Dynamic risk aversion, stochastic discount factor and asset pricing

Zhenlong Zheng and Hai Lin*

Abstract: The paper analyzes the consumption-based asset-pricing problems on a dynamic risk aversion base. Different from Campbell and Cochrane (1998) and Brandt and Wang (2001), which both hypothesize the steady state, this paper suppose that the dynamic risk aversion follows a unit root process of. Also, this paper does not suppose the relevant factors of the forming of consumption habit. So the result is a general form of the relationship between the asset pricing and dynamic risk aversion. This generalization can help to explain the problems of equity premium puzzle and the people's decision in a whole economic condition. Finally, this paper shows some econometric methods for future empirical tests of this model.

Keywords: Dynamic risk aversion, Stochastic discount factor, Asset pricing

Since the foundation of Capital Asset Pricing Model (CAPM) by Markowitz (1959), Sharpe (1964) and Lintner (1965), asset pricing has always been one focus of debate and research. Black (1972) derived a more general version of CAPM. Since CAPM is concluded from one risk factor analysis of one-period optimization, it is reasonable to say that it is too simple to explain the capital world[1]. With the consideration of more than one risk factor, Ross (1976) introduces the Arbitrage Pricing Theory (APT). Considering multi-period decision problem, Merton (1973a) develops the Intertemporal Capital Asset Pricing Model (ICAPM) based on consumption. From then on, consumption-based CAPM has been developed very quickly to get the conclusion of stochastic discount factor. Campbell (2000) makes a review of asset pricing by a stochastic discount factor, and Cochrane (2000) includes all types of asset pricing in a general framework of stochastic discount factors, which are related to the utility functions with consumption[2].

Under the traditional hypothesis of the utility function, relative risk aversion is a constant. Mehra and Prescott (1985) find the equity premium puzzle under this condition. Brennan and Yihong Xia (2000) develop a dynamic general equilibrium model to solve the puzzle. Under the condition of time varying risk aversion, Campbell and Cochrane (1998) explain the puzzle on a habit-formed consumption base. Brandt and Wang (2001) introduce the factor of inflation on the base of Campell and Cochrane (1998). Normandin and Pascal (1998) explain the puzzle by introducing a taste shock into the consumption.[3]

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[1] One of which is "small company effect".

[2] In general form, it is $m_{t+1} = b\,\dfrac{u'(c_{t+1})}{u'(c_t)}$, m_{t+1} is the stochastic discount factor, $u(c)$ is the utility function.

[3] The other papers trying to explain the equity premium puzzle include the hypothesis of different distribution, such as Fama and French (1988), Heaton (1993); and the consideration of market imperfections, such as Heaton and Lucas

But Campbell and Cochrane (1998) and Brandt and Wang (2001) both suppose there exists one steady state for representative agent and the log relative aversion follows AR (1) mean reversion process. In my opinion, the representative agent doesn't know the steady state and doesn't concern the steady state either. He only cares for the uncertainty of the world and only changes with uncertainty. If the world is certain, he will make optimal decision to make risk aversion constant. So it is reasonable to hypothesize the log relative risk aversion follows a unit root process. The AR (1) coefficient of near 1 in the two papers also shows proof for the hypothesis of unit root process. This unit root process method doesn't need the hypothesis of the steady state[4] and can make the estimation far easier without losing the explaining ability.

In the field of risk aversion research, Pratt (1964) made an important research on the measure of risk aversion and its character. Amihud (1980) divide the general risk averse into consumption part and volatility part and analyzed their respective characters. Stapleton and Subrahmanyam (1990) study the relationship between risk aversion and the intertemporal behavior of asset prices and got the necessary and sufficient conditions of the random walk of asset price. Zeckhauser and Keeler (1970) show another kind of risk aversion related to the size, which is due to the marginal law in essence. Green and Srivastava (1985) do a good research on the relationship between risk aversion and arbitrage demonstrating that no arbitrage not only means a positive price of payoff state[5], but also can show the positive price of consumption state. Using the method of Merton (1990), Sundaresan (1983) analyzes the relationship between the constant absolute risk aversion and the equilibrium interest rate. Merton (1980) and Klock and Philips (1999) derive the relationship between the price volatility and risk averse.

In this paper, we will make a generalization of dynamic risk aversion on the base of habit-formed consumption-based CAPM, thus can explain the equity premium puzzle in a general way. The paper is constructed as follows: section one is the introduction of the problem; in section two, we set up the model; in section three, we make the general analysis; section four is a summary analysis of the econometric methods; and section five is a simple conclusion.

I. Model

Suppose a representative agent wants to maximize the lifetime utility function

$$\sum_{t=0}^{\infty} b^t U(C_t - X_t),$$

Subject to the condition,

$$W_{t+1} = (W_t - C_t) R_{t+1},$$

Where W denotes the wealth state, C denotes the consumption level, R denotes asset return. The utility function is

$$U(C_t - X_t) = \frac{(C_t - X_t)^{1-a} - 1}{1-a}, \text{ if } a \succ 1,$$

(1995).

[4] In Brandt and Wang (2001), the steady relative risk aversion is supposed to equal the initial relative risk aversion, which is unreasonable.

[5] This can also refer to Cochrane (2000).

$$U(C_t - X_t) = \ln(C_t - X_t), \text{ if } \alpha = 1.$$

In this utility function, X_t denotes the agent's habit, which is a function of history data, is exogenous and changes with time. Then $\frac{\partial X_t}{\partial C_t} = 0$.

A. The Dynamic Optimal Problem

To get the dynamic optimal decisions, we can construct the Bellman dynamic function

$$V(W_t, C_t) = U(C_t - X_t) + bE_t V((W_t - C_t)R_{t+1}, C_{t+1}).$$

The first order condition is

$$U'(C_t - X_t) = bE_t R_{t+1} V_1((W_t - C_t)R_{t+1}, C_{t+1}).$$

The envelope condition is

$$V_1(W_t, C_t) = U'(C_t - X_t).$$

So

$$U'(C_t - X_t) = bE_t R_{t+1} U'(C_{t+1} - X_{t+1}).$$

$$E_t\left(b \frac{U'(C_{t+1} - X_{t+1})}{U'(C_t - X_t)} R_{t+1}\right) = 1,$$

The stochastic discount factor M_{t+1} is written as

$$M_{t+1} = b \frac{U'(C_{t+1} - X_{t+1})}{U'(C_t - X_t)},$$

Which is different from the traditional hypothesis.

The N-period stochastic discount factor is

$$M_{t+N} = b^N \frac{U'(C_{t+N} - X_{t+N})}{U'(C_t - X_t)}. \tag{1}$$

And relative risk aversion is

$$RRA_t = -\frac{C_t U''(C_t - X_t)}{U'(C_t - X_t)} = \alpha \frac{C_t}{C_t - X_t} = \frac{\alpha}{S_t}, S_t = \frac{C_t - X_t}{C_t} \tag{2}$$

Log relative risk aversion

$$g_t = \ln RRA_t = \ln \alpha - \ln\left(1 - \frac{X_t}{C_t}\right) \approx \ln \alpha + \frac{X_t}{C_t}$$

During the first period, the agent has no idea about the habit X, so we suppose

$$g_0 = \ln a \ .$$

So log relative risk aversion changes with the habit-consumption ratio $\frac{X_t}{C_t}$.

B. Unit Root Process and New Information

In general, people tend to keep their old traditions and like to smooth their consumption actions during their life. So if there is no uncertainty, people will make the optimal decision to keep the habit-consumption ratio constant. The change of habit-consumption ratio is related to the uncertainty. And the uncertainty is also referred to the future, which means that the future will or will not be what we expect according to the available information. With time passing, our expectations will be justified by the reality. If the reality is the same as our expectations, we will keep our actions. And if the reality differs from our expectations, then it will include some new information and we should adjust our optimal decisions according to it. In this way, the habit-consumption ratio follows a unit root process; the log relative risk aversion also follows a unit root process.

$$g_{t+1} = g_t - e_{t+1}, g_0 = \ln a \ ,$$

e_{t+1} is a martingale process, denoting the new information.

The minus demonstrates the reverse movement of the relative risk aversion with the new information. When the news is good, risk aversion will decrease. When the news is bad, risk aversion will increase.

This hypothesis differs from Campbell and Cochrane (1998)[6] in two ways. First, we did not suppose the steady state, since the agent doesn't know where the state is. Neither does him concern it. He only makes the dynamic optimal decision according the available information. Secondly, they suppose the AR (1) process, but we suppose the unit root process. But their AR (1) coefficient being near 1 shows the similarity. In a word, this hypothesis is much more simple since it does not want to make supposition about the steady state.

Furthermore, Campbell and Cochrane (1998) hypothesize that e_{t+1} is the new information about the consumption growth. In Brandt and Wang (2001), it is the new information about consumption growth and inflation. In our paper, we will generalize the supposition, hypothesizing e_{t+1} is the new information about the relevant factors.

Let G be a $N \times 1$ vector, denoting the N relevant factors, H be a $N \times 1$ vector, denoting the sensitivity function. Then

$$e_{t+1} = H_t^{'}(G_{t+1} - E_t G_{t+1}) \ ,$$

[6] In their paper, the log relative risk aversion follows $g_{t+1} = \bar{g} + f(g_t - g) + e_{t+1}, \bar{g}$ denotes the steady sate relative risk aversion, $f \prec 1$ means AR coefficient. But they also suppose that f should be near 1 to make riskfree rate stable.

$H_t^{'}$ is the transpose of H_t. In general, H_t is related to g_t, so

$$g_{t+1} = g_t - H_t(g_t)'(G_{t+1} - E_t G_{t+1})$$

$E_t G_{t+1}$ can be estimated by $VAR(m)$ process.

$$G_t = A + \sum_{i=1}^{m} B_i G_{t-i} + x_t$$

A is a $N \times 1$ vector, B_i is a $N \times N$ matrices.

Ⅱ. A General Analysis

We can derive the general relationship between the relevant factors from the above-mentioned model.

A. Stochastic Discount Factor, Dynamic Risk Aversion and Consumption Growth

From (1),

$$M_{t+N} = b^N \frac{U'(C_{t+N} - X_{t+N})}{U'(C_t - X_t)} = b^N \frac{(C_{t+N} - X_{t+N})^{-a}}{(C_t - X_t)^{-a}}$$

$$= b^N \exp(a[\ln(C_t - X_t) - \ln(C_{t+n} - X_{t+n})]) \quad (3)$$

From (2), we get

$$\ln(C_t - X_t) = \ln a + \ln C_t - g_t .$$

Then (3) becomes

$$b^N \exp(a(g_{t+N} - g_t - (\ln C_{t+N} - \ln C_t))) \quad (4)$$

Suppose

$$g_{t+1} = \ln C_{t+1} - \ln C_t ,$$

We get

$$M_{t+N} = b^N \exp(a(g_{t+N} - g_t - (g_{t+1} + g_{t+2} + ... + g_{t+N}))) \quad (5)$$

$$= b^N \exp(-a(\sum_{i=1}^{N} e_{t+i} + \sum_{j=1}^{N} g_{t+j}) \quad (6)$$

It is the generalization of the relationship between stochastic discount factor, dynamic risk aversion and consumption growth.

Especially, when $N = 1$,

$$M_{t+1} = b \exp(a(g_{t+1} - g_t - g_{t+1})) = b \exp(-a(e_{t+1} + g_{t+1}))$$

From (5) and (6), we can know that the stochastic discount factor moves in the same direction with the change of risk aversion, move in reverse direction with the new information and consumption growth. These are reasonable with the economic view. When risk aversion increases, people like "now" and dislike "future" more and more. The future value will decrease then the stochastic discount increases. With consumption increasing, people will feel more and more optimistic about the future. The value of "future" will increase then the stochastic discount factor decreases.

B. Stochastic Discount Factor, Asset Pricing and Dynamic Risk Aversion

According to the stochastic discount factor model,

$$E_t(M_{t+1}R_{t+1}) = 1,$$

$$E_t R_{t+1} = \frac{1}{E_t M_{t+1}} - \frac{\mathrm{cov}_t(M_{t+1}, R_{t+1})}{E_t M_{t+1}} \tag{7}$$

To simplify (7), we can use Stein's Lemma[7]. Suppose the relevant factors and R_{t+1} are conditionally jointly normal,

$$\mathrm{cov}_t(M_{t+1}, R_{t+1}) = \mathrm{cov}_t(b\ \exp(-a(e_{t+1} + g_{t+1})), R_{t+1})$$

$$= E_t(b\ \exp(-a(e_{t+1} + g_{t+1})))(-a\ \mathrm{cov}_t(e_{t+1}, R_{t+1}) - a\ \mathrm{cov}_t(g_{t+1}, R_{t+1}))$$

So

$$E_t R_{t+1} = \frac{1}{E_t M_{t+1}} + a\ \mathrm{cov}_t(e_{t+1}, R_{t+1}) + a\ \mathrm{cov}_t(g_{t+1}, R_{t+1}) \tag{8}$$

Furthermore, since

$$\frac{1}{E_t M_{t+1}} = R_f^t \text{ [8]},$$

R_f^t means risk-free gross return.

$$E_t R_{t+1} = R_f^t + a\ \mathrm{cov}_t(g_{t+1}, R_{t+1}) + a\ \mathrm{cov}_t(e_{t+1}, R_{t+1}) \tag{9}$$

We get the generalization of the relationship between asset pricing, riskfree interest rate, consumption growth and dynamic risk aversion.

If

$$\mathrm{cov}_t(e_{t+1}, R_{t+1}) = 0,$$

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[7] Stein's Lemma states that if two random variables x and y are jointly normal, and f is a differential function of one variable, such that $E(f'(x))$ exists, then $\mathrm{cov}(f(x), y) = E(f'(x))\,\mathrm{cov}(x, y)$.

[8] Since $E_t(M_{t+1}R_f^t) = 1$, and M_{t+1} is independent from R_f^t, then $E_t M_{t+1} = \frac{1}{R_f^t}$.

We return to the standard consumption-based CAPM under the power utility function. a denotes the relative risk averse level. It cannot explain the existence of equity premium puzzle and risk free interest rate puzzle. But if we consider the dynamics of relative risk averse on a habit-formed utility function, we add the conditional covariance between the dynamic risk aversion and expected return. Then we can get reasonable R_f^t and a .

From (9) we get the counter-cycle decision of representative agent. When the covariance between consumption growth and expected return is more than zero, the asset's price moves in the same direction with the whole economic conditions. People dislike this kind of asset and will need more returns to make them invest in it. When the covariance is less than zero, the asset's price moves in the reverse direction with the whole economic conditions, people like this kind of asset and want to invest in it even if it can only give little returns.

Since g_{t+1} is also one of the relevant factors, and

$$\mathrm{cov}_t(g_{t+1}, R_{t+1}) = \mathrm{cov}_t(g_{t+1} - E_t g_{t+1}, R_{t+1})$$

(9) can be simplified as

$$E_t R_{t+1} = R_f^t + a\,\mathrm{cov}_t((H_t(g_t) + I)'(G_{t+1} - EG_{t+1}), R_{t+1})$$

I is a $N \times 1$ vector. The row on consumption factor is 1, others are 0.

III. Econometric Methods

From VAR (m) process of the relevant factors, we can get the parameters $y = (A, B_i), i = 1,2,...m$. With the series of the real data of G, we want to estimate $q = (b, a, H)$.

Then from the value and function of H_t , we can estimate the sensitivity of the risk aversion to every relevant factor.

A. Bonds

For the discounted bonds, bond's value at expiration is certain, and has no dividend[9], so we can make their final value equal one, then the price of bond with N periods to expiration should be $E_t M_{t+N}$ [10]. For this, we can estimate the parameter q by minimize the following equation:

$$\sum_{t=1}^{T}(P_t - P_t(q))' W \sum_{t=1}^{T}(P_t - P_t(q)) = Q(q) \qquad (10)$$

9 For those bonds with interest cash flow at certain time, since the interest is also certain, we can reduce the bond's price by the present value of the interest and get the equivalent discounted bond.

10 Since $E_t(M_{t+N} R_{t+N}) = E_t(M_{t+N} \times \frac{P_{t+N}}{P_t}) = \frac{E_t M_{t+N}}{P_t} = 1$, $E_t M_{t+N} = P_t = \frac{1}{E_t R_{t+N}}$.

P_t is a $K \times 1$ vector, denoting the real price of the bonds with periods of expiration from N1 to Nk.

$$P_t(q) = E_t(M_{t+N1}(q), M_{t+N2}(q), ... M_{t+Nk}(q) \mid y)$$

denotes the theoretical price on the conditional information. W is the weighted average matrice.

B. Stocks

The stock's future cash flow is uncertain, so we cannot use the same method as the bonds. We can use the general moments method (GMM) to estimate the parameter.

Since

$$E_t(M_{t+1}(q) R_{t+1}) = 1,$$

$$E_t(M_{t+1}(q) R_{t+1} \otimes Z_t) = 1 \otimes Z_t,$$

Z_t is a conditional instrument set[11], $\otimes$ is Kronecker product[12].

$$E_t((M_{t+1}(q) R_{t+1} - 1) \otimes Z_t) = 0,$$

So the unconditional mean

$$E((M_{t+1}(q) R_{t+1} - 1) \otimes Z_t) = 0,$$

Let $h_{t+1}(q) = (M_{t+1}(q) R_{t+1} - 1) \otimes Z_t$, we can estimate the parameter q by minimize

$$\left(\frac{1}{T}\sum_{t=1}^{T} h_{t+1}(q)\right)' W \left(\frac{1}{T}\sum_{t=1}^{T} h_{t+1}(q)\right) \tag{11}$$

Ⅳ. Conclusion

This paper make a simple but reasonable hypothesis on the dynamic risk aversion and derive a general relationship between the dynamic risk aversion, asset pricing and the stochastic discount factor. This generalization can help to explain the problems of equity premium puzzle and the people's decision in a whole economic condition. Finally, this paper shows some econometric methods for future empirical tests of this model.

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[11] For a detail discussion of the GMM method, please refer to Cochrane (2000).
[12] For a statement of the kronecker product, please refer to Green (1993).

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