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Prepayment and the Valuation of Mortgage-Backed Securities

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ABSTRACT

This paper puts forward a valuation framework for mortgage-backed securities. Rather than imposing an optimal, value-minimizing call condition, we assume that at each point in time there exists a probability of prepaying; this conditional probability depends upon the prevailing state of the economy. To implement our valuation procedure, we use maximum-likelihood techniques to estimate a prepayment function in light of recent aggregate GNMA prepayment experience. By integrating this empirical prepayment function into our valuation framework, we provide a complete model to value mortgage-backed securities.

GIVEN AN OPTIMAL, VALUE-MINIMIZING call policy, a mortgage should never be called when its market value is less than its call price. Similarly, a mortgage should be called if it is worth more than its call price. However, mortgagors often call their loans when the prevailing refinancing rate exceeds the contract rate on the loan (Dunn and McConnell (1981)). In addition, some mortgagors do not call their loans when the loan's contract rate exceeds the prevailing refinancing rate.

The purpose of this paper is to put forward a valuation framework for mortgage-backed securities consistent with these stylized facts associated with mortgage prepayments. The mortgagor's prepayment decision is integral to our valuation framework. However, we do not impose an optimal, value-minimizing call condition to price these securities. Rather, we assume that at each point in time there exists a probability of prepaying, this conditional probability depending upon the prevailing state of the economy. By integrating this prepayment function into our valuation framework, we provide a complete model to value mortgage-backed securities.

To implement our valuation procedures, we estimate a prepayment function given recent GNMA prepayment experience. We follow Green and Shoven (1986) by using a proportional-hazards model to estimate the influence of various explanatory variables or covariates on the mortgagor's prepayment decision. Distinct from Green and Shoven, we explicitly model the effects of seasoning, as well as investigating the influence of interest cost savings from refinancing. In addition, we also consider the effects of lagged refinancing rates, heterogeneity

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in mortgagors, and seasonality. We provide maximum-likelihood estimates of our proportional-hazards model given limited prepayment data over the period January 1978 to November 1987 for a number of GNMA thirty-year Single-Family pools.

The point of departure for our mortgage-backed security valuation model is Brennan and Schwartz's (1982, 1985) two-factor model for valuing default-free interest-dependent claims. This model assumes that all information about the term structure of interest rates can be summarized by two state variables: the instantaneous riskless rate of interest and the yield on a default-free consol bond. We add state variables underlying the posited prepayment function, thereby allowing us to integrate the prepayment function into this valuation framework, resulting in a complete model to value mortgage-backed securities. Monte Carlo simulation methods are used to solve the resultant second-order partial differential equation subject to the boundary and terminal conditions which characterize the particular mortgage-backed security.

We apply our valuation procedures to the pricing of default-free, fully amortizing mortgages. A majority of the mortgage loans which back GNMA securities are fully amortizing. We highlight the importance of prepayment behavior by comparing mortgage values assuming that prepayments occur according to our estimated prepayment function with mortgage values assuming an optimal, value-minimizing call policy and mortgage values assuming that prepayments occur according to FHA experience.

The plan of this paper is as follows. Since the mortgage prepayment function is central to our valuation procedures, Section I carefully details our proportional-hazards model. We discuss its maximum-likelihood estimation and investigate the significance of various covariates in influencing a mortgagor's prepayment decision given recent GNMA prepayment experience. Section II presents our mortgage-backed securities valuation model. We illustrate the application of the model to the valuation of default-free, fully amortizing mortgages in Section III. By integrating the estimated prepayment function into this valuation framework, we help explain a number of the stylized facts associated with the pricing of mortgages. Section IV presents our summary and conclusions.

I. Prepayment

A variety of economic, demographic, and geographic factors influence a mortgagor's prepayment decision. In this section, we model this propensity to prepay in light of actual GNMA prepayment experience. Statistical estimation of the resultant prepayment function allows us to investigate the significance of a number of these factors in influencing a mortgagor's prepayment decision. We first describe our prepayment data as this motivates both the specification of our prepayment function and its statistical estimation. We conclude this section with our empirical results.

A. Data

Our data are annualized monthly conditional prepayment rates over the period January 1978 to November 1987 for a number of GNMA thirty-year Single-

Table I

Characteristics of Sampled GNMA Single-Family Pools

We provide coupon rates, issue years, and remaining term-to-maturity of our sampled GNMA Single-Family pools. For each pool we also tabulate corresponding maximum and minimum annualized monthly prepayment rates over our sample period. Data were compiled by Salomon Brothers.

Coupon (%)	Issue Year	Remaining Term-to-Maturity (in years, as of December 1987)	Maximum Annualized Prepayment	Minimum Annualized Prepayment
7.5	1977	19.4	0.101	0.005
8.0	1976	18.8	0.111	0.007
8.0	1977	19.8	0.101	0.007
8.0	1978	20.2	0.104	0.004
8.25	1978	20.5	0.107	0.003
8.5	1976	18.2	0.140	0.012
9.0	1978	20.9	0.118	0.001
9.0	1979	21.3	0.112	0.003
9.5	1979	21.7	0.134	0.003
10.0	1979	21.9	0.183	0.002
11.0	1980	22.5	0.356	0.002
11.5	1980	22.7	0.384	0.002
11.5	1983	25.4	0.488	0.004
12.0	1983	25.6	0.512	0.006
12.5	1980	22.7	0.462	0.003
12.5	1983	25.9	0.552	0.004
13.0	1981	23.3	0.462	0.003
13.0	1982	24.8	0.626	0.001
13.0	1983	25.7	0.555	0.009
13.5	1981	23.4	0.522	0.002
14.0	1982	24.5	0.699	0.001
15.0	1982	24.1	0.467	0.003
16.0	1982	25.0	0.563	0.004
12.0	1984	26.3	0.468	0.003
12.5	1984	26.6	0.554	0.005
13.0	1984	26.8	0.551	0.001
13.5	1984	26.7	0.550	0.001

Family pools. The data are compiled by Salomon Brothers. Single-Family pools, which comprise the largest number of GNMA pools and have the largest outstanding balance, contain long-term fixed rate fully amortizing mortgages on residential properties. A conditional prepayment rate gives the proportion of principal outstanding at the beginning of a particular period that prepays during that period.

Table I characterizes our sample of Single-Family pools. The sample contains only those pools which, if issued prior to January 1978, have prepayment data available from January 1978 or, if issued subsequent to January 1978, have prepayment data available from their issue date. Coupons on the pools range from 7.5 percent to 16 percent, while their remaining terms until maturity as of December 1987 range from 18.2 years to 26.8 years. Table I also gives for each sampled pool its maximum and minimum annualized monthly prepayment rates. Prepayment rates varied considerably within our sample period, especially for high coupon pools.

We calculate monthly prepayment rates for each pool. A monthly prepayment rate, m, is obtained from a given annualized monthly prepayment rate, a, according to

$$m = 1 - (1 - a)^{1/12}. \tag{1}$$

Multiplying a pool's monthly prepayment rate by its outstanding balance as of the end of the preceding month gives the dollar amount of mortgages which prepaid during a particular month. In addition, we calculate the dollar amount prepaid prior to January 1978 for pools issued prior to that date. For these cases, prepayment occurs prior to our observation period. Our prepayment data also give the dollar amount outstanding of each pool as of the end of November 1987. Therefore, for each pool, there exists a possibility of prepayment beyond our observation period.

B. Prepayment Function

A prepayment function gives the probability of a mortgagor prepaying a mortgage during a particular period, conditional on the mortgage not having been prepaid prior to that period. By expressing this conditional probability as a function of various explanatory variables or covariates, we may assess statistically the significance of these covariates in influencing a mortgagor's prepayment decision.

Let T be a continuous random variable representing the time until prepayment of a mortgage, and let t denote its realization. Let $\underline{v} = (v_1, v_1, \cdots, v_s)$ be a vector of explanatory variables or covariates upon which the time until prepayment may depend, while $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_k)$ is a vector of parameters to be estimated. The prepayment function $\pi(t; \underline{v}, \underline{\theta})$ is defined by

$$\begin{aligned} \pi(t; \underline{v}, \underline{\theta}) &= \lim_{\Delta t \to 0^+} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} \\ &= \frac{f(t; \underline{v}, \underline{\theta})}{F(t; \underline{v}, \underline{\theta})}, \end{aligned} \tag{2}$$

where $F(t; \underline{v}, \underline{\theta})$ represents the survivor function

$$F(t; \underline{v}, \underline{\theta}) = P(T \geq t \mid \underline{v}, \underline{\theta}) \tag{3}$$

and $f(t; \underline{v}, \underline{\theta})$ is the probability-density function of T:

$$\begin{aligned} f(t; \underline{v}, \underline{\theta}) &= \lim_{\Delta t \to 0^+} \frac{P(t \leq T < t + \Delta t)}{\Delta t} \\ &= -\frac{dF(t)}{dt}. \end{aligned} \tag{4}$$

The prepayment function $\pi(t; \underline{v}, \underline{\theta})$ specifies the instantaneous rate of prepayment at $T = t$ conditional upon the mortgage not having been prepaid prior to time t.

Our GNMA prepayment data do not include mortgages of every possible age. Recall that the minimum term to maturity as of December 1987 of a pool included in our sample is 18.2 years. Consequently, our posited prepayment function must be sufficiently flexible so as to allow inferences on the prepayment of mortgages with maturities differing from observed maturities. By contrast, Green and Shoven's sample of 3938 mortgages issued by large California Savings and Loan Associations over the period 1947–1976 includes mortgages of every possible age. Green and Shoven then employ observations on mortgages of a particular age to make inferences on the probability of prepaying at that age.

We model the prepayment function by a proportional-hazards model:

$$\pi(t;\underline{v}, \underline{\theta}) = \pi_0(t; \gamma, p)\exp(\underline{\beta}\underline{v}), \tag{5}$$

where the base-line hazard function $\pi_0(t; \gamma, p)$ is given by the log-logistic hazard function

$$\pi_0(t; \gamma, p) = \frac{\gamma p(\gamma t)^{p-1}}{1 + (\gamma t)^p}. \tag{6}$$

The base-line hazard function measures the probability of prepayment under homogeneous conditions, $\underline{v} = \underline{0}$. The log-logistic hazard function admits a variety of relationships between the probability of prepayment and the age of the mortgage. In particular, for $p > 1$, the probability of prepayment increases from zero to a maximum at

$$t^* = (p - 1)^{1/p}/\gamma \tag{7}$$

and decreases to zero thereafter. For this specification, there exists a mortgage age at which the probability of prepayment is maximum. This is consistent with the observation that, all other things being equal, conditional prepayment rates are typically low in the early years of a mortgage, increase as the age of the mortgage increases, and then diminish with further seasoning (Askin (1985)).[1] By modeling the base-line hazard function, as opposed to employing an arbitrary specification, we incorporate our prior knowledge of seasoning's influence on mortgage prepayments. This improves our prepayment function's predictive ability and, as a result, improves the pricing accuracy of our resultant mortgage-backed securities valuation model.

However, the probability of prepayment does not depend solely upon a mortgage's age. Our posited prepayment function, expression (5), takes into account the fact that various explanatory variables, $\underline{v}$, influence the prepayment decision. These covariates may include the cost of refinancing, demographic variables characterizing mortgagors, and geographic factors particular to local mortgage markets. According to the proportional-hazards model, these explanatory variables have an equiproportional impact at all mortgage ages. That is, if covariates $\underline{v}$ make prepayment more likely at a particular mortgage age, they make prepay-

[1] FHA survivalship tables indicate that eventually prepayments increase with increasing mortgage age. However, in constructing these tables, mortgage loans issued at different points in time with different contract rates are combined, so that the incentives to prepay mortgages with the same number of elapsed years are vastly different. Furthermore, these tables are based on few, if any observations of mortgages over twenty years old. For example, the 1984 FHA survivalship table is based on mortgages at most fifteen years old.

ment more likely at any other mortgage age. The vector of regression coefficients, $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_s)$, measures the effect of the covariates upon the prepayment decision.

To empirically implement our prepayment function requires that we specify explicitly the covariates influencing the mortgagor's prepayment decision. The fact that we have aggregate prepayment data precludes our use of demographic or geographic explanatory variables. We therefore restrict our attention to covariates with observed values that are not particular to a specific mortgagor or geographic location and, further, can be embedded within a partial-equilibrium valuation framework.

A mortgagor's prepayment decision is dependent upon the relationship between rates at which the mortgage may be refinanced and the contract rate on the mortgage, c. If an available refinancing rate is less than the contract rate, there exists an incentive to prepay. We proxy refinancing rates by long-term Treasury rates, l.[2] Mortgages included in GNMA Single-Family pools have FHA, VA, or FmHA default guarantees and, as such, may be viewed as long-term default-free securities. We rely on long-term Treasury rates in our empirical analysis since they do not vary across geographical locations. To investigate the effect of refinancing costs on the mortgagor's prepayment decision, we employ the covariate $v_1(t)$, where

$$v_1(t) = c - l(t - s), \quad s \geq 0. \tag{8}$$

Since preparing a mortgage requires time, current prepayment decisions may be influenced by past refinancing rates. Our later empirical analysis will determine the extent to which lagged refinancing costs, $s > 0$, affect current prepayment decisions. Notice that $v_1(t) \gtreqless 0$ if and only if $c \gtreqless l(t - s)$. If $v_1(t) > 0$, there exists an incentive to prepay which, by the proportional-hazards model, is assumed to be equiproportional across mortgages of all ages. The larger $v_1(t)$ is, the greater is this incentive to prepay. We therefore expect that $\beta_1 > 0$.

To allow the possibility that prepayments may further accelerate when refinancing rates are sufficiently lower than the mortgage's contract rate, we also consider the covariate:

$$v_2(t) = (c - l(t - s))^3, \quad s \geq 0. \tag{9}$$

The further acceleration in prepayments reflects transaction costs which make prepayment less profitable when interest cost savings are small. This covariate allows the possibility that, for sufficiently low refinancing rates, the resultant prepayment speed may be greater than the prepayment speed predicated by the difference $c - l(t - s)$ only. Since for $c > l(t - s)$ there is an incentive to prepay, we expect $\beta_2 > 0$.

The proportion of a GNMA Single-Family pool previously prepaid may also influence the probability of further prepayments. With greater past prepayment

[2] Long-term Treasury rates approximate prevailing long-term interest rate conditions and hence approximate refinancing rates. Since our goal is to integrate the empirical prepayment function into a partial-equilibrium valuation framework, we do not employ prevailing mortgage rates as refinancing rates. To do so would take as given the price of a security we wish to value. Further, the empirical fit of the prepayment model is not improved by using prevailing mortgage rates.

activity, mortgagors less prone to prepay remain in the pool. We define this covariate, $v_3(t)$, by

$$v_3(t) = \ln(AO_t/AO_t^*), \tag{10}$$

where AO_t represents the dollar amount of the pool outstanding at time t, while AO_t^* is the pool's principal which would prevail at t in the absence of prepayments but reflecting the amortization of the underlying mortgages. The greater the amount previously prepaid, and hence the smaller $v_3(t)$ is, the less likely are further prepayments across all mortgage ages. We therefore expect that $\beta_3 > 0$.

Finally, seasonality may influence prepayment activity. We represent this covariate by the dummy variable, $v_4(t)$, defined by

$$v_4(t) = \begin{cases} +1 & \text{if} \quad t = \text{May–August}, \\ 0 & \text{if} \quad t = \text{September–April}. \end{cases} \tag{11}$$

More residential real estate transactions occur in the spring and summer than in the fall and winter. Hence, we expect greater prepayment activity across all mortgage ages in the spring and summer, implying that $\beta_4 > 0$.

C. Maximum-Likelihood Estimation

Given the assumed prepayment function and available GNMA prepayment data, we can estimate statistically the significance of seasoning as well as the posited covariates in influencing a mortgagor's prepayment decision. We employ the method of maximum likelihood. That is, we determine the prepayment function's parameter values that are most plausible in light of the observed prepayment activity.

The likelihood function gives the probability of the observed GNMA prepayment data conditional upon parameter values of the assumed prepayment function, $\underline{\theta} = (\gamma, p, \beta_1, \beta_2, \beta_3, \beta_4)$. It is important to recognize that the assumed prepayment function involves time-varying covariates, $\underline{v}(t) = (v_1(t), v_2(t), v_3(t), v_4(t))$. In general, the entire path of a time-varying covariate influences the probability of prepayment. (See Kalfleisch and Prentice (1980).) For example, the probability of a mortgagor prepaying a mortgage today may depend not only upon the past history of mortgage rates but also upon the mortgagor's expectation of the future course of mortgage rates. However, for empirical tractability, we follow Green and Shoven (1986) and assume that a mortgagor considers only current values of the covariates, as opposed to their past or future values, in deciding whether to prepay. Nevertheless, current values of certain covariates do provide information regarding a mortgagor's past environment. In particular, v_1 and v_2 include lagged refinancing costs while the proportion of a pool outstanding, v_3, summarizes the history of past prepayments.

For the jth mortgage within the ith pool, $i = 1, \cdots, I$, $j = 1, \cdots, J_i$, we have data of the form $(\underline{\delta}_{ij}, t_{ij}, h(t_{ij}))$. Here $\underline{\delta}_{ij} = (\delta_{ij0}, \delta_{ij1}, \cdots, \delta_{ijK}, \delta_{ijK+1})$ is a vector of indicator variables where $\delta_{ij0} = 1$ only if the jth mortgage within the ith pool is prepaid prior to the beginning of our observation period (left-censored) and $\delta_{ij0} = 0$ otherwise; $\delta_{ijk} = 1$ only if the jth mortgage within the ith pool is prepaid during the kth period ($k = 1, \cdots, K$) and $\delta_{ijk} = 0$ otherwise; while $\delta_{ijK+1} = 1$ only if the jth mortgage within the ith pool is not prepaid prior to the end of our observation period (right-censored) and $\delta_{ijK+1} = 0$ otherwise. The variable t_{ij} represents the number of months from the issue date of the jth mortgage within the ith pool to the beginning of our observation period, to the end of our observation period, or to the mortgage's prepayment month according to whether this mortgage is left-censored, right-censored, or not censored. As before, $\underline{v}(t_{ij})$ is a vector of the s posited covariates evaluated at t_{ij}.

Assuming the conditional independence of prepayment decisions across time and across mortgages (given the posited covariates $\underline{v}(t_{ij})$ $\mathrm{i} = 1, \cdots, \mathrm{I}$, $\mathrm{j} = 1, \cdots, \mathrm{J}_i$), the resultant logarithmic likelihood function is given by[3]

$$\begin{aligned}
\ln L(\underline{\theta} \mid \underline{\delta}_0, \underline{\delta}_1, \cdots, \underline{\delta}_K, \underline{\delta}_{K+1}) \\
= \sum_{i=1}^{I} \sum_{j=1}^{J_i} \Big[& \delta_{ij0} \ln(1 - (1 + \gamma t_{ij})^p) - \exp(\textstyle\sum_{h=1}^{s} \beta_h v_h(t_{ij})) \\
& + \textstyle\sum_{k=1}^{K} \delta_{ijk} (\ln \gamma + \ln p + (p-1) \ln(\gamma t_{ij}) - \ln(1 + (\gamma t_{ij})^p) \\
& + \textstyle\sum_{h=1}^{s} \beta_h v_h(t_{ij}) - \exp(\sum_{h=1}^{s} \beta_h v_h(t_{ij})) \ln(1 + (\gamma t_{ij})^p)) \\
& - \delta_{ijK+1} \exp(\textstyle\sum_{h=1}^{s} \beta_h v_h(t_{ij})) \ln(1 + (\gamma t_{ij})^p) \Big],
\end{aligned} \tag{12}$$

where $\underline{\delta}_k = (\delta_{ijk})$, $i = 1, \cdots, I$, $j = 1, \cdots, J_i$, $k = 1, \cdots, K$.

Notice that the logarithmic likelihood function requires as input the *number* of mortgages which prepaid during a particular time period. However, as mentioned earlier, we have data only on the *dollar values* of mortgages which prepaid. In order to operationalize our estimation procedures, we assume that all mortgage principals are equal. The values of the resultant maximum-likelihood estimates are unaffected by the assumed common principal; however, the statistical significance of these estimates, as given either by the square root of the corresponding diagonal elements of the negative of the inverse of the resultant logarithmic likelihood function's Hessian matrix or the corresponding likelihood ratio test statistic, is indeterminate.[4] To the extent that we do not know the number of

[3] If a mortgage is left-censored, that is, prepaid prior to the beginning of our observation period, its contribution to the likelihood function is $1 - F(t; \underline{v}, \underline{\theta})$. If a mortgage is prepaid within our observation period, its contribution to the likelihood function is $f(t; \underline{v}, \underline{\theta})$. If a mortgage is potentially prepayable beyond our observation period, its contribution to the likelihood equation is $F(t; \underline{v}, \underline{\theta})$.

[4] To see this, notice that, by increasing by a factor of $\omega > 0$ the number of mortgages which prepaid during a particular time period, we increase the corresponding logarithmic likelihood function by a factor of ω:

$$\ln L(\underline{\theta} \mid \omega\underline{\delta}_0, \cdots, \omega\underline{\delta}_{k+1}) = \omega \ln L(\underline{\theta} \mid \underline{\delta}_0, \cdots, \underline{\delta}_{k+1}).$$

Clearly, the parameter values which maximize the logarithmic likelihood function are unaffected by the choice of ω. However, since the value of the logarithmic likelihood function increases by a factor of ω, likelihood ratio statistics to test the statistical significance of these maximum-likelihood estimates will increase by a factor of ω. Also, as partial derivatives of the logarithmic likelihood function increase by a factor of ω, corresponding asymptotic standard errors decrease by a factor of the square root of ω. Intuitively, by increasing ω, we increase the number of mortgages which prepaid

Table II

Maximum-Likelihood Estimates of Prepayment Function

We provide maximum-likelihood estimates, with jackknifed standard deviation estimates in parentheses, of the prepayment function $\pi = ((\gamma p(\gamma t)^{p-1})/(1 + (\gamma t)^p))\exp(\sum_{h=1}^{s} \beta_h v_h)$. The estimated age at which the base-line hazard function is maximized, $t^* = (p - 1)^{1/p}/\gamma$, is also tabulated. We also provide sample statistics—$Q3$, the upper quantile, the median, and $Q1$, the lower quantile—of the resultant distribution of prepayment errors.

	A: With Seasonality	B: Without Seasonality
γ	0.01496(0.00110)	0.01572(0.00187)
p	2.31217(0.13919)	2.35014(0.12103)
β_1	0.38089(0.06440)	0.39678(0.04346)
β_2	0.00333(0.00134)	0.00356(0.00126)
β_3	3.57673(0.34504)	3.74351(0.44697)
β_4	0.26570(0.32870)	
t^*	6.265 years	6.0234 years
$Q3$	0.01518	0.01629
median	−0.02177	−0.02069
$Q1$	−0.07663	−0.07668

mortgages which actually prepaid during a particular time period, we cannot use these parametric methods to assess the statistical significance of our maximum-likelihood parameter estimates.

Jackknifing (Efron (1982)) provides a nonparametric means of assessing the variability of our maximum-likelihood parameter estimates.[5] The jackknife variance estimate tends to be conservative in the sense that its expectation is greater than the true variance. Furthermore, since jackknifing requires only the independence of prepayment decisions across time periods, any dependence in prepayment decisions between the various pools will be taken into account when calculating jackknife estimates of variance.

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during a particular time period for a given economic environment, thereby increasing our confidence in the parameter estimates.

[5] The jackknife estimate of the variance of $\hat{\theta}_\iota$ is given by

$$\hat{\mathrm{VAR}}_\iota = ((K - 1)/K) \textstyle\sum_{k=1}^{K} (\hat{\theta}_{\iota(k)} - \hat{\theta}_{\iota(\cdot)})^2,$$

where

$$\hat{\theta}_{\iota(\,)} = \textstyle\sum_{k=1}^{K} \hat{\theta}_{\iota(k)}/K$$

and $\hat{\theta}_{\iota(k)}$ is that value of θ_ι which maximizes

$$\ln L(\underline{\theta} \mid \underline{\delta}_0, \cdots, \underline{\delta}_{k-1}, \underline{\delta}_{k+1}, \cdots, \underline{\delta}_{K+1}).$$

In words, $\hat{\theta}_{\iota(k)}$ is that value of θ_ι which maximizes the likelihood of the observed prepayment data once prepayment data are excluded for all pools during the kth time period. Assuming the independence of prepayment decisions across time periods, this resampling plan allows us to derive estimates of the variability of our parameter estimates. Extensive Monte Carlo evidence confirms that $\hat{\mathrm{VAR}}$ tends to be biased moderately upward. (See Efron (1982), especially Chapter 4.)

D. Empirical Results

Necessary conditions for the existence of maximum-likelihood estimators $\underline{\hat{\theta}} = (\hat{\gamma}, \hat{p}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4)$ are provided by

$$\frac{\partial \ln L(\underline{\theta})}{\partial \theta_i} = 0. \tag{13}$$

Since these likelihood equations are nonlinear, we employ a multidimensional Newton-Raphson procedure to solve (13).[6]

Resultant maximum-likelihood parameter estimates of the full model together with jackknifed estimates of their standard deviations are presented in Panel A of Table II. Our empirical analysis is based on refinancing rates lagged three months.[7] Notice that all the posited covariates affect a mortgagor's prepayment decision in the expected directions. In particular, the conditional probability of prepayment increases significantly when refinancing rates are less than the mortgage's contract rate. The proportion of a pool outstanding also significantly influences the conditional probability of prepayment. As the size of the pool decreases, this probability decreases significantly as mortgagors more likely to prepay have already done so. In addition, prepayments accelerate significantly when refinancing rates are sufficiently lower than the mortgage's contract rate. However, while the conditional probability of prepayment increases during the summer months, this covariate does not appear to be significant in our sample. The resultant estimated base-line conditional probability of prepayment initially increases with the mortgage's age, reaches a maximum at $t^* = 6.265$ years, and diminishes thereafter with age. Panel B presents maximum-likelihood parameter estimates together with jackknifed estimates of their standard deviations when seasonality is excluded. As expected, the results are similar to those presented in Panel A.

We assess the fit of the prepayment model at the maximum-likelihood parameter estimates by examining corresponding prepayment errors, actual less model annualized monthly prepayment rates, for each pool and each month for which we have data. Several sample statistics—the upper quantile, $Q3$, the median, and the lower quantile, $Q1$—of the resultant error distribution are also tabulated in Table II. With or without seasonality, these empirical results are consistent with the prepayment model at the maximum-likelihood estimates slightly overestimating the actual conditional prepayment rate. Of course, the model's fit could be improved by including additional covariates, for example, the housing turnover

[6] The logarithmic likelihood function maximized assumes that mortgages have a common principal of \$100,000. As noted earlier, this assumed value will not affect the values of the maximum-likelihood estimates.

[7] For a given common mortgage principal, the maximized value of the logarithmic likelihood function is larger for $s = 3$ than for $s = 0, 1, 2, 4, 5$, or 6. That is, the observed prepayment data are more likely given that mortgagors' base current prepayment decisions on refinancing rates lagged three periods, as opposed to basing current prepayment decisions on contemporaneous refinancing rates or refinancing rates lagged one, two, four, five, or six periods. Using Akaike's criterion, our subsequent empirical analysis takes $s = 3$.

rate, to capture macroeconomic effects.[8] However, to the extent that our purpose is to estimate a prepayment function which can be used in the valuation of mortgage-backed securities, we have restricted our attention to those covariates which can be embedded within a partial-equilibrium valuation framework.

II. Valuation of Mortgage-Backed Securities

The valuation of mortgage-backed securities must accurately reflect a mortgagor's option to prepay the underlying mortgage. In this section, we derive a continuous-time valuation model of mortgage-backed securities which succinctly takes into account the nature of mortgagors' prepayment decisions. Throughout we assume no taxes, transaction costs, or short-selling constraints.

Without loss of generality, we couch our discussion in terms of the valuation of a default-free fixed-rate fully amortizing mortgage. We assume that at origination the mortgage has a principal of $P(0)$, a continuous contract rate of c, and a term to maturity of T years. As a result, the total payout rate is

$$A = cP(0)/(1 - \exp(-cT)), \tag{14}$$

and the principal oustanding at time t is

$$P(t) = (A/c)(1 - \exp(-c(T - t))). \tag{15}$$

A. Valuation Model

We make the following assumptions to develop a model to value mortgage-backed securities.

(A1) All information about the term structure of interest rates can be summarized by two state variables. Following Brennan and Schwartz (1979), we take these state variables to be r, the instantaneous risk-free rate of interest, and l, the yield on a default-free consol.

(A2) Dynamics of r and l are assumed to be described by

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r dz_1, \tag{16}$$

$$dl = (a_2 + b_2 l + c_2 r)dt + \sigma_2 l dz_2, \tag{17}$$

where z_1 and z_2 are standardized Wiener processes. Increments to z_1 and z_2 are assumed to be instantaneously correlated:

$$dz_1 dz_2 = \rho dt, \tag{18}$$

where ρ denotes the instantaneous correlation coefficient.

This specification assumes that unanticipated changes in both r and l are proportional to their respective levels. Consistent with expectations hypotheses

[8] In addition, our prepayment model is incomplete because it ignores the possibility that the mortgagor may default on a loan. Since our sampled mortgages are guaranteed by GNMA, a default is equivalent to a loan prepayment. However, given aggregate prepayment data, we cannot isolate prepayments due to default.

of the term structure, the drift of the instantaneous risk-free interest rate process posits that r reverts to the current value of l, which itself varies stochastically through time. In addition, we include the coefficient a_1 to take into account possible liquidity premia in l. By contrast, for full generality, the drift of the consol yield process is assumed to be linear in both r and l. For further details regarding this specification, see Brennan and Schwartz (1982).

(A3) Mortagages are prepaid at the instantaneous rate of prepayment:

$$\pi = \pi(l, x, y, t; c). \tag{19}$$

For full generality, we assume only that π depends upon the prevailing consol yield, $l(t)$, relative to the mortgage's contract rate, c, as well as the history of past interest rates, summarized by the state variable $x(t)$, the relative proportion of the pool previously prepaid, given by the state variable $y(t)$, and time, t.

In particular, following Ramaswamy and Sundaresan (1986), the state variable $x(t)$ is defined by

$$x(t) = \alpha \int_{\infty}^{0} \exp(-as) l(t - s)\, ds, \quad \alpha > 0, \tag{20}$$

an exponential average of past consol yields. Within our continuous-time framework, this state variable captures the effects of past refinancing rates on current prepayment decisions. The dynamics of $x(t)$ are given by

$$dx = \alpha(l - x)dt. \tag{21}$$

We denote by $y(t)$ the fraction of a pool of these mortgages currently outstanding relative to their principal which would prevail in the absence of prepayments but reflecting amortization. This state variable captures any heterogeneity in mortgagors. The dynamics of $y(t)$ are given by

$$dy = -y(\pi + AP^{-1}(t) - c)dt. \tag{22}$$

Time, t, affects the instantaneous rate of prepayment by determining both the age of the mortgage and the season of the year.

Given the above assumptions, the value of any mortgage-backed security can be expressed as

$$B = B(r, l, x, y, t). \tag{23}$$

Standard arbitrage arguments give the following partial differential equation which the value of the mortgage must satisfy:

$$\begin{aligned} \tfrac{1}{2} r^2 \sigma_1^2 B_{rr} + rl\rho\sigma_1\sigma_2 B_{rl} + \tfrac{1}{2} l^2 \sigma_2^2 B_{ll} + (a_1 + b_1(l - r) - \lambda_1 \sigma_1 r) B_r \\ + l(\sigma_2^2 + l - r)B_l + \alpha(l - x)B_x - y(\pi + AP^{-1}(t) - c)B_y \\ + B_t - (r + \pi)B + \pi P(t) + A = 0, \end{aligned} \tag{24}$$

where λ_1 is the market price of short-term interest rate risk. Since the mortgage is fully amortizing, the following terminal boundary condition must be satisfied:

$$B(r, l, x, 0, T) = 0. \tag{25}$$

Table III

Maximum-Likelihood Estimates of Interest Rate Process Parameters

We provide maximum-likelihood estimates, with standard errors in parentheses, of the parameters of the interest rate processes $dr = (a_1+b_1(l-r))dt + \sigma_1 r dz_1$ and $dl = (a_2+b_2 l+c_2 r)dt + \sigma_2 l dz_2$ with $(dz_1)(dz_2) = \rho dt$.

a_1	b_1	a_2	b_2	c_2	σ_1	σ_2	ρ
−0.800	0.0382	−0.0033	−0.0007	0.0008	0.0262	0.0173	0.3732
(0.0359)	(0.0174)	(0.0063)	(0.0019)	(0.0016)			

Different mortgage-backed securities differ in their specifications of $P(t)$ and A.

Notice that we do not impose an optimal, value-minimizing call condition to value the mortgage-backed security. Rather, the value of the mortgage-backed security reflects the fact that at each point in time there exists a probability of prepaying, this probability depending upon the current state of the economy as summarized by the model's state variables.

B. Estimation of Interest Rate Processes

The coefficients of the partial differential equation, expression (24), depend upon the parameters of the interest rate processes. Estimating these parameters and the market price of short-term interest rate risk (to be estimated later) allows us to implement our mortgage-backed security valuation model given the previously estimated prepayment function.

To estimate the parameters of the interest rate processes requires data on r and l. The instantaneous risk-free interest rate is approximated by the annualized one-month CD rate. The consol yield is approximated by the annualized running coupon yield on long-term U.S. Treasury bonds. The running coupon yield at a particular point in time is the coupon rate on a newly issued U.S. Treasury bond if the bond is then issued; otherwise, it is the yield on the most recently issued U.S. Treasury bond. Weekly observations on r and l were collected from the week ending December 29, 1982 until the week ending April 1, 1987, for a total of 223 observations. Salomon Brothers provided the interest rate data.

For empirical purposes, we estimate the parameters given discrete approximations to the interest rate processes. An iterative Aitken procedure is applied to the resultant system of equations to yield maximum-likelihood estimates. (For further details, see Brennan and Schwartz (1982).) The results are tabulated in Table III. The estimated parameters a_1 and b_1 from the drift of the short-term interest rate process are statistically significant, while the estimated parameters a_2, b_2, and c_2 from the drift of the long-term interest rate processes are statistically insignificant. This is to be expected since, in the absence of arbitrage, long-term interest rates follow a random walk. The estimated standard deviation of proportional changes in short-term interest rates exceeds the estimated standard deviation of proportional changes in long-term interest rates, $\hat{\sigma}_1 > \hat{\sigma}_2$. That is, short-term interest rates were more volatile than long-term interest rates over our sample period. Finally, the estimated correlation coefficient is consistent with unanticipated proportional changes in r and l being positively correlated.

C. Monte Carlo Simulation Methods

Monte Carlo simulation methods (Boyle (1977)) are employed to solve the partial differential equation, expression (24), subject to the terminal boundary condition, expression (25). The fact that we have five state variables—$r(t)$, $l(t)$, $x(t)$, and $y(t)$, as well as the deterministic state variable, t—implies that solution by finite-difference methods will be complicated. Furthermore, Monte Carlo simulation methods allow us to easily integrate our estimated prepayment function into the contingent-claims valuation framework.

Monte Carlo simulation methods require that r and l are generated by the following correlated risk-adjusted processes:

$$dr = (a_1 + b_1(l - r) - \lambda_1 \sigma_1 r)dt + \sigma_1 r dz_1, \tag{26}$$

$$dl = l(\sigma_2^2 + l - r)dt + \sigma_2 l dz_2. \tag{27}$$

To value a mortgage-backed security, we generate correlated normal random variables corresponding to r and l, at every month during the life of the security. Given the probability that the pool will be prepaid during that month, we determine the cash flows—contractually obligated and prepayments—to the mortgage-backed security holder. The present value of these cash flows gives a particular realization of the mortgage-backed security's value. By repeating this procedure, the average of the corresponding realizations gives the solution of the partial differential equation.

III. Valuation Results

In this section, we illustrate our valuation procedures by pricing under various prepayment assumptions a twenty-five-year, default-free fully amortizing eleven percent mortgage originated five years ago with ninety percent of the relative principal of a pool of these mortgages currently outstanding. In particular, we assume that prepayments occur according to our estimated prepayment function[9] and according to one hundred percent of FHA experience.[10] Alternatively, we assume that an optimal, value-minimizing prepayment policy is followed, and, for comparison purposes, the mortgage is also assumed not to be prepayable.[11] By simulating these prices for various interest rates, we investigate the critical role prepayment plays in mortgage-backed security pricing.

We use parameter estimates of the underlying short and long interest rate processes tabulated in Table III to implement our valuation procedures. Given these parameter estimates, together with interest rate conditions prevailing at the end of November 1987, we specify the market price of short-term interest

[9] The insignificant effect of seasonality is excluded. Also, to minimize problems associated with extrapolation, our Monte Carlo solution procedure takes the minimum value of AO_t/AO_t^* to be 0.11, the minimum value observed in our prepayment sample.

[10] In other words, conditional prepayment rates depend only upon the age of the mortgage and correspond to the historical FHA experience. See Dunn and McConnell (1981).

[11] Mortgage values assuming optimal prepayments and prepayments according to one hundred percent of FHA experience, as well as nonprepayable mortgage values, are obtained by the solution to the appropriate partial differential equation using numerical methods rather than Monte Carlo simulation procedures. See Brennan and Schwartz (1985).

Table IV

Default-Free Mortgage Prices

For a given short rate r and a given long rate l we provide for varying prepayment assumptions the price per $100 principal of an eleven percent, twenty-five-year default-free fully amortizing mortgage originally issued five years ago with ninety percent of its relative principal currently outstanding. In particular, we assume that the mortgage is not prepayable, that mortgagors follow an optimal, value-minimizing call policy, that prepayments occur according to one hundred percent of FHA experience, and that prepayments occur according to our estimated prepayment function.

r	l	Nonprepayable	Optimal Prepayment	100% FHA Prepayment	Empirical Prepayment
7%	7%	144.95	100.00	132.67	117.97
	9%	119.79	100.00	116.66	110.49
	11%	101.18	95.67	103.88	99.55
	13%	86.89	85.19	93.45	88.88
	15%	75.48	75.03	84.70	78.08
	17%	65.94	66.01	77.15	68.92
11%	7%	144.14	100.00	131.36	116.10
	9%	119.34	99.92	115.67	109.22
	11%	100.94	94.84	103.13	98.99
	13%	86.80	84.86	92.87	87.85
	15%	75.53	74.97	84.28	78.58
	17%	66.17	66.16	76.89	68.18
15%	7%	143.31	99.50	130.06	114.91
	9%	118.87	99.05	114.69	108.06
	11%	100.69	93.97	102.38	97.83
	13%	86.70	84.50	92.29	87.57
	15%	75.56	74.88	83.83	78.01
	17%	66.36	66.29	76.60	68.56

rate risk by determining iteratively that value of λ_1 such that a thirty-year default-free nonprepayable fully amortizing mortgage with an eleven percent contract rate is priced at par for $r = l = 11$ percent. The resultant estimate of the market price of short-term interest rate risk is $\hat{\lambda}_1 = -0.01$.[12]

Table IV provides our pricing results. For a given short interest rate r—seven, eleven, and fifteen percent—we provide corresponding mortgage values for varying long rates l—seven, nine, eleven, thirteen, fifteen, and seventeen percent. Mortgage values are insensitive to the prevailing short rate r. However, it is clear from Table IV that mortgage values are very sensitive to the prepayment assumption.

As expected, for sufficiently low long rates, the nonprepayable mortgage's value can be quite high. For example, for r = eleven percent and l = seven percent, this mortgage is priced at $144.14. By contrast, assuming an optimal, value-minimizing prepayment policy, the mortgage will be prepaid for sufficiently low long rates. For example, for r = eleven percent and l = seven percent, the mortgage is

[12] To empirically implement our valuation procedures, we would choose that value of λ_1 which minimizes the mean squared error between model prices and actual prices of a sample of default-free Treasury bonds.

prepaid. By contrast, if l is sufficiently high and, as such, the probability of prepayment is sufficiently low, the mortgage allowing optimal prepayments behaves like the nonprepayable mortgage. For r = eleven percent and l = fifteen percent, the value of the mortgage assuming optimal prepayments is \$74.97, while the nonprepayable mortgage's value is \$75.53.

Assuming that mortgages are prepaid according to either our estimated prepayment function or one hundred percent of FHA experience gives results more consistent with the stylized facts associated with mortgage pricing than when assuming that an optimal, value-minimizing prepayment policy is followed. By using our estimated prepayment function or assuming one hundred percent of FHA experience, we see that for low long rates the mortgage sells for more than \$100 since some mortgagors will not refinance, which is beneficial to premium security holders. For high long rates, the mortgage again sells for more than the price corresponding to an optimal, value-minimizing prepayment policy since some mortgagors will now prepay, which is beneficial to discount security holders.

Assuming that prepayments occur according to one hundred percent of FHA experience is consistent with some mortgagors calling their loans when the prevailing refinancing rate exceeds their contract rate and some mortgagors not calling their loans when their contract rate exceeds the refinancing rate. However, this prepayment assumption is not consistent with the fact that mortgagors' prepayment decisions exhibit interest rate sensitivity. For low long rates, mortgage prices assuming that prepayments occur according to our estimated prepayment function are closer to corresponding mortgage prices assuming an optimal, value-minimizing prepayment policy since at these refinancing rates most mortgagors prepay. Similarly, for high long rates we see that mortgage prices given our estimated prepayment function are also closer both to mortgage prices allowing optimal prepayments and to nonprepayable mortgage prices since at these refinancing rates most mortgagors in fact do not prepay.

Figure 1 graphically summarizes our pricing results for r = eleven percent and varying long rates. Mortgage prices given our estimated prepayment function reflect the fact that if long rates are sufficiently low all mortgages will be prepaid. These mortgage prices initially increase in response to increases in the refinancing rate. Intuitively, while increasing the long rate decreases the present value of future mortgage payments, this is more than offset by the fact that the probability of prepayment also decreases, which is beneficial to a premium security holder. Eventually, mortgage prices given our estimated prepayment function decrease with subsequent increases in the refinancing rate as the resultant decrease in the present value of future mortgage payments more than offsets the pricing effects of the resultant dampening in prepayment behavior. However, for sufficiently high long rates, these mortgage prices still exceed mortgage prices assuming an optimal, value-minimizing prepayment policy since some mortgagors will prepay even though the prevailing refinancing rate exceeds the loan's contract rate.

These simulation results clearly illustrate the importance of prepayment to the valuation of mortgage-backed securities. To the extent that our estimated prepayment function more accurately models mortgagors' prepayment behavior, the resultant mortgage-backed security values more accurately reflect the pricing effects of this prepayment behavior.

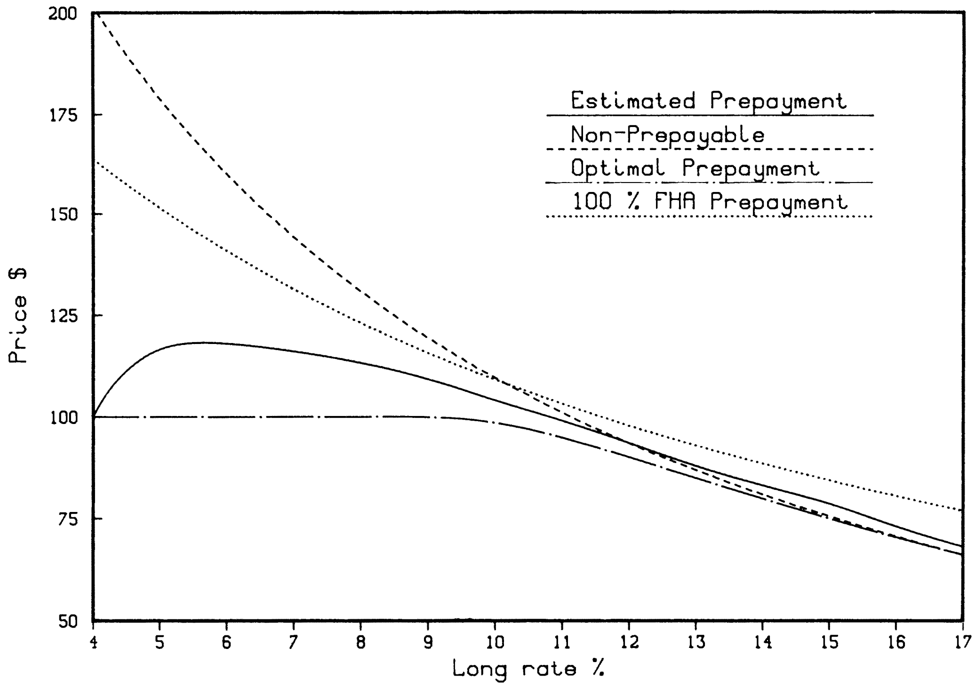

Figure 1. Prices per $100 principal of an eleven percent twenty-five-year default-free fully amortizing mortgage issued five years ago with ninety percent of its relative principal currently outstanding as a function of the long rate, *l*, for various prepayment assumptions.

IV. Summary and Conclusions

It is well known that mortgage-backed security valuation models assuming an optimal, value-minimizing prepayment policy cannot explain a number of the stylized facts associated with the pricing of mortgages. In particular, many mortgagors prepay their loans when the prevailing refinancing rate exceeds their loan's contract rate and, conversely, other mortgagors do not prepay even when the contract rate on their loan exceeds the prevailing refinancing rate.

This paper puts forward a valuation framework for mortgage-backed securities which can help explain these stylized facts. We assume that the value of a mortgage-backed security reflects the fact that at each point in time there exists a probability of prepaying, this conditional probability depending upon the prevailing state of the economy. Given limited publicly available prepayment data, we provide maximum-likelihood estimates of our posited prepayment function. By integrating this estimated prepayment function into our valuation framework, the resultant mortgage prices are consistent with the premiums at which mortgage-backed securities often trade.

The pricing of other mortgage-backed securities such as "stripped" mortgage-backed securities and CMOs can be explored with this valuation framework. In addition, the hedging of these and other mortgage-backed securities can be investigated. Finally, if sufficient price data on mortgage-backed securities become available, future research should include the empirical testing of our valuation procedures.

REFERENCES

Askin, D., 1985, Forecasting prepayment rates for mortgage securities, in F. Fabozzi, ed.: *The Handbook of Mortgage-Backed Securities* (Probus Publishing, Chicago, IL).

Boyle, P., 1977, Options: A Monte Carlo approach, *Journal of Financial Economics* 4, 323–338.

Brennan, M. and E. Schwartz, 1979, A continuous time approach to the pricing of bonds, *Journal of Banking and Finance* 3, 133–155.

——— and E. Schwartz, 1982, An equilibrium model of bond pricing and a test of market efficiency, *Journal of Financial and Quantitative Analysis* 17, 201–229.

——— and E. Schwartz, Determinants of GNMA mortgage prices, *AREUEA Journal* 13, 209–228.

Dunn, K. and J. McConnell, 1981, Valuation of GNMA mortgage-backed securities, *Journal of Finance* 36, 599–616.

Efron, B., 1982, *The Jackknife, the Bootstrap, and Other Resampling Plans* (SIAM, Philadelphia, PA).

Green, J. and J. Shoven, 1986, The effects of interest rates on mortgage prepayments, *Journal of Money, Credit and Banking* 18, 41–59.

Kalbfleisch, J. and R. Prentice, 1980, *The Statistical Analysis of Failure Time Data* (John Wiley, New York).

Ramaswamy, K. and S. Sundaresan, 1986, The valuation of floating rate instruments: Theory and evidence, *Journal of Financial Economics* 17, 251–272.