL Classification: G11

Key Words: Short sale constraint; Transaction cost; Resale restriction; Utility indifference discount (UID), Markov decision process (MDP); Consumption-portfolio choice.

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1 Introduction

Many stocks are subject to some kinds of resale restrictions in the financial markets.One example is the restricted stocks issued in USA, which are issued by a company but not registered with the Securities and Exchange Commision (SEC) and can be sold via private placement to sophicticated investors but cannot resold in the open market for one-year holding period[2], and limited amounts can be sold after that. Another example is Chinese markets, where most listed companies have two classes of shares: Restritce Institutional Shares (RIS) which are almost completely illiquid[3] and common shares that are traded on stock exchanges. Most of the RIS are held by the state with a purpose to control the economy. More broadly, sale restrictions are usually included in executive stock or stock-option based compensation contracts. The average waiting period before the options can be exercised is 23.6 months (Kole (1997)). Similar lockups are also usually part of the contract between the issuer and the underwriter in the vast majority of IPOs. Most lockups do not allow the company insiders (officers, directors, employers, their friends and family, and venture capitalists) to sell their shares for a period of 180 days. Ibbotson and Ritter (1995) reported that Morgan Stanley agreed to a two year lockup period in its IPO. The holders of these restricted stocks, therefore, bear some costs of holding an illiquid portfolio for some periods.

Since these restricted stocks have the common stocks alongside, the discounts of the value of restricted stocks with respect to the value of an otherwise identical class of common stocks measure the impact of restriction on the stock value. Some empirical studies looked at the magnitude of this discount. Maher (1976) examined restricted stock purchase made by four mutual funds in the period 1969-1973 and concluded that they traded an average discount of 35.43% on publicly traded stock in the same companies. Silber (1991) examined 310 restricted stock issues from

[2]The holding period was two year prior to 1997.

[3]RIS have to be transacted through private placement. Starting in 2000, the Chinese government has allowed for auctions of these shares. Now, in order to further push the market reform, the government is considering to give up the resale restriction and make the RIS publicly traded so that she could sell them and quit some competitive areas. Since it will change the value of the restricted stocks immediately and deliver a large impact on the market, how to price the restircted shares with correspondence to the common shares remains a key problem with a lot of concerns.

1981 to 1988 and found the price discount was average 33.75%. Johnson (1999) found a smaller discount of 20%. However, just as Damodaran (2005) pointed out, there are reasons to be skeptical about these findings. These studies are based on small sample sizes, spread out over long time periods and the standard errors in the estimates are substantial. Moreover, the investors with whom equity is privately placed may be providing other services to the firm, for which the discount is compenstion.

Other studies compared unregistered private placement that represent the restricted stock issues to registered private placement and found much smaller illiquidity discounts. Wruck (1989) estimated a difference of 17.6% in average discounts and only 10.4% in the mean discount. Hertzel and Smith (1993) concluded that discount of restricted stock is 13.5% higher than that of registered stock. Bajaj el. (2001) attribute only 7.23% to be the illiquidity discount. In Chinese market, Chen and Xiong (2001) compared the market prices of traded common stock in 258 listed companies with the auction and private placement price of the RIS shares and conclude that the discount was 78% for the auctions and almost 86% for private placements.

In the meantime, a lot of researchers have tried to propose some models to study the impact of resale restriction on the asset value during the last decades. Longstaff (1995) regarded the liquidity as a look-back option and presented an upper bound for the options by considering an investor with perfect market timing abitilites who owns an asset on which she is not allowed to trade for a period. In the absence of trading restrictions, this investor would sell at the maximum price that an asset reaches during the time period and the value of the look-back option estimated using this maximum price should be the upper bound for the value of illiquidity. However, since it is the upper bounds of illiquidity, the value of illiquidity in practice will be lower if the investors are unsure about when an asset has reached its maximum price.

Longstaff (2001) studied the impact of illiquidity on the portfolio choice in a framework of two assets under continuous time. One asset is risk free asset, the other is risky and not allowed to trade for some period. The resale restriction influenced the portfolios of the investors significantly, and resulted in the decrease of value of the risky asset.In a more general model with three assets, Kahl, Liu and

Longstaff (2003) studied the impact of resale restriction on the asset allocation of the investors. In the three assets, one is riskfree asset, one is market portfolio which is traded in market, another is the restricted asset with its common unrestricted asset alongside. Under the assumptions that: (i) there is no short sales constraint; (ii) the optimal weight of restricted asset is zero, they also demonstrated that the resale restriction made a big impact on the asset value. However, they fail to consider the case that the investor constructs the portfolio himself rather than just choose the market portfolio, which is important in studying the impact of restriction.

In a recent work, Longstaff (2005) studied the general equalibrium pricing for the two assets (one is liquid, the other is restricted) with two heterogeneous investors. The pricing results showed that even if the dividends of the two assets are the same, the price of liquid asset could be 25% higher than that of illiquid asset. Other relevant works include Mayers (1972,1973, 1976), Stapleton and Subrahmanyam (1979), Vayanos (1998), Hong and Wang (2000), Subramanian and Jarrow (2001),Lo, Mamaysky and Wang (2004), Shwartz and Tebaldi (2004),G$\widehat{a}$rleanu (2005),etc.

Both the empirical evidence and theoretical analysis suggest that the resale restriction matters and the restricted asset should be discounted to otherwise similar liquid asset. However, the magnitude of the discounts remains unknown. While most researches show discount of restricted asset value, it is widely held practitioner view that restricted stock has only a minor cost to the recipient[4]. In 2003 court case between McCord and Commisioner, the taxpayer's expert argued for a discount of 35% based on the restricted stock studies. The Internal Revenue Service (IRS), however, argued for only 7%, on the basis tha a big portion of the observed discount in restricted stock and IPO studies reflects factors other than liquidity.

The factors that affect the discounts is another interesting question that one would ask. Most of the researches on the discount in restricted stock focus on the firm specific characteristics. Silber (1991) noted that the discount varies directly with the amount of restricted stock relative to the publicly traded stock and inversely with the credit-worthiness of the issuing company. In Chinese stock

[4] Wall Street Journal, April 12, 2001.

market, Chen and Xiong (2001) found that the discount in RIS vary across firms with smaller discounts at larger, less volatile firms. Longstaff (1995) applied the general lessons of option pricing and argued that the cost of illiquidity, stated as a percent of firm value, will be greater for more volatile assets and will increase with the length of the period for which trading is restricted.

One of the main costs of restriction for the investor is the failure to adjust the portfolio if necessary. Therefore, when considering the discount in restricted stocks with common stocks alongside, it is very intuitive to argue that the discounts make sense only under the short sale constraints, since the holders can always adjust his portfolio by short selling the common stocks if the weight of restricted stocks is too high. In this sense, short sale constraints give the economic sense of discount in restricted stocks.

There is a considerable literature that looks at the impact of short sales restriction on the equity markets (see, e.g., Miller (1977), Jarrow (1981), Detemple and Murphy (1997), Cuoco (1997), D'Avolio (2002),Geczy et al (2002), Duffie et al (2002), Hong and Stein (2002), Jones and Lamont (2002), among others) and options market (see, e.g. Figlewski and Web(1993), Danielson and Sorescu (2001), Lamont and Thaler (2003) , and Ofek et al (2004),Bai, Chang and Wang (2006), etc). Most of the reseaches evidenced the existence of impact of short sale constraint.

However, there has been much less attention paid to understand the direct links between the short sale constraints and the discounts of restricted stocks. Although the magnitudes of short sale constraints are different for the different markets, their existence of is quite well known. For example, the short sale is prohibited in China. In USA, although short sale is permitted in general, two major additional transaction costs would arise: (i) the short seller should pay fee, which is embedded in the level of "rebate rate". This rebate rate is the interest rate that investors earn on their required cash deposit to the proceeds of short sale. (ii) The stocks can be recalled at any time. Although the recall rarely happens, the fee does exist and is different for different stocks[5]. Moreover, investors are generally unwilling to sell stock short (Chen et al. (2002)). The existence of short sale constraint thus

[5] see D'Avolio (2002) and Geczy et al (2002) for an empirical description of short selling market in USA and showed that short sales restrictions exist and are not uncommon.

make the investors reluctant to hold the restricted stocks with respect to otherwise similar common stocks, which results in the discounts of restricted stocks.

This paper would like to introduce the short sale constraints into a three asset consumption-portfolio choice model that is similar to Kahl, Liu and Longstaff (2003) and apply the Utility Indifference Pricing (UIP) theory to study the Utility Indifference Discount (UID) of the restricted stocks. In order to consider the case that the investor could construct the portfolio himself, we relax the assumption in Kahl, Liu and Longstaff (2003) that one risky asset is market portfolio so the best weight of restricted stock is zero to be that two risky assets are both stocks. One is totally common stock and the other is restricted stock with its common stock alongside. Therefore, the best weight of restricted stock is decided by the investor and is not zero any more. The short sale constraints limit the ability of the investor in adjusting the weight in restricted stocks by short selling their common stocks alongside.

This analysis has some important implications for us to understand the economic sense of short selling constraints in determining the discounts of restricted stocks. The results indicate that if short sale is permitted, the resale restriction has no impact on the consumption-portfolio choice of the investor and thus has no impact on the restricted stock value. Moreover, the conclusion still holds even if we take the transaction cost that is not related to short sale constraints[6] into consideration. Therefore, the resale restriction only makes sense under the short sale constraints if we are interested in the consumption-portfolio decision of the investor. Our simulated results also suggested that the UID are affected by the restricted terms, the correlation of the risky assets and the risk aversion degree of the investor, which are all quite intuitive in finance theory.

Our other differences from Kahl, Liu and Longstaff (2003) are: (i) Kahl, Liu and Longstaff (2003) assume the short sale is permitted, while we would like to introduce the short sale constraints and consider different cases; (ii) they applied the continuous time framework, while we would like to use discrete Markov Decision Process (MDP) theory. One reason for the preference of discrete time is that we could add the constraints more easily to the action sets to consider different

[6] for example, fixed transaction cost (Lo, Mamaysky and Wang(2004))and proportional transaction cost (Vayanos (1998))

case; (iii) This research can also constributs to study the impact of short sales constraints on asset allocations and pricing.

The remainder of this paper is organized as follows. Section 2 describles the economic model we consider. Section 3 presents the economic model into Markov Decision Process (MDP). Section 4 uses the backward induction algorithm to do with the stochastic dynamic programming problem. Section 5 introduce the Utility Indifference Pricing (UIP) theory and the concept of Utility Indifference Discount (UID). Section 6 examines the impact of resale restriction by simulation. Section 7 makes the concluding remark.

2 Economic Model

In this section, we model the consumption-portfolio choice of an representative investor where some portion of his wealth is in restricted stocks that he cannot sell for a given period of time. We describe the model in details.

(1) There is a representative investor with initial wealth W, where the liquid part is W_1, and restricted part is W_2.

(2) The utility function for the investor is represented by Costant Relative Risk Aversion (CRRA) function:

$$U(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma}, & C \geq 0 \\ -\infty, & C < 0 \end{cases}$$

where γ is the relative risk aversion parameter.

(3) There are three assets in the market. one is riskfree asset B_t with constant return r

$$B_{t+\Delta t} = B_t \exp(r\Delta t)$$

The other two assets are risky stocks. stock 1 is totally common stock S_t with price dyanmics following

$$S_{t+\Delta t} = S_t \exp(y_{1t}(\Delta t))$$

where $y_{1t}(\Delta t)$ is the log return during the Δt period and can be written by:

$$y_{1t}(\Delta t) = \left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_1$$

where μ is the risk premium, σ is the volatility, ε_1 is the standard normal distribution random variable. This assumption is consistent with the log-normal distribution of stock price and the Geometric Brownian Motion assumption of stock price in continuous time.

Stock 2 denoted by P_t, is common stock with restricted stock alongside. The price dynamics similarly follows:

$$P_{t+\Delta t} = P_t \exp\left(y_{2t}(\Delta t)\right)$$

where $y_{2t}(\Delta t)$ is the log return during the Δt period and can be written by:

$$y_{2t}(\Delta t) = \left(r + \lambda - \frac{v^2}{2}\right)\Delta t + v\sqrt{\Delta t}\varepsilon_1$$

where λ, v are the risk premium and the volatility, ε_2 is the standard normal distribution random variable. ρ is the correlation between $\varepsilon_1, \varepsilon_2$.

(4) the investor is to maximize the expected utility in a finite horizon T by choosing the consumption C, the investment in the totally common stock (X_1) and restricted stock (X_2) [7].

$$V(W_1, W_2) = \max_{\{C, X_1, X_2\}} E\left(\sum_{t=0}^{T-1} \exp(-\beta t)\, U(C_t) + \exp(-\beta T)\, U(W_{1T}, W_{2T})\right)$$

where β is the rate of time preference.

(5) The period of resale restriction τ is less than the investment horizon.

$$\tau \leq T$$

this implies

$$U(W_{1T}, W_{2T}) = U(W_{1T} + W_{2T}) = U(W_T)$$

[7] thus the investment in riskfree asset is $W_1 - C - X_1 - X_2$ if there is no transaction cost.

(6)

$$U\left(W_T\right) = \begin{cases} \frac{W_T^{1-\gamma}}{1-\gamma}, \ W_T \geq 0 \\ -\infty, \ W_T < 0 \end{cases}$$

3 Markov Decision Process (MDP)

It is clear that the objective in the economic model could be stated as the discounted reward maximization under a finite horizon. We could then formulate the model as a Markov Decision Process.

(1) Decision Epoch (t):

$$t = \{0, \Delta t, ..., n\Delta t\}$$

where $n\Delta t = T - \Delta t$.

(2) State (S_t):

(i) if there is no transaction cost, the state is just the value of liquid asset and restricted asset:

$$S_t = (W_{1t}, W_{2t})$$

with dynamics

$$W_{1t+\Delta t} = X_{1t} \exp\left(y_{1t}\left(\Delta t\right)\right) + \left(W_{1t} - C_t - X_{1t} - X_{2t}\right) \exp\left(r\Delta t\right) + X_{2t} \exp\left(y_{2t}\left(\Delta t\right)\right)$$

$$W_{2t+\Delta t} = W_{2t} \exp\left(y_{2t}\left(\Delta t\right)\right)$$

(ii) If there is transaction cost, the wealth dynamics are related with the portfolio structure, then the state vector S_t was the portfolio rather than the wealth level, i.e.,

$$S_t = (Z_{0t}, Z_{1t}, Z_{2t}; W_{2t})$$

where Z_{0t}, Z_{1t}, Z_{2t} are the market value of risk free asset and two risky stocks respectively, $W_{1t} = \sum_{j=1}^{3} Z_{jt}$.The state dynamics are:

$$Z_{0t+\Delta t} = X_{0t} \exp(r)$$

$$\begin{aligned} Z_{1t+\Delta t} &= X_{1t} \exp(y_{1t}(\Delta t)) \\ Z_{2t+\Delta t} &= X_{2t} \exp(y_{2t}(\Delta t)) \end{aligned}$$

$$W_{2t+\Delta t} = W_{2t} \exp(y_{2t}(\Delta t))$$

and

$$X_{0t} = W_{1t} - C_t - X_{1t} - X_{2t} - c_1(X_{1t}, Z_{1t}) - c_2(X_{2t}, Z_{2t})$$

where $c_j(X_{jt}, Z_{jt}), j = 1, 2$ is the transaction cost function for the two stocks respectively. It can be fixed transaction cost

$$c_j(X_{jt}, Z_{jt}) = k_j$$ [8]

or the proportional transaction cost:

$$c(X_{jt}, Z_{jt}) = k_j |X_{jt} - Z_{jt}|$$ [9]

(iii) We can also consider the transaction cost related to the short sales, one simple example is the generalization of proportional cost to consider the short sale

[8] or more generally

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j^+, \text{if } X_{jt} - Z_{jt} > 0 \\ k_j^{-1}, \text{if } X_{jt} - Z_{jt} < 0 \\ 0, \text{ if } X_{jt} - Z_{jt} = 0 \end{cases}$$

[9] or more generally,

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j^+(X_{jt} - Z_{jt}), \text{if } X_{jt} - Z_{jt} \geq 0 \\ -k_j^{-1}(X_{jt} - Z_{jt}), \text{if } X_{jt} - Z_{jt} < 0 \end{cases}$$

constraints:

$$c_j(X_{jt}, Z_{jt}) = \begin{cases} k_j(X_{jt} - Z_{jt}), & \text{if } X_{jt} - Z_{jt} \geq 0 \\ k_j(Z_{jt} - X_{jt}), & \text{if } X_{jt} \geq 0, X_{jt} - Z_{jt} < 0 \\ (k_j + h_j)(Z_{jt} - X_{jt}), & \text{if } X_{jt} < 0, X_{jt} - Z_{jt} < 0 \end{cases}$$

where $h_j > 0$ is the transaction cost induced by short selling.

(3) Action set (A_t):

$$A_t = (C_t, X_{1t}, X_{2t})$$

(i) if short sale is permitted then

$$C_t \in (0, \infty), X_{1t} \in (-\infty, \infty), X_{2t} \in (-\infty, \infty)$$

(ii) if short sale is not permitted, then

$$C_t \in (0, \infty), X_t \in [0, \infty), X_{2t} \in [0, \infty), C_t + X_{1t} + X_{2t} \leq W_{1t}$$

(4) Reward function $(r(s_t, a_t))$

$$r(s_t, a_t) = U(C_t)$$

$$r_T(s_T) = U(W_{1T} + W_{2T})$$

where

$$U(X) = \begin{cases} \frac{X^{1-\gamma}}{1-\gamma}, & X \geq 0 \\ -\infty, & X < 0 \end{cases}$$

(5) Transition Probability function $(f(s_{t+\Delta t}|(s_t, a_t)))$: If there is no transaction cost, the state transition probability function $f(s_{t+\Delta t}|(s_t, a_t))$ can be written by[10]:

[10] the transition probability function under transaction cost can be computed in similar way.

$$
\begin{aligned}
& f\left(\left(W_{1t+\Delta t}, W_{2t+\Delta t}\right)=\left(w_1, w_2\right) \mid \left(W_{1t}, W_{2t}\right),\left(C_t, X_{1t}, X_{2t}\right)\right) \\
= \; & f\left(\exp\left(y_{1t}(\Delta t)\right)=w_1-D_t-\frac{X_{2t} w_2}{W_{2t}}, \exp\left(y_{2t}(\Delta t)\right)=\frac{w_2}{W_{2t}}\right) \\
= \; & f\left(\varepsilon_1=\frac{\ln\left(w_1-A_t-\frac{X_{2t} w_2}{W_{2t}}\right)-E_t}{\sigma\sqrt{\Delta t}}, \varepsilon_2=\frac{\ln w_2-\ln W_{2t}-F_t}{v\sqrt{\Delta t}}\right)
\end{aligned}
$$

where

$f\left(\varepsilon_1=z_1, \varepsilon_2=z_2\right)$ is the density function of bivariate normal distribution with correlation ρ, and

$$D_t=\left(W_{1t}-C_t-X_{1t}-X_{2t}\right)\exp(r\Delta t)$$

$$E_t=\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t;$$

$$F_t=\left(r+\lambda-\frac{v^2}{2}\right)\Delta t$$

.

Remark: if $W_{2t}=0$, the above model return to the case with no restricted assets, and the similar MDP model can be set up.

4 Backward Induction Algorithm

In this section, we introduce the backward induction algorithm to do with the above MDP problem. In order to make the introduction simple, we assume (i) there is no transaction cost, (ii) $\tau=T$,$\Delta t=1$.

The basic ideas of backward induction algorithm are as follows:

(1) $t=T$, the resale restriction disappears. Then

$$V_T\left(W_{1T}, W_{2T}\right)=U\left(W_{1T}+W_{2T}\right)$$

(2) $t = t - 1$, compute

$$V_t\left(W_{1t}, W_{2t}\right) = \max_{a_t \in A_t}\left(r\left(s_t, a_t\right) + \exp\left(-\beta\right)\int\int f\left(j_1, j_2|s_t, a_t\right) V_{t+1}\left(j_1, j_2\right) dj_1 dj_2\right)$$

Let

$$A^*_{s_t,t} = \arg\max_{a_t \in A_t}\left(r\left(s_t, a_t\right) + \exp\left(-\beta\right)\int\int f\left(j_1, j_2|s_t, a_t\right) V_{t+1}\left(j_1, j_2\right) dj_1 dj_2\right)$$

(3) if $t = 0$,stop the induction. Otherwise return to (2).

It should be noted that this backward induction methodology is a general way to do with the discount reward maximization. We could consider the transaction cost by adjusting the transition probability function $f\left(j_1, j_2|s_t, a_t\right)$ .We could also consider the impact of short sale constraints by imposing the restriction on action set A_t.

If $\tau < T$, we could modify the backward induction a little to fit the case. The backward induction with no restricted stock is used between τ and T to get $V_\tau\left(W_{1\tau}, W_{2\tau}\right) = V_\tau\left(W_{1\tau} + W_{2\tau}\right)$ .Then the above algorithm applies to the period from 0 to τ.

If there is transaction cost, we may modify the state vector to be the portfolio and apply the similar backward induction to do with the dynamic programming problem.

Based on the backward induction algorithm, we could propose some propositions on the discounts in restricted stocks.

propsition 1. Suppose $\widetilde{V}_t\left(W_{1t}, W_{2t}\right)$ is the value function for the investor with liquid asset W_{1t} and restricted asset W_{2t}, and $V_t\left(W_t\right)$ is the value function for the investor with liquid asset W_t and no restricted asset. The resale restriction period is τ. If short sale is permitted and there is no transaction cost,

$$\widetilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_{1t} + W_{2t}\right), \quad \forall t \leq \tau$$

i.e., the restriction has no impact on the maximum expected utility level that the investor could achieve.

Proof.

(1) $t=\tau$,the restriction period expires and the restricted stock becomes its common stock, then

$$\widetilde{V}_{\tau}\left(W_{1\tau},W_{2\tau}\right)=V_{\tau}\left(W_{1\tau}+W_{2\tau}\right)$$

the proposition holds at time τ.

(2) $t=\tau-1$,suppose the optimal strategy for investor with no restricted wealth $W_t=W_{1t}+W_{2t}$ is

$$a_t^*=\left(C_t^*,X_{1t}^*,X_{2t}^*\right)$$

where X_{it}^* denotes the value of stock i the investor holds optimally at time t. By budget constraint, the investment in riskfree asset X_{0t}^* is

$$X_{0t}^*=W_t-C_t^*-\sum_{j=1}^{2}X_{1t}^*$$

then,

$$V_t\left(W_t\right)=U\left(C_t^*\right)+\exp\left(-\beta\right)\int f\left(j|s_t,a_t^*\right)V_{t+1}\left(j\right)dj$$

where

$$j=X_{0t}^*\exp\left(r\right)+\sum_{i=1}^{2}X_{it}^*\exp\left(y_{it}\right)=\left(W_t-C_t^*\right)\exp\left(r\right)+\sum_{i=1}^{2}X_{it}^*\left(\exp\left(y_{it}\right)-\exp\left(r\right)\right)$$

and

$$s_t=W_t$$

Then we consider the restricted case.

It is obvious that

$$\widetilde{V}_t\left(W_{1t},W_{2t}\right)\leq V_t\left(W_t\right)\qquad(1)$$

consider the strategy

$$\widetilde{a}_t^*=\left(\widetilde{C}_t^*,\widetilde{X}_{1t}^*,\widetilde{X}_{2t}^*\right)$$

where

$$\widetilde{C}_t^* = C_t^*, \widetilde{X}_{1t}^* = X_{1t}^*$$

$$\widetilde{X}_{2t}^* = X_{2t}^* - W_{2t}$$

By budget constraint,

$$\widetilde{X}_{0t}^* = X_{0t}^{*}\,[11] \tag{0}$$

The expected utility of this strategy is

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = U\left(C_t^*\right) + \exp\left(-\beta\right) \int\int f'\left(\left(j_1, j_2\right) | \widetilde{s}_t, \widetilde{a}_t^*\right) \widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2$$

where

$$j_1 = X_{0t}^* \exp\left(r\right) + X_{1t}^* \exp\left(y_{1t}\right) + \left(X_{2t}^* - W_{2t}\right) \exp\left(y_{2t}\right) = j - W_{2t} \exp\left(y_{2t}\right)$$

$$j_2 = W_{2t} \exp\left(y_{2t}\right)$$

and

$$\widetilde{s}_t = \left(W_{1t}, W_{2t}\right)$$

therefore,

$$f'\left(\left(j_1, j_2\right) | \widetilde{s}_t, \widetilde{a}_t^*\right) = f\left(j = j_1 + j_2 | s_t, a_t^*\right) g\left(j_1, j - j_1 | j\right)$$

and

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[11]

$$\begin{aligned} \widetilde{X}_{0t}^* &= W_{1t} - \widetilde{C}_t^* - \widetilde{X}_{1t}^* - \widetilde{X}_{2t}^* \\ &= W_{1t} - C_t^* - X_{1t}^* - X_{2t}^* + W_{2t} \\ &= W_t - C_t^* - X_{1t}^* - X_{2t}^* \\ &= X_{0t}^* \end{aligned}$$

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = V_t\left(W_t\right)^{12} \qquad (2) \qquad (0)$$

Combine (1) and (2), we can easily get

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = \widetilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_t\right)$$

(3) if $t = 0$,stops. Otherwise $t = t - 1$ and return to (2).

Q.E.D.

Collary 1. Suppose $a_t^* = \left(C_t^*, X_{1t}^*, X_{2t}^*\right)$ is the optimal consumption-portfolio choice with no restricted assets at time t, W_{2t} is the value of restricted asset at time t, then if short sales is permitted and there is no transaction cost, the investor with restricted asset will choose the optimal strategy $\widetilde{a}_t^* = \left(\widetilde{C}_t^*, \widetilde{X}_{1t}^*, \widetilde{X}_{2t}^*\right)$ according to:

$$\widetilde{C}_t^* = C_t^*, \widetilde{X}_{1t}^* = X_{1t}^*$$
$$\widetilde{X}_{2t}^* = X_{2t}^* - W_{2t}$$

i.e., the investor will adjust his portfolio to keep the same optimal portfolio as with no restricted assets.

We then consider the transaction cost that is not related to short sales, i.e., fixed transaction cost and proportional transaction cost. Under these conditions,

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$$\begin{aligned}
& \widetilde{v}_t\left(\widetilde{a}_t^*\right) \\
= & U\left(C_t^*\right) + \exp\left(-\beta\right) \int\int f'\left(\left(j_1, j_2\right) | \widetilde{s}_t, \widetilde{a}_t^*\right) \widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2 \\
= & U\left(C_t^*\right) + \exp\left(-\beta\right) \int\int f\left(j = j_1 + j_2 | s_t, a_t^*\right) g\left(j_1, j - j_1 | j\right) \widetilde{V}_{t+1}\left(j_1, j - j_1\right) dj_1 dj \\
= & U\left(C_t^*\right) + \exp\left(-\beta\right) \int f\left(j = j_1 + j_2 | s_t, a_t^*\right) \int g\left(j_1, j - j_1 | j\right) V_{t+1}\left(j\right) dj_1 dj \\
= & U\left(C_t^*\right) + \exp\left(-\beta\right) \int f\left(j | s_t, a_t^*\right) V_{t+1}\left(j\right) dj \\
= & V_t\left(W_t\right)
\end{aligned}$$

the transaction cost function $c_j(X_{jt}, Z_{jt})$ can be written as

$$c_j(X_{jt}, Z_{jt}) = c_j(X_{jt} - Z_{jt})$$

Proposition 2. Suppose $\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t})$ is the value function for the investor with liquid asset Z_{0t}, Z_{1t}, Z_{2t} and restricted stock W_{2t}, and $V_t(Z_{0t}, Z_{1t}, Z_{2t})$ is the value function for the investor with liquid asset Z_{0t}, Z_{1t}, Z_{2t} and no restricted asset. The resale restriction period is τ.If short sale is permitted, then under the fixed transaction cost or proportional transaction cost,

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}) = V_t(Z_{0t}, Z_{1t}, Z_{2t} + W_{2t}) \quad \forall t < \tau$$

i.e., the restriction has no impact on the maximal expected utility level that the investor could achieve.

Proof.

(1) $t = \tau$,the restriction period expires and the restricted stock becomes its common stock, then

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}) = V_t(Z_{0t}, Z_{1t}, Z_{2t} + W_{2t})$$

the proposition holds at time τ.

(2) $t = t-1$,suppose the optimal strategy for investor with portfolio $(Z'_{0t}, Z'_{1t}, Z'_{2t})$ where $Z'_{0t} = Z_{0t}, Z'_{1t} = Z_{1t}, Z'_{2t} = Z_{2t} + W_{2t}$ and no restricted stock is

$$a^*_t = (C^*_t, X^*_{1t}, X^*_{2t})$$

By budget constraint,

$$\begin{aligned} X^*_{0t} &= \sum_{j=0}^{2} Z'_{jt} - C^*_t - \sum_{j=1}^{2} X^*_{jt} - \sum_{j=1}^{2} c_j\left(X^*_{jt} - Z'_{jt}\right) \\ &= Z'_{0t} - C^*_t + \sum_{j=1}^{2}\left(Z'_{jt} - X^*_{jt} - c_j\left(X^*_{jt} - Z'_{jt}\right)\right) \end{aligned}$$

then,

$$V_t\left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right) = U\left(C^*_t\right) + \exp\left(-\beta\right) \int_{d=3} f\left(j|s_t, a^*_t\right) V_{t+1}\left(j\right) dj$$

where

$$j = \begin{pmatrix} X^*_{0t} \exp\left(r\right) \\ X^*_{1t} \exp\left(y_{1t}\right) \\ X^*_{2t} \exp\left(y_{2t}\right) \end{pmatrix}$$

$$s_t = \left(Z'_{0t}, Z'_{1t}, Z'_{2t}\right)$$

Then we consider the restricted case.

Similarly, it is obvious that

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) \leq V_t\left(Z_{0t}, Z_{1t}, Z'_{2t}\right) \qquad (3)$$

consider the strategy

$$\widetilde{a}^*_t = \left(\widetilde{C}^*_t, \widetilde{X}^*_{1t}, \widetilde{X}^*_{2t}\right)$$

where

$$\widetilde{C}^*_t = C^*_t, \widetilde{X}^*_{1t} = X^*_{1t}$$

$$\widetilde{X}^*_{2t} = X^*_{2t} - W_{2t}$$

Based on the budget constraint,

$$\widetilde{X}_{0t}^{*} = X_{0t}^{*}\,^{13} \tag{0}$$

The expected utility of this strategy is

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = U\left(C_t^*\right) + \exp\left(-\beta\right)\int_{d_1=3}\int_{d_2=3} f'\left(j_1, j_2|\widetilde{s}_t, \widetilde{a}_t^*\right)\widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2$$

where

$$j_1 = \begin{pmatrix} X_{0t}^*\exp\left(r\right) \\ X_{1t}^*\exp\left(y_{1t}\right) \\ \left(X_{2t}^* - W_{2t}\right)\exp\left(y_{2t}\right) \end{pmatrix}$$

$$j_2 = \begin{pmatrix} 0 \\ 0 \\ W_{2t}\exp\left(y_{2t}\right) \end{pmatrix} = j - j_1$$

$$\widetilde{s}_t = \left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right)$$

therefore,

$$f'\left(\left(j_1, j_2\right)|\widetilde{s}_t, \widetilde{a}_t^*\right) = f\left(j = j_1 + j_2|s_t, a_t^*\right) g\left(j_1, j_2|j\right)$$

and similarly,

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$$\begin{aligned}
\widetilde{X}_{0t}^* &= \sum_{j=1}^{3} Z_{jt} - \widetilde{C}_t^* - \sum_{j=1}^{2}\left(\widetilde{X}_{jt}^* + c_j\left(\widetilde{X}_{jt}^* - Z_{jt}\right)\right) \\
&= Z_{0t} - C_t^* + \left(Z_{1t} - X_{1t}^*\right) + \left(Z_{2t} - X_{2t}^* + W_{2t}\right) \\
&\quad - c_1\left(X_{1t}^* - Z_{1t}\right) - c_2\left(X_{2t}^* - Z_{2t} - W_{2t}\right) \\
&= Z_{0t}' - C_t^* + \sum_{j=1}^{2}\left(Z_{jt}' - X_{jt}^* - c_j\left(X_{jt}^* - Z_{jt}'\right)\right) \\
&= X_{0t}^*
\end{aligned}$$

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = V_t\left(Z_{0t}', Z_{1t}', Z_{2t}'\right)^{14} \qquad (4) \qquad (0)$$

Combine (3) and (4), we can easily get

$$\widetilde{v}_t\left(\widetilde{a}_t^*\right) = \widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) = V_t\left(Z_{0t}', Z_{1t}', Z_{2t}'\right)$$

(3) if $t = 0$, stop. Otherwise $t = t - 1$ and return to (2).

Q.E.D.

Collary 2. Suppose $a_t^* = (C_t^*, X_{1t}^*, X_{2t}^*)$ is the optimal consumption-portfolio choice with no restricted assets at time t, and W_{2t} is the value of restricted asset at time t, then if short sales is permitted and the transaction cost is fixed or proportional, the investor with restricted asset will choose the optimal strategy $\widetilde{a}_t^* = (\widetilde{C}_t^*, \widetilde{X}_{1t}^*, \widetilde{X}_{2t}^*)$ according to:

$$\widetilde{C}_t^* = C_t^*, \widetilde{X}_{1t}^* = X_{1t}^*$$

$$\widetilde{X}_{2t}^* = X_{2t}^* - W_{2t}$$

i.e., the investor will adjust his portfolio to keep the same optimal portfolio as with no restricted stock.

The propositions and collaries indicate that if short sale is permitted, the restriction has no effect on the consumption-portfolio behavior of the investor even

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$$\begin{aligned}
& \widetilde{v}_t\left(\widetilde{a}_t^*\right) \\
= \; & U\left(C_t^*\right) + \exp\left(-\beta\right) \int\int f'\left(\left(j_1, j_2\right)|\widetilde{s}_t, \widetilde{a}_t^*\right) \widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2 \\
= \; & U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j = j_1 + j_2|s_t, a_t^*\right) \int_{d_2=3} g\left(j_1, j - j_1|j\right) \widetilde{V}_{t+1}\left(j_1, j - j_1\right) dj_1 dj \\
= \; & U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j = j_1 + j_2|s_t, a_t^*\right) \int_{d_2=3} g\left(j_1, j - j_1|j\right) V_{t+1}\left(j\right) dj_1 dj \\
= \; & U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=3} f\left(j|s_t, a_t^*\right) V_{t+1}\left(j\right) dj \\
= \; & V_t\left(Z_{0t}', Z_{1t}', Z_{2t}'\right)
\end{aligned}$$

if there is transaction cost such as fixed transaction cost and proportional transaction cost. Assuming (a_t^*, X_{0t}^*) is optimal if there is no restriction, then $\left(\widetilde{a_t^*}, X_{0t}^*\right)$ is always feasbile under the restricted case and the investors could achieve the same expected level by simply choosing $\left(\widetilde{a_t^*}, X_{0t}^*\right)$. Thus the investor automatically adjusts his portfolio to consider his weight in restricted stock and trades as if the restriction dose not exist. In this sense, the resale restriction has minor effect on the asset value. It is also worthwhile to note that the propositions could be generalized to consider N restricted stocks with their common stocks alongside[15].

However, if the short sale is not permitted or is permitted but with higher transaction cost, the restriction does make sense. The main reason is that $\left(\widetilde{a_t^*}, X_{0t}^*\right)$ will not be feasible sometimes so that the investor with restricted stock fail to achieve the same expected utility level as with no restricted stock. Therefore, the restricted stock should be discounted with respect to its common stock to consider such failure. In other words,

$$\widetilde{V}_t\left(W_{1t}; W_{2t}\right) \leq V_t\left(W_t\right)$$

if there is no transaction cost and

$$\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}; W_{2t}\right) \leq V_t\left(Z_{0t}, Z_{1t}, Z_{2t}'\right)$$

if there is transaction cost.

5 Utility Indifference Discount

The idea of Utility Indifference Pricing (UIP) comes form the concept of certainty equivalent amount, which is the certain amount of money that makes them indifferent between the return from the gamble and this amount. Recently, this concept has been adapted for derivative security pricing, espeically in an imcomplete financial markets, such as transaction costs (Hodges and Neuberger (1989), Davis and Norman (1990), Davis et al (1993), Davis and Zariphopoulou (1995) , Constantinides and Zariphopoulou (1999)), portfolio constraints (Munk (1999),

[15] the proof is given in Appendix

Detemple and Sundaresan (1999)) and non-traded assets (Tepl$\overline{a}$(2000), Henderson (2002), Kahl, Liu and Longstaff (2003)).

The UIP is economiclally intuitive in the sense that it measures the amount the investor is willing to pay today for some claim or right such that she is no worse off in expected utility terms than she would have been without them. Let the claim or right be k and assume the initial wealth of the investor be x,then the UIP for the claim or right is the amount p that satisfies

$$V(x-p,k)=V(x,0)$$

if the action is buying or

$$V\left(x,k\right)=V(x+p,0)$$

if the action is selling.

Then, the UIP is based on a comparison between optimal behavior under the alternative scenarios. As such, it is possible to incorporate a host of features into the model: the risk aversion of the agent, his initial wealth and his prior exposure to the non-replicable risk at the moment (Henderson and Hobson (2004)).In its economic sense, UIP is also known as reservation price (Munk (1999)). Tepl$\overline{a}$(2000), Detemple and Sundaresan (1999) called it "private valuation" to emphasize that the proposed price is for an individual with particular risk preference.

We may apply the UIP framework to study the discount of restricted stock value. Let H_t be UIP for her illiquid asset W_{2t},then

$$\widetilde{V}_t\left(W_{1t};W_{2t}\right)=\widetilde{V}_t\left(W_{1t}+H_t;0\right)=V_t\left(W_{1t}+H_t\right)$$

[16]

it is obvious that

$$W_{1t}+H_t\leq W_{1t}+W_{2t}$$

[16] The other equivalent method is to assume the return of restricted stock to be $y_{2t}+l\left(\tau-t\right)$ and choose the illiquidity premium $l\left(\tau-t\right)$ to make

$$\widetilde{V}_t\left(W_{1t},W_{2t}\right)=V_t\left(W_t\right)$$

the impact of restriction is represented by the illiquidity premium $l\left(\tau-t\right)$.

Then, the Utility Indifference Discount (UID), d_t is defined as the percentage

$$d_t = \frac{W_{2t} - H_t}{W_{2t}}$$

It can represent the discount ratio of the restricted stock due to the resale restriction. It is clear that $d_t = 0$ if there is no short sale constraint.

6 Simulated Results

In this section, we study the effect of restriction by some simulated results. We first examine how the restriction affect the optimal consumption-portfolio decision of the investor, then compute the utility indifference discounts of the restricted stock with respect to its common stock.

In order to simulate the dynamics of asset price[17], we follow Longstaff (2005) binominal method, i.e.

$$\begin{aligned}
S_{t+\Delta t} &= S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t \pm \sigma\sqrt{\Delta t}\right) \\
P_{t+\Delta t} &= P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t \pm v\sqrt{\Delta t}\right)
\end{aligned}$$

and the probabilities are

$$\begin{aligned}
& p\begin{pmatrix} S_{t+\Delta t} = S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\right) \\ P_{t+\Delta t} = P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t + v\sqrt{\Delta t}\right) \end{pmatrix} \\
= \; & p\begin{pmatrix} S_{t+\Delta t} = S_t \exp\left(\left(r + \mu - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}\right) \\ P_{t+\Delta t} = P_t \exp\left(\left(r + \lambda - \frac{v^2}{2}\right)\Delta t - v\sqrt{\Delta t}\right) \end{pmatrix} \\
= \; & (1 + \rho)/4
\end{aligned}$$

[17] Merton (1990) showed that the MDP problem has closed form solution if there are no transaction costs and no restricted stocks. Here we apply numerical methods to all cases in order to compare them on the same benchmark.

$$
\begin{aligned}
& p\left(\begin{array}{c} M_{t+\Delta t}=M_t\exp\left(\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}\right) \\ S_{t+\Delta t}=S_t\exp\left(\left(r+\lambda-\frac{v^2}{2}\right)\Delta t-v\sqrt{\Delta t}\right) \end{array}\right) \\
= \;& p\left(\begin{array}{c} M_{t+\Delta t}=M_t\exp\left(\left(r+\mu-\frac{\sigma^2}{2}\right)\Delta t-\sigma\sqrt{\Delta t}\right) \\ S_{t+\Delta t}=S_t\exp\left(\left(r+\lambda-\frac{v^2}{2}\right)\Delta t+v\sqrt{\Delta t}\right) \end{array}\right) \\
= \;& (1-\rho)/4
\end{aligned}
$$

Thus it can ensure the correlation of two asset price dynamics to be ρ. $\Delta t = 1$.

The simulated parameters are: $r = 5\%$, $\mu = 8\%$,$\lambda = 10\%$, $\sigma = 25\%$, $v = 30\%$, $T = 8$. we let $\rho = -0.5$, 0, 0.5; $W_1 = 0, 10\%$,30%,50%,70%, 90% ; the restriction period $\tau = 1, 2, 3$; $\gamma = 2, 4$. Three market conditons are considered in simulation: (i) no transaction cost ($k_j = 0$); (ii) fixed proportional transaction cost 0.05 (h_j) is induced by short sale; (iii) short sale is not permitted.

Table 1,2,3 report the optimal consumption-portfolio choice with varying fractions wealth held in restricted stocks under the above mentioned three conditions. Firstlt, the results in table 1 illustrate the conclusion in collary 1, i.e., $\widetilde{C}_t^* = C_t^*, \widetilde{X}_{1t}^* = X_{1t}^*, \widetilde{X}_{2t}^* = X_{2t}^* - W_{2t}$. Specifically, the resale restriction has no economic impact on the optimal consumption-portfolio choices of the investors if there is no transaction cost and short sale is permitted. Secondly, the existence of transaction cost with respect to short sales decrease the demand of short selling. The absolute value of $\widetilde{X}_{2t}^*$ with proportional short sale transaction costs is no as high as that with no transaction costs. Thirdly, if short sale is not permitted, the optimal weight in the second risky stock is mostly zero when its restricted part is beyond the optimal weight with no restriction.This indicates the limit of short sale constraints on the investor behavior. However, this is not true if the fraction of restricted stock is very high (higher than 70% in table 3) and the restricted term is long (more than 1 year in table 3). Under these circumstances, although the restricted fraction is beyond the optimal weight, the investors will still choose a positive weight on the second risky asset. This is reasonable since if the restricted term is long, the investors may consider the optimal consumption-portfolio choice just from his liquid wealth without paying attentions to the weight in restricted stocks.

The results in table 1,2,3 also reveal some reasonable relationship between the optimal consumption-portfolio choice and risk aversion degree. The change of risk aversion degree has little impact on the optimal consumption. The consumption rates under $\gamma = 2$ are mostly equal to those under $\gamma = 4$.However, the optimal portfolio changes a lot from $\gamma = 2$ to $\gamma = 4$.when the risk aversion degree increases form 2 to 4, the optimal weights for the risky assets decreases dramatically, which means the investors with higher risk aversion degree put more weight on the riskfree assets. For example, the optimal weights of two risky assets under $\gamma = 2, \rho = 0.5$ with no restricted stock is 35% and 33% respectively. This weights decrease to 17% and 16% respectively if $\gamma = 4, \rho = 0.5$.

The results in table 1,2,3 also indicates the impact of return correlation on the optimal consumption-portfolio choice. The change of correlation has little impact on the optimal consumption. However, the optimal portfolio changes a lot when ρ changes. The optimal weights in the two risk assets are negatively related with the correlation. For example, the optimal weights of two risky assets under $\gamma = 2, \rho = 0.5$ with no restricted stock is 35% and 33% respectively. When ρ decreases to -0.5, the optimal weights increase to 89% and 75% dramatically. Such increase could be explained by the more significant diversification effect from the negative correlation of asset returns and is quite consistent with the finance theory.

[INSERT TABLE 1]
[INSERT TABLE 2]
[INSERT TABLE 3]

Based on the results in table 1, we can easily conclude that $d_t = 0$ with no transaction costs. Their utility indifference discount results are thus neglected. Table 4,5 reports the UID results with proportional short sale transaction cost and short sale prohibition. It is obvious that the UID increase with fraction of illiquid wealth. The UID with short sale prohibition are much higher than those witn proportional transaction costs. The discounts with short sale prohibitions could reach as high as 76.21%, which is consistent with the empirical results of Chen and Xiong (2001).

It is interesting to find that although the results in table 4 show the existence of UID with proportional short sale transaction costs, the impact of short sale transaction costs on the UID is quite limited. All the discounts are less than 5%.This partly indicates that the discount of restricted stock in USA is not as much as we expect. This is also some consistent with the argument of IRS (7%) in 2003 court case and Bajaj el. (2001,7.23%). One possible reason for the empirical tests that give higher discounts may be that they fail to detect other factors that may affect the discounts. Since the difference of the two stocks in this paper lies only on the resale restriction, the UID results are more robust.

[INSERT TABLE 4]

[INSERT TABLE 5]

Figure 1 and figure 2 plot the discounts with respect to different correlation and restricted term respectively under proportional short sale transaction costs. Figure 3 and figure 4 plot the discounts with respect to different correlation and restricted term respectively under short sale prohibition. It is reasonable to find that the discounts increase with the τ under both conditions. However, the relationship between UID and ρ is quite different. With proportional short sale transaction cost, the UID are positively related with ρ.The is mainly due to the smaller short sale transaction cost payout (smaller UID) with respect to negative ρ.If short sale is prohibited, however, the UID are negatively related with ρ.This indicates the more signifance of diversification effect that could not be realized under short sale prohibition due to the negative ρ.

[INSERT FIG. 1]
[INSERT FIG. 2]
[INSERT FIG. 3]
[INSERT FIG. 4]

7 Conclusion:

Many stocks are subject to some kinds of resale restrictions in the financial markets.This paper introduces the short sale constraints into a three asset consumption-portfolio choice model and studies the Utility Indifference Discount (UID) of the restricted stocks. The short sale constraints limit the abilities of the investors in adjusting the weight in restricted stock by short selling its common stock alongside. Two short sale constraints are considered in this paper: (i) 5% proportional short sale transaction costs; (ii) short sale is not permitted. No transaction cost case is also considered as a benchmark.

Our propositions and collaries show that the resale restriction has no impact of the optimal consumption-portfolio choice of the investors if there is no transaction cost or transaction cost is not related with short sales. This help us to understand the economic sense of short sale constraints on the discounts of restricted stock value.

Our simulated results indicate the impact of short sale constraints on UID. The impact of proportional short sale transaction cost on the UID is quite limited. One possible reason for those empirical tests that give higher discounts may be that they fail to detect other factors that may affect the discounts. Therefore, this finding could help resolve the argument on the magnitude of discount due to resale restriction. The simulated results also suggeste that the UID are affected by the restricted terms, the correlations of the risky returns and the risk aversion degree of the investor, which are all quite intuitive in finance theory.

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APPENDIX

Generalization of Proposition 2.

Proposition. Suppose $\widetilde{V}_t\left(Z_{0t}, Z_{1t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}\right)$ is the value function for the investor with liquid asset $Z_{1t}, Z_{2t}, ..., Z_{Nt}$ (asset 0 is riskfree) and restricted asset $K_{1t}, ..., K_{Nt}$ and $V_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt})$ is the value function for the investor liquid asset $Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}$ and no restricted asset. The restriction period is τ.The short sale is permitted.Then under fixed or proportional transaction cost,

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}) = V_t(Z_{0t}, Z_{1t} + K_{1t}, ..., Z_{Nt} + K_{Nt}), \ \forall t < \tau$$

i.e., the restriction has no impact on the expected utility that the investor could achieve.

Proof.

(1) $t = \tau$,the restriction period expires and the restricted stocks becomes their correspondent common stocks, then

$$\widetilde{V}_t(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}) = V_t(Z_{0t}, Z_{1t} + K_{1t}, ..., Z_{Nt} + K_{Nt})$$

the proposition holds at time τ.

(2) $t = \tau - 1$,suppose the optimal strategy for investor with with portfolio $(Z'_{0t}, Z'_{1t}, ..., Z'_{Nt})$ where $Z'_{0t} = Z_{0t}, Z'_{jt} = Z_{jt} + K_{jt}, \ j = 1, ..., N$ and no restricted stock is

$$a_t^* = (C_t^*, X_{1t}^*, ..., X_{Nt})$$

where X_{it}^* means the value of asset i the investor holds optimally at time t. By budget constraint,

$$\begin{aligned} X_{0t}^* &= \sum_{j=0}^{N} Z'_{jt} - C_t^* - \sum_{j=1}^{N} X_{jt}^* - \sum_{j=1}^{N} c_j\left(X_{jt}^* - Z'_{jt}\right) \\ &= Z'_{0t} - C_t^* + \sum_{j=1}^{N}\left(Z'_{jt} - X_{jt}^* - c_j\left(X_{jt}^* - Z'_{jt}\right)\right) \end{aligned}$$

where $c_j\left(X_{jt}^* - Z_{jt}\right)$ is the transaction cost function.

then,

$$
\begin{aligned}
& V_t\left(Z_{0t}', Z_{1t}', Z_{2t}', ...Z_{Nt}'\right) \\
= \;& U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d=N+1} f\left(j|s_t, a_t^*\right) V_{t+1}\left(j\right) dj
\end{aligned}
$$

where

$$
j = \begin{pmatrix} X_{0t}^* \exp\left(r\right) \\ X_{1t}^* \exp\left(y_{1t}\right) \\ . \\ X_{Nt}^* \exp\left(y_{Nt}\right) \end{pmatrix}
$$

$$
s_t = \left(Z_{1t}, ..., Z_{Nt}\right)
$$

Then we consider the restricted case.

It is obvious that

$$
\widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}\right) \leq V_t\left(Z_{0t}', Z_{1t}', ..., Z_{Nt}'\right) \tag{5}
$$

consider the strategy

$$
\widetilde{a}_t^* = \left(\widetilde{C}_t^*, \widetilde{X}_{1t}^*, \widetilde{X}_{2t}^*, ..., \widetilde{X}_{Nt}^*\right)
$$

where

$$
\widetilde{C}_t^* = C_t^*
$$

$$
\widetilde{X}_{jt}^* = X_{jt}^* - K_{jt}, \quad j = 1, ..., N
$$

and the riskfree asset

$$
\widetilde{X}_{0t}^* = X_{0t}^*
$$

Since

$$\begin{aligned}
& \widetilde{C}_t^* + \sum_{j=0}^{N} \widetilde{X}_{jt}^* + \sum_{j=1}^{N} c_j \left(\widetilde{X}_{jt}^* - Z_{jt} \right) \\
= \; & C_t^* + \sum_{j=0}^{N} \left(X_{jt}^* - K_{jt} \right) + \sum_{j=1}^{N} c_j \left(X_{jt}^* - K_{jt} - Z_{jt} \right) \\
= \; & C_t^* + \sum_{j=0}^{N} X_{jt}^* + \sum_{j=1}^{N} c_j \left(X_{j,T-1}^* - Z'_{jt} \right) - \sum_{j=1}^{N} K_{jt} \\
= \; & \sum_{j=0}^{N} Z'_{jt} - \sum_{j=1}^{N} K_{jt} \\
= \; & \sum_{j=1}^{N} Z_{jt}
\end{aligned}$$

therefore,then it is admissible under the restricted case.
The expected utility of this strategy is

$$\widetilde{v}_t \left(\widetilde{a}_t^* \right) = U \left(C_t^* \right) + \exp \left(-\beta \right) \int \int f' \left(\left(j_1, j_2 \right) | \widetilde{s}_t, \widetilde{a}_t^* \right) \widetilde{V}_{t+1} \left(j_1, j_2 \right) dj_1 dj_2$$

where

$$j_1 = \begin{pmatrix} X_{0t}^* \exp \left(r \right) \\ \left(X_{1t}^* - K_{1t} \right) \exp \left(y_{1t} \right) \\ . \\ \left(X_{Nt}^* - K_{Nt} \right) \exp \left(y_{Nt} \right) \end{pmatrix}$$

$$j_2 = \begin{pmatrix} 0 \\ K_{1t} \exp \left(y_{1t} \right) \\ . \\ K_{Nt} \exp \left(y_{Nt} \right) \end{pmatrix} = j - j_1$$

$$\widetilde{s}_t = \left(Z_{0t}, Z_{1t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt} \right)$$

therefore,

$$
\begin{aligned}
& f'\left((j_1, j_2)\,|\widetilde{s}_t, \widetilde{a}_t^*\right) \\
= \;& f\left(j = j_1 + j_2 | s_t, a_t^*\right) g\left(j_1, j_2 | j\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \widetilde{v}_t\left(\widetilde{a}_t^*\right) \\
= \;& U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=N+1} \int_{d_2=N+1} f'\left((j_1, j_2)\,|\widetilde{s}_t, \widetilde{a}_t^*\right) \widetilde{V}_{t+1}\left(j_1, j_2\right) dj_1 dj_2 \\
= \;& U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=N+1} f\left(j = j_1 + j_2 | s_t, a_t^*\right) \int_{d_2=N+1} g\left(j_1, j - j_1 | j\right) \widetilde{V}_{t+1}\left(j_1, j - j_1\right) dj_1 dj \\
= \;& U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=N+1} f\left(j = j_1 + j_2 | s_t, a_t^*\right) \int_{d_2=N+1} g\left(j_1, j - j_1 | j\right) V_{t+1}\left(j\right) dj_1 dj \\
= \;& U\left(C_t^*\right) + \exp\left(-\beta\right) \int_{d_1=N+1} f\left(j | s_t, a_t^*\right) V_{t+1}\left(j\right) dj \\
= \;& V_t\left(W_t\right) \qquad (6)
\end{aligned}
$$

Combine (5) and (6), we can easily get

$$
\begin{aligned}
\widetilde{v}_t\left(\widetilde{a}_t^*\right) &= \widetilde{V}_t\left(Z_{0t}, Z_{1t}, Z_{2t}, ..., Z_{Nt}; K_{1t}, ..., K_{Nt}\right) \\
&= V_t\left(Z'_{0t}, Z'_{1t}, ..., Z'_{Nt}\right)
\end{aligned}
$$

(3) if $t = 0$, stop. Otherwise $t = t - 1$ and return to (2).

Q.E.D.

Table 1. Optimal Consumption-Portfolio Rates with Illiquid Wealth: No Transaction Cost

			Fraction of Illiquid Wealth					
γ	ρ	τ	**0%**	**10%**	**30%**	**50%**	**70%**	**90%**
2	0.5	1		(0.14,0.35,0.23)	(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)	(0.14,0.35,-0.57)
		2	(0.14,0.35,0.33)	(0.14,0.35,0.23)	(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)	(0.14,0.35,-0.57)
		3		(0.14,0.35,0.23)	(0.14,0.35,0.03)	(0.14,0.35,-0.17)	(0.14,0.35,-0.37)	(0.14,0.35,-0.57)
2	0	1		(0.15,0.53,0.36)	(0.15,0.53,0.16)	(0.15,0.53,-0.04)	(0.15,0.53,-0.24)	(0.15,0.53,-0.44)
		2	(0.15,0.53,0.46)	(0.15,0.53,0.36)	(0.15,0.53,0.16)	(0.15,0.53,-0.04)	(0.15,0.53,-0.24)	(0.15,0.53,-0.44)
		3		(0.15,0.53,0.36)	(0.15,0.53,0.16)	(0.15,0.53,-0.04)	(0.15,0.53,-0.24)	(0.15,0.53,-0.44)
2	-0.5	1		(0.16,0.89,0.65)	(0.16,0.89,0.45)	(0.16,0.89,0.25)	(0.16,0.89,0.05)	(0.16,0.89,-0.15)
		2	(0.16,0.89,0.75)	(0.16,0.89,0.65)	(0.16,0.89,0.45)	(0.16,0.89,0.25)	(0.16,0.89,0.05)	(0.16,0.89,-0.15)
		3		(0.16,0.89,0.65)	(0.16,0.89,0.45)	(0.16,0.90,0.24)	(0.16,0.89,0.05)	(0.16,0.89,-0.15)
4	0.5	1		(0.14,0.17,0.06)	(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)	(0.14,0.17,-0.74)
		2	(0.14,0.17,0.16)	(0.14,0.17,0.06)	(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)	(0.14,0.17,-0.74)
		3		(0.14,0.17,0.06)	(0.14,0.17,-0.14)	(0.14,0.17,-0.34)	(0.14,0.17,-0.54)	(0.14,0.17,-0.74)
4	0	1		(0.14,0.27,0.13)	(0.14,0.27,-0.07)	(0.14,0.27,-0.27)	(0.14,0.27,-0.47)	(0.14,0.27,-0.67)
		2	(0.14,0.27,0.23)	(0.14,0.27,0.13)	(0.14,0.27,-0.07)	(0.14,0.27,-0.27)	(0.14,0.27,-0.47)	(0.14,0.27,-0.67)
		3		(0.14,0.27,0.13)	(0.14,0.27,-0.07)	(0.14,0.27,-0.27)	(0.14,0.27,-0.47)	(0.14,0.27,-0.67)
4	-0.5	1		(0.15,0.47,0.29)	(0.15,0.47,0.09)	(0.15,0.47,-0.11)	(0.15,0.47,-0.31)	(0.15,0.47,-0.51)
		2	(0.15,0.47,0.39)	(0.15,0.47,0.29)	(0.15,0.47,0.09)	(0.15,0.47,-0.11)	(0.15,0.47,-0.31)	(0.15,0.47,-0.51)
		3		(0.15,0.47,0.30)	(0.15,0.47,0.09)	(0.15,0.47,-0.11)	(0.15,0.47,-0.31)	(0.15,0.47,-0.51)

This table reports the ratio of optimal consumption, investment in totally liquid stock and investment in restricted stock for an investor with varying fractions of wealth held in the form of stock that is restricted for a period of τ years if short sale is permitted and there is no transaction cost. The risk aversion coefficient is γ. The correlation between liquid stock and restricted stock is ρ. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate.

Table 2. Optimal Consumption-Portfolio Rates with Illiquid Wealth: Proportional Short Sales Transaction Cost

			Fraction of Illiquid Wealth					
γ	ρ	τ	**0%**	**10%**	**30%**	**50%**	**70%**	**90%**
2	0.5	1		(0.14,0.35,0.23)	(0.14,0.34,0.03)	(0.14,0.25,0.00)	(0.14,0.16,-0.04)	(0.14,0.16,-0.24)
		2	(0.14,0.35,0.33)	(0.14,0.35,0.23)	(0.14,0.36,0.02)	(0.14,0.25,0.00)	(0.14,0.27,-0.25)	(0.14,0.27,-0.45)
		3		(0.14,0.34,0.24)	(0.14,0.34,0.03)	(0.14,0.32,-0.15)	(0.14,0.34,-0.35)	(0.14,0.36,-0.57)
2	0	1		(0.15,0.53,0.36)	(0.15,0.53,0.16)	(0.15,0.53,0.00)	(0.15,0.50,0.00)	(0.14,0.50,-0.19)
		2	(0.15,0.53,0.46)	(0.15,0.52,0.36)	(0.15,0.54,0.16)	(0.15,0.52,0.00)	(0.14,0.49,-0.16)	(0.14,0.48,-0.37)
		3		(0.15,0.52,0.36)	(0.15,0.50,0.15)	(0.14,0.49,0.00)	(0.14,0.50,-0.19)	(0.14,0.50,-0.40)
2	-0.5	1		(0.16,0.89,0.65)	(0.16,0.89,0.45)	(0.16,0.89,0.25)	(0.16,0.89,0.05)	(0.16,0.91,0.00)
		2	(0.16,0.89,0.75)	(0.16,0.90,0.65)	(0.16,0.88,0.45)	(0.16,0.88,0.25)	(0.16,0.83,0.05)	(0.15,0.83,-0.07)
		3		(0.16,0.90,0.65)	(0.16,0.88,0.44)	(0.16,0.89,0.25)	(0.15,0.84,0.05)	(0.15,0.84,-0.09)
4	0.5	1		(0.14,0.17,0.06)	(0.14,0.10,0.00)	(0.14,0.08,-0.18)	(0.14,0.08,-0.38)	(0.14,0.08,-0.59)
		2	(0.14,0.17,0.16)	(0.14,0.18,0.06)	(0.14,0.14,-0.06)	(0.14,0.14,-0.27)	(0.14,0.13,-0.47)	(0.14,0.14,-0.69)
		3		(0.14,0.18,0.06)	(0.14,0.13,-0.11)	(0.14,0.14,-0.31)	(0.14,0.15,-0.50)	(0.14,0.14,-0.70)
4	0	1		(0.14,0.27,0.13)	(0.14,0.26,0.00)	(0.14,0.26,-0.15)	(0.14,0.25,-0.35)	(0.14,025,-0.56)
		2	(0.14,0.27,0.23)	(0.14,0.27,0.13)	(0.14,0.27,-0.04)	(0.14,0.25,-0.22)	(0.14,0.26,-0.43)	(0.14,0.25,-0.63)
		3		(0.14,0.26,0.13)	(0.14,0.28,-0.07)	(0.14,0.25,-0.24)	(0.14,0.25,-0.45)	(0.14,0.25,-0.65)
4	-0.5	1		(0.15,0.47,0.29)	(0.15,0.47,0.09)	(0.15,0.50,0.00)	(0.15,0.50,-0.18)	(0.15,0.50,-0.39)
		2	(0.15,0.47,0.39)	(0.15,0.47,0.29)	(0.15,0.48,0.10)	(0.15,0.47,-0.07)	(0.15,0.45,-0.27)	(0.15,0.45,-0.47)
		3		(0.15,0.47,0.29)	(0.15,0.43,0.11)	(0.15,0.47,-0.09)	(0.15,0.46,-0.28)	(0.15,0.46,-0.48)

This table reports the ratio of optimal consumption, investment in totally liquid stock and investment in restricted stock for an investor with varying fractions of wealth held in the form of stock that is restricted for a period of τ years if short sale is permitted but there is 5% proportional transaction cost. The risk aversion coefficient is γ. The correlation between liquid stock and restricted stock is ρ. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate.

Table 3. Optimal Consumption-Portfolio Rates with Illiquid Wealth: Short Sale is not permitted

			Fraction of Illiquid Wealth					
γ	ρ	τ	**0%**	**10%**	**30%**	**50%**	**70%**	**90%**
2	0.5	1		(0.14,0.35,0.23)	(0.14,0.35,0.03)	(0.14,0.25,0.00)	(0.14,0.14,0.00)	(0.10,0.00,0.00)
		2	(0.14,0.35,0.33)	(0.14,0.35,0.23)	(0.14,0.34,0.03)	(0.14,0.25,0.00)	(0.13,0.16,0.00)	(0.05,0.02,0.02)
		3		(0.14,0.35,0.23)	(0.14,0.34,0.04)	(0.14,0.24,0.00)	(0.10,0.09,0.05)	(0.03,0.04,0.02)
2	0	1		(0.15,0.45,0.30)	(0.15,0.45,0.10)	(0.15,0.35,0.00)	(0.14,0.16,0.00)	(0.10,0.00,0.00)
		2	(0.15,0.45,0.40)	(0.15,0.45,0.30)	(0.15,0.45,0.10)	(0.14,0.36,0.00)	(0.13,0.17,0.00)	(0.05,0.03,0.02)
		3		(0.15,0.45,0.30)	(0.15,0.44,0.12)	(0.14,0.35,0.01)	(0.11,0.11,0.08)	(0.03,0.04,0.02)
2	-0.5	1		(0.15,0.44,0.31)	(0.15,0.44,0.11)	(0.15,0.35,0.00)	(0.14,0.16,0.00)	(0.10,0.00,0.00)
		2	(0.15,0.44,0.41)	(0.15,0.44,0.31)	(0.15,0.45,0.10)	(0.15,0.35,0.00)	(0.13,0.17,0.00)	(0.05,0.02,0.02)
		3		(0.15,0.44,0.31)	(0.15,0.42,0.13)	(0.14,0.32,0.04)	(0.11,0.09,0.10)	(0.04,0.04,0.03)
4	0.5	1		(0.14,0.17,0.06)	(0.14,0.09,0.00)	(0.14,0.01,0.00)	(0.13,0.00,0.00)	(0.10,0.00,0.00)
		2	(0.14,0.17,0.16)	(0.14,0.18,0.06)	(0.14,0.10,0.00)	(0.13,0.00,0.00)	(0.13,0.00,0.00)	(0.05,0.01,0.01)
		3		(0.14,0.17,0.06)	(0.14,0.11,0.00)	(0.13,0.03,0.00)	(0.10,0.05,0.02)	(0.04,0.01,0.01)
4	0	1		(0.14,0.27,0.13)	(0.14,0.26,0.00)	(0.14,0.25,0.00)	(0.14,0.16,0.00)	(0.10,0.00,0.00)
		2	(0.14,0.27,0.23)	(0.14,0.27,0.11)	(0.14,0.26,0.00)	(0.14,0.23,0.00)	(0.13,0.17,0.00)	(0.05,0.01,0.01)
		3		(0.14,0.27,0.13)	(0.14,0.28,0.00)	(0.13,0.21,0.00)	(0.10,0.08,0.04)	(0.04,0.02,0.02)
4	-0.5	1		(0.15,0.46,0.29)	(0.15,0.46,0.09)	(0.18,0.32,0.00)	(0.15,0.16,0.00)	(0.10,0.00,0.00)
		2	(0.15,0.46,0.39)	(0.15,0.45,0.29)	(0.15,0.45,0.11)	(0.15,0.35,0.00)	(0.13,0.17,0.00)	(0.05,0.03,0.02)
		3		(0.15,0.46,0.29)	(0.15,0.48,0.06)	(0.15,0.31,0.03)	(0.11,0.13,0.07)	(0.04,0.03,0.02)

This table reports the ratio of optimal consumption, investment in totally liquid stock and investment in restricted stock for an investor with varying fractions of wealth held in the form of stock that is restricted for a period of τ years if no short sale is permitted. The risk aversion coefficient is γ. The correlation between liquid stock and restricted stock is ρ. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate.

Table 4. Utility Indifference Discount of Restricted Stock: Proportional Short Sale Transaction Cost

			Fraction of Illiquid Wealth					
γ	ρ	τ	**0% (%)**	**10% (%)**	**30% (%)**	**50% (%)**	**70% (%)**	**90% (%)**
2	0.5	1		0.00	0.00	0.45	1.43	2.20
		2	0.00	0.02	0.08	1.31	2.29	2.86
		3		0.00	0.41	1.96	2.80	3.26
2	0	1		0.00	0.00	0.04	0.81	1.72
		2	0.00	0.00	0.00	0.51	1.72	2.42
		3		0.16	0.20	1.33	2.32	2.88
2	-0.5	1		0.00	0.00	0.00	0.00	0.26
		2	0.00	0.00	0.00	0.00	0.58	1.36
		3		0.00	0.00	0.57	1.61	2.12
4	0.5	1		0.00	1.00	2.50	3.16	3.53
		2	0.00	0.01	2.01	3.12	3.59	3.86
		3		0.18	2.61	3.45	3.83	4.04
4	0	1		0.00	0.34	2.03	2.83	3.27
		2	0.00	0.04	1.24	2.66	3.28	3.62
		3		0.12	2.50	3.07	3.56	3.84
4	-0.5	1		0.00	0.00	0.42	1.68	2.39
		2	0.00	0.03	0.11	1.40	2.38	2.93
		3		0.07	0.86	2.12	2.88	3.31

The table reports the utility indifference discount of restricted stock with respect to its non-restricted stock when short sale is permitted but there is a 5% proportional short sale transaction cost. The discount ratio d_t is computed by $d_t = \frac{W_{2t} - H_t}{W_{2t}}$, where H_t is the utility indifference price of W_{2t} and satisfies $\tilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_{1t} + H_t\right)$.

Table 5. Utility Indifference Discount of Restricted Stock: Short Sale is not Permitted

			Fraction of Illiquid Wealth					
γ	ρ	τ	**0% (%)**	**10% (%)**	**30% (%)**	**50% (%)**	**70% (%)**	**90% (%)**
2	0.5	1		0.00	0.00	0.45	1.45	4.41
		2	0.00	0.00	0.07	1.31	3.64	30.56
		3		0.00	0.41	2.80	11.76	56.94
2	0	1		0.00	0.00	0.30	2.05	6.15
		2	0.00	0.00	0.07	1.47	5.47	33.18
		3		0.00	0.43	3.93	15.50	59.17
2	-0.5	1		0.00	0.00	0.41	2.98	8.15
		2	0.00	0.00	0.06	2.24	7.77	36.09
		3		0.00	0.91	5.98	19.85	61.07
4	0.5	1		0.00	0.99	3.37	5.89	10.64
		2	0.00	0.02	2.59	7.36	12.29	50.74
		3		0.08	4.85	11.99	23.71	73.75
4	0	1		0.00	0.34	2.77	5.62	11.81
		2	0.00	0.06	1.41	6.36	12.31	52.20
		3		0.01	3.39	10.97	25.34	74.18
4	-0.5	1		0.00	0.00	3.47	5.93	14.93
		2	0.00	0.04	0.55	4.42	13.59	55.80
		3		0.00	3.43	11.92	29.62	76.21

The table reports the utility indifference discount of restricted stock with respect to its non-restricted stock if short sale is not permitted. The discount ratio d_t is computed by $d_t = \frac{W_{2t} - H_t}{W_{2t}}$, where H_t is the utility indifference price of W_{2t} and satisfies $\tilde{V}_t\left(W_{1t}, W_{2t}\right) = V_t\left(W_{1t} + H_t\right)$.

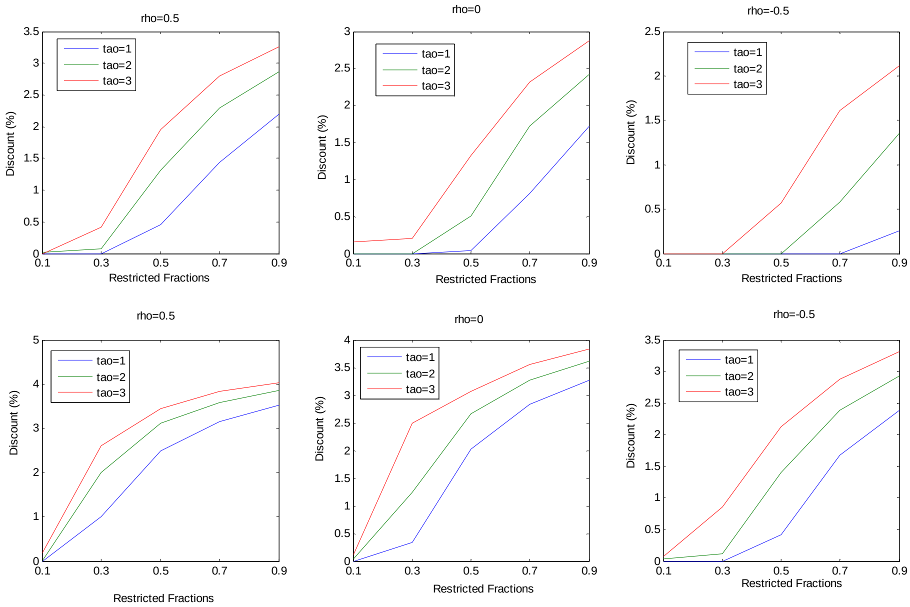

Fig.1. This figure plots the utility indifference discounts of restricted stock value with respect to different correlation as a function of the fraction of restricted wealth when short sale is permitted but there is a 5% proportional short sale transaction cost. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate. In the top panel, risk aversion coefficient γ =2; in the bottom panel, γ =4.

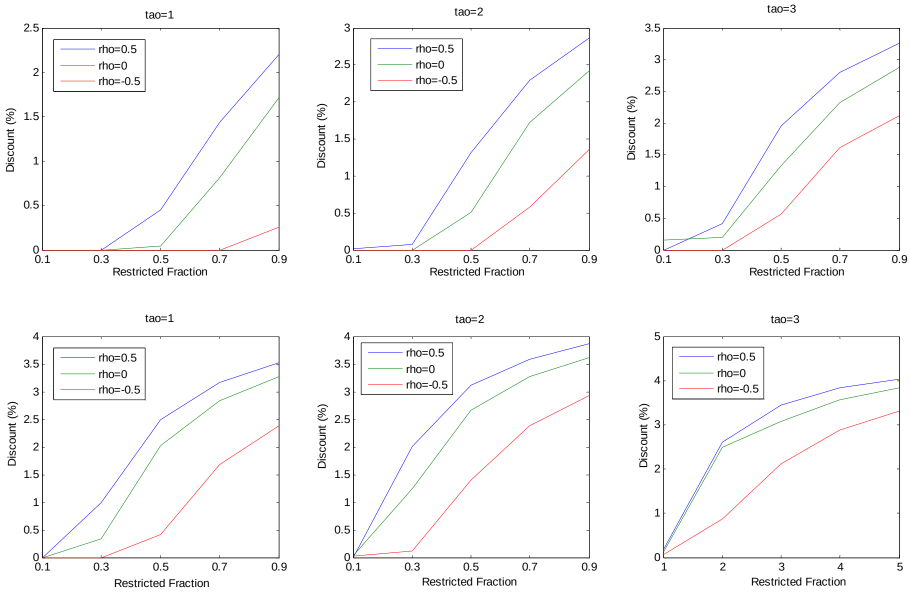

Fig 2. This figure plots the utility indifference discounts of restricted stock value with respect to different restricted terms as a function of the fraction of restricted wealth when short sale is permitted but there is a 5% proportional short sale transaction cost. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate. In the top panel, risk aversion coefficient γ =2; in the bottom panel, γ =4.

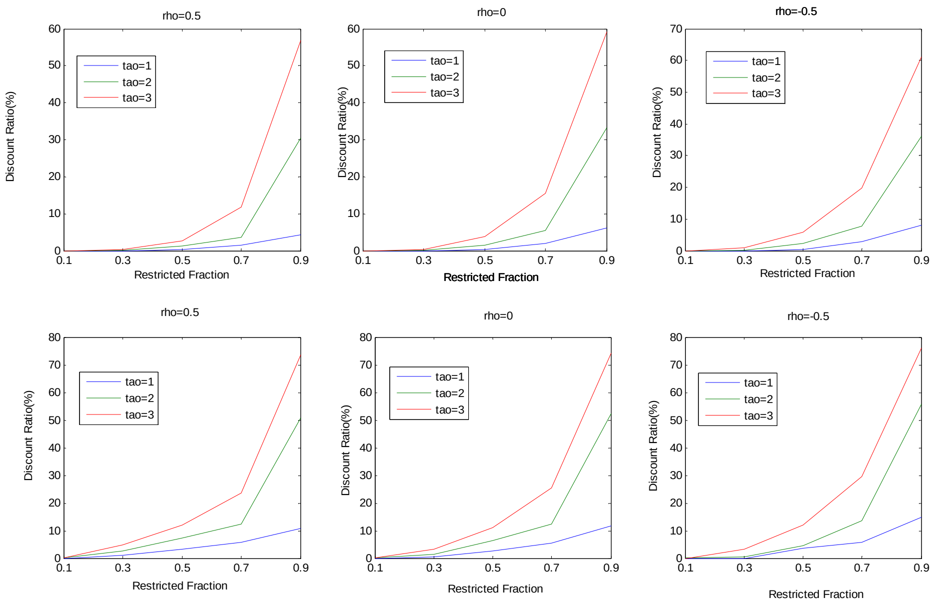

Fig.3. This figure plots the utility indifference discounts of restricted stock value with respect to different correlation as a function of the fraction of restricted wealth if no short sale is permitted. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate. In the top panel, risk aversion coefficient $\gamma=2$; in the bottom panel, $\gamma=4$.

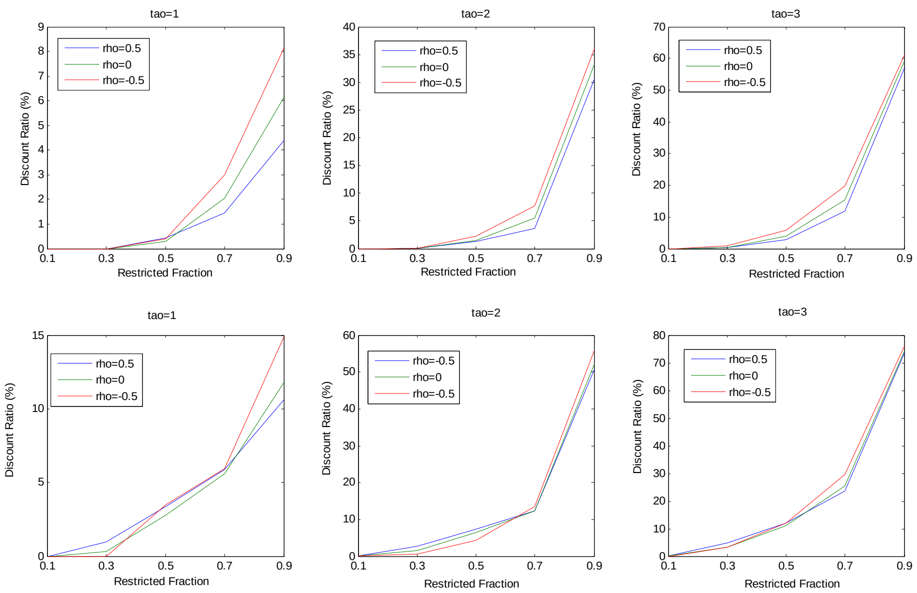

Fig 4. This figure plots the utility indifference discounts of restricted stock value with respect to different restricted terms as a function of the fraction of restricted wealth if no short sale is permitted. The riskless rate is 5%, the volatility of liquid stock and restricted stock are 25% and 30%, the risk premium on the liquid stock and restricted stock are 8% and 10%, and the investment horizon is 8 years. The rate of time preference equals the riskless rate. In the top panel, risk aversion coefficient γ =2; in the bottom panel, γ =4.