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Hersh Shefrin and Meir Statman*

Abstract

We develop a positive behavioral portfolio theory (BPT) and explore its implications for portfolio construction and security design. The optimal portfolios of BPT investors resemble combinations of bonds and lottery tickets, consistent with Friedman and Savage's (1948) observation. We compare the BPT efficient frontier with the mean-variance efficient frontier and show that, in general, the two frontiers do not coincide. Optimal BPT portfolios are also different from optimal CAPM portfolios. In particular, the CAPM two-fund separation does not hold in BPT. We present BPT in a single mental account version (BPT-SA) and a multiple mental account version (BPT-MA). BPT-SA investors integrate their portfolios into a single mental account, while BPT-MA investors segregate their portfolios into several mental accounts. BPT-MA portfolios resemble layered pyramids, where layers are associated with aspirations. We explore a two-layer portfolio where the low aspiration layer is designed to avoid poverty while the high aspiration layer is designed for a shot at riches.

I. Introduction

We develop behavioral portfolio theory (BPT) as a positive portfolio theory on the foundation of SP/A theory (Lopes (1987)) and prospect theory (Kahneman and Tversky (1979)), two theories of choice under uncertainty. Both SP/A theory and prospect theory emerged from the literature addressing Friedman and Savage's (1948) puzzle, the observation that people who buy insurance policies often buy lottery tickets as well.

Markowitz's (1952a) mean-variance portfolio theory is one of three portfolio theories introduced in 1952 and the only one inconsistent with the Friedman-Savage puzzle. The two other portfolio theories, Markowitz's (1952b) customary wealth theory and Roy's (1952) safety-first theory, are consistent with the puzzle. Indeed, Markowitz (1952b) introduced customary wealth theory to deal with some unrealistic implications of the Friedman-Savage framework.

Embedded within BPT is an efficient frontier. We compare the BPT efficient frontier with the mean-variance efficient frontier and show that, in general, the two

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frontiers do not coincide; portfolios on the BPT efficient frontier are generally not on the mean-variance efficient frontier. Mean-variance investors choose portfolios by considering mean and variance. In contrast, BPT investors choose portfolios by considering expected wealth, desire for security and potential, aspiration levels, and probabilities of achieving aspiration levels.

The optimal portfolios of BPT investors are different from those of CAPM investors as well. The optimal portfolios of CAPM investors combine the market portfolio and the risk-free security. In contrast, the optimal portfolios of BPT investors resemble combinations of bonds and lottery tickets.

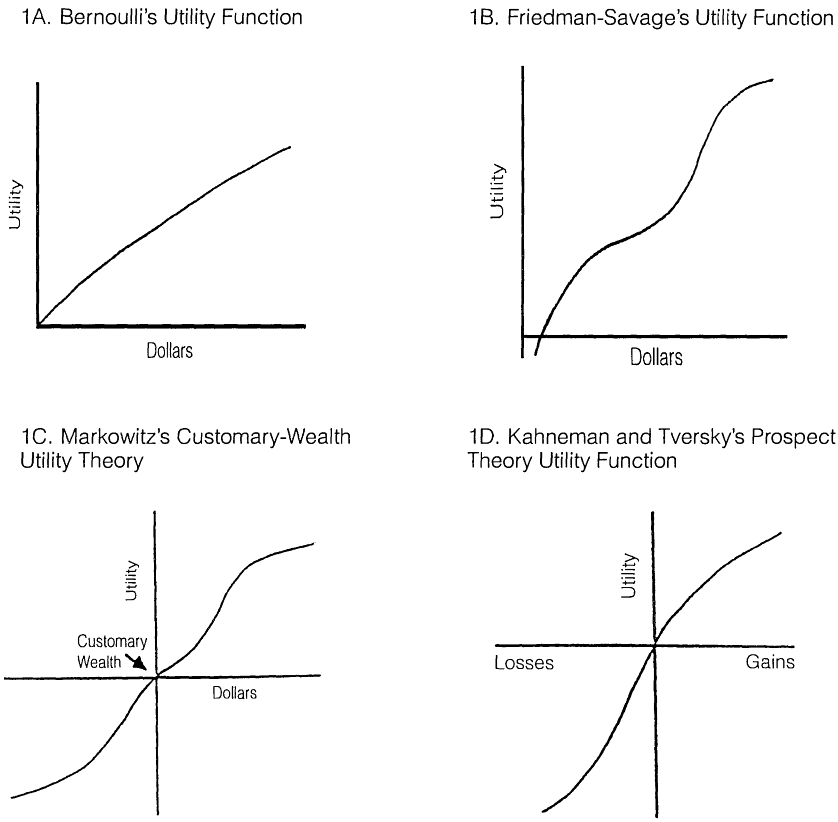
We present BPT in two versions: a single mental account BPT version (BPT-SA) and a multiple mental account version (BPT-MA). BPT-SA investors, like mean-variance investors, integrate their portfolios into a single mental account; they do so by considering covariance. In contrast, BPT-MA investors segregate their portfolios into mental accounts and overlook covariance among mental accounts. Note that BPT-MA differs from both Markowitz's mean-variance theory (1952a) and Markowitz's customary wealth theory (1952b). For example, BPT-MA investors might place foreign stocks in one mental account and domestic stocks in another. They might consider foreign stocks highly risky because they overlook the effect that the covariance between foreign and domestic stocks exerts on the risk of the portfolio, viewed as an integrated single account.

The Friedman-Savage puzzle is a thorn in the side of conventional expected utility theory, which is based upon concave utility functions defined over final asset position. Von Neumann and Morgenstern (1944) developed expected utility theory on the foundation of Bernoulli's utility theory, a theory consistent with uniform attitude toward risk; the Bernoulli utility function is concave throughout (see Figure 1, A). However, the Friedman-Savage puzzle is inconsistent with a uniform attitude toward risk.

Friedman and Savage offer a solution to the insurance lottery puzzle based on a utility function that features both concave and convex portions. The concave portions are consistent with the purchase of insurance policies and the convex portion is consistent with the purchase of lottery tickets (see Figure 1, B). Markowitz (1952b) points out that only a few Friedman-Savage investors will buy both insurance policies and lottery tickets. Specifically, buyers of both insurance and lotteries are those whose wealth levels fall in a narrow region defined by the location of the inflection points in their utility functions. Moreover, Markowitz points out that the Friedman-Savage utility function implies that poor people never purchase lottery tickets, and middle income people never insure themselves against modest losses. To address these points, Markowitz (1952b) modified Friedman and Savage's function by locating one of the inflection points of the utility function at customary wealth (see Figure 1, C). Customary wealth is status quo wealth, usually current wealth.

Economists, especially Quiggen (1982) and Yaari (1987), generalized expected utility theory to accommodate the Allais' paradoxes (Allais (1979)). Psychologists, especially Kahneman and Tversky (1979), constructed prospect theory on the foundation of Markowitz's customary wealth theory and Allais' work. Lopes, in effect, built SP/A theory on the safety-first model and the work of Quiggen and Yaari (Lopes (1987) and Lopes and Oden (1998)).

FIGURE 1



Source: Lopes (1987)

The paper is organized as follows: in Section II, we review the safety-first portfolio framework from Roy to Kataoka, Telser, and Arzac and Bawa. In Section III, we present Lopes' SP/A theory. SP/A theory focuses on the twin desires for security (S) and potential (P) and on the aspiration levels associated with security and potential. Section IV develops BPT. We characterize the BPT-SA efficient frontier and show that, in general, BPT-SA efficient portfolios are not mean-variance efficient. Section V describes the return distributions of efficient BPT-SA portfolios. In Section VI, we address the misconception that Tchebyshev's inequality implies that optimal safety-first portfolios always lie on the mean-variance efficient frontier. In Section VII, we describe BPT-SA portfolios when returns are normally distributed and show that constraints on short sales can lead these portfolios to lie off the mean-variance efficient frontier. Section VIII develops BPT-MA for the case where investors segregate portfolios into two mental accounts, one designed for safety and the other designed for potential. In Section IX, we discuss several structural issues concerning BPT portfolios. In particular, we show that BPT-MA portfolios are neither BPT-SA efficient in general, nor mean-variance efficient. Section X discusses the evolutionary implications of

BPT. We show that BPT-SA efficient portfolios are fitter than some, but not all, mean-variance efficient portfolios. In Section XI, we compare real life portfolios to BPT portfolios, and in Section XII, we offer conclusions and directions for future research.

II. Safety-First Portfolio Theory

Investors in Roy's (1952) safety-first portfolio theory aim to minimize $\Pr\{W < s\}$, the probability of ruin. An investor is *ruined* when his terminal wealth W falls short of a subsistence level s.

Let P be an arbitrary portfolio with corresponding return mean μ_P and return standard deviation σ_P. Roy focuses on the case when there is no risk-free security ($\sigma_P > 0$ for all P) and the subsistence level s is low ($s < \mu_P$ for all P). In the special case when all portfolio return distributions are constrained to be normal, minimizing the probability of ruin is equivalent to minimizing the number of standard deviations in which s lies below μ_P. That is, in Roy's safety-first model, with normally distributed returns, an investor chooses a portfolio P to minimize the objective function $(s - \mu_P)/\sigma_P$.

Suppose returns are not constrained to be normally distributed. Roy uses Tchebyshev's inequality to argue that the same objective function applies. His argument seems to imply that all optimal safety-first portfolios lie on the mean-variance frontier. However, we argue that this is not the case. In general, optimal safety-first portfolios are not mean-variance efficient.

Elton and Gruber (1995) discuss two generalizations of safety-first, one by Kataoka (see Elton and Gruber (1995), p. 237) and two others by Telser (1955). Kataoka dispenses with Roy's notion of a predetermined subsistence level, s. The objective of Kataoka's safety-first investors is to maximize the subsistence level subject to the constraint that the probability that wealth (W) falls below the subsistence level (s) does not exceed a predetermined α.[1]

Telser (1955) developed a model that features both a fixed subsistence level s and a ruin probability α. In Telser's model, a portfolio is considered safe if the probability of ruin does not exceed α. Telser suggested that an investor choose a portfolio to maximize expected wealth $E(W)$ subject to $\text{Prob}\{W \leq s\} \leq \alpha$.

Arzac (1974) analyzes the nature and existence of an optimal solution in Telser's safety-first model. Arzac and Bawa (1977) extend Telser's model by allowing α, the probability of ruin, to vary. Arzac-Bawa safety-first investors maximize an objective function V defined over expected wealth, $E(W)$, and the probability of ruin, α. In particular, choice over $(E(W), \alpha)$ pairs is accommodated within an expected utility framework, when the utility function u is defined by

$$\begin{aligned} u(W) &= W \quad \text{if } \text{Prob}\{W \leq s\} \leq \alpha, \\ u(W) &= W - c \quad \text{if } \text{Prob}\{W \leq s\} \geq \alpha, \end{aligned} \tag{1}$$

for $c > 0$. In this case, expected utility takes the form $E(W) - c\text{Prob}\{W \leq s\}$. Indeed, Markowitz (1959) established that this is the only functional form consis-

[1]That is, the objective is to choose a portfolio featuring the maximum wealth W_α defining the lower 100α percentile.

tent with the principles of expected utility and Telser's version of the safety-first problem.[2]

There is now a large literature demonstrating that people systematically violate the axioms of expected utility (Kahneman and Tversky (1979)). In the remainder of the paper, we focus on choice theories where the axioms of expected utility are violated.

III. SP/A Theory

Lopes (1987) developed SP/A theory, a psychological theory of choice under uncertainty. SP/A theory is a general choice framework rather than a theory of portfolio choice. However, SP/A theory can be regarded as an extension of Arzac's version of the safety-first portfolio model.

In SP/A theory, the S stands for *security*, P for *potential*, and A for *aspiration*. Lopes' notion of security is analogous to safety in safety-first, a general concern about avoiding low levels of wealth. Her notion of aspiration relates to a goal, and generalizes the safety-first concept of reaching a specific target value, such as s. There is no counterpart to potential in the safety-first framework. Potential relates to a general desire to reach high levels of wealth.

In the Arzac-Bawa and Telser models, danger means the possibility that wealth might fall below a particular minimum level s. They measure the probability of safety as $\text{Prob}\{W \geq s\}$. This probability is a decumulative probability, meaning that it has the form $D(x) = \text{Prob}\{W \geq x\}$: D is called a *decumulative distribution function.*

In Lopes' framework, two emotions operate on the willingness to take risk: fear and hope. Both emotions function by altering the relative weights attached to decumulative probabilities. As in Arzac-Bawa and Telser's formulation, Lopes' choice model focuses on an expected wealth function.

Lopes uses a discrete-state formulation, similar to that in Arzac (1974). In this regard, consider a two-date framework, where the dates are labeled zero and one. Let there be n states associated with date one, where $p_i = \text{Prob}\{W_i\}$, $i = 1, 2, \ldots, n$, and wealth levels are ranked $W_1 \leq W_2 \leq \ldots \leq W_n$.

Lopes notes that expected wealth, $E(W) = \Sigma p_i W_i$, can be expressed as $\Sigma D_i\,(W_i - W_{i-1})$, where the summation is from $i = 1$ to n and W_0 is zero. In this expression for $E(W)$, the individual receives W_1 with certainty (note that $D_1 = 1$), receives the increment $W_2 - W_1$ (that is, an amount over W_1) with probability D_2, receives the further increment $W_3 - W_2$ with probability D_3, and so on.

Lopes contends that fear operates through an overweighting of the probabilities attached to the worst outcomes relative to the best outcomes. She postulates that fear leads individuals to act as if they computed $E(W)$ using a value for p_1 that is excessively high, and a value for p_n that is excessively low. In other words, they act as if they were unduly pessimistic when computing $E(W)$. Lopes also postulates that hope leads individuals to act as if they were unduly optimistic when computing $E(W)$, using a value for p_1 that is excessively low, and a value for p_n that is excessively high.

[2]This utility function can be considered a limiting case of the utility function developed in Markowitz (1952b). We thank Harry Markowitz for this point.

In Lopes' framework, fear underlies the concern for security, and hope underlies the concern for potential. Formally, fear affects attitude toward a risky outcome through a reweighting of the decumulative probabilities. Specifically, Lopes computes $E(W)$ using the decumulative function $h_s(D) = D_s^{1+q}$ where the subscript s stands for security. For $q_s > 0$, this function attaches disproportionate weight to higher values of D. As a result, D stochastically dominates $h_s(D)$, since it effectively shifts probability weight from the right of the distribution's support to the left.

Hope operates like fear, but it induces higher weighting of lower values of D, the ones that attach to higher outcomes. Lopes' counterpart to $h_s(D)$ is $h_p(D)$, where the p stands for potential: $h_p(D)$ has the form $1-(1-D)^{1+q_p}$.

Lopes argues that the emotions of fear and hope reside within all individuals, and that each emotion serves to modify the decumulative weighting function. She suggests that the final shape of the decumulative transformation function is a convex combination of h_s and h_p, reflecting the relative strength of each. Specifically, the transformation function $h(D)$ has the form,

$$h(D) = \delta h_s(D) + (1-\delta)h_p(D). \tag{2}$$

In SP/A theory, the investor substitutes $E(W)$ with $E_h(W)$, replacing the probability p_i attached to the ith event with the difference $r_i = h(D_{i+1}) - h(D_i)$. SP/A departs from the expected utility framework by using a rank dependent formulation, replacing D with $h(D)$. Note that the rank ordering of states is typically the inverse of the rank associated with the price per unit probability ratio, v_i/p_i, this being the pricing kernel. Note also that consistency requires that the ranking of states by v_i/r_i be the same as the original ranking by v_i/p_i.

Lopes postulates that risky outcomes are evaluated in terms of two variables. The first variable is $E_h(W)$, the expected value of W under the transformed decumulative function $h(D)$. The second variable is $D(A)$, the probability that the payoff will be A or higher. Note that these two variables are virtual analogues of the arguments used in the safety-first model, $E(W)$ and $\text{Prob}\{W \leq s\}$. In fact, the criterion function used to evaluate alternative risky outcomes is a monotone increasing function $U(E_h(W), D(A))$. Note also that this conforms to the structure suggested by Arzac-Bawa (1977), wherein individuals do not maximize $E(W)$ subject to a probability constraint, but instead maximize a function $V(E(W), \alpha)$.

Formally, the mechanics underlying the optimization in SP/A theory can be viewed as an adaptation of the Arzac-Bawa (1977) characterization of safety-first theory, where the major difference is in the interpretation of the variables. Specifically, the subsistence level s is replaced by a more general aspiration level A. The probability α is a cumulative probability, but can be cast in decumulative form $1-\alpha$. Finally, $E(W)$ is replaced by $E_h(W)$. In the special case where $q_s = q_p = 0$, Lopes' model collapses to Arzac-Bawa's model.

Note that $E_h(W)$ will be lower than $E(W)$ for individuals who are strongly driven by fear. The greater the fear, the lower the value of $E_h(W)$. Similarly, hope operates by increasing $E_h(W)$ relative to $E(W)$.

IV. Portfolio Selection in BPT

We develop behavioral portfolio theory (BPT) by combining Lopes' SP/A theory with the mental accounting structure from Kahneman-Tversky's prospect theory. We begin with a single account version of BPT (BPT-SA), building on SP/A. Later in the paper, we move to a multiple account version of BPT (BPT-MA), by adding the mental accounting structure of prospect theory.

Both BPT-SA and mean-variance investors consider the portfolio as a whole, namely as a single mental account. They do so by considering covariances, as Markowitz (1952a) prescribed. Portfolio selection in a BPT-SA framework is similar to portfolio selection in a mean-variance framework, up to a point.

The cornerstone of mean-variance theory is the mean-variance efficient frontier in (μ, σ)-space. The BPT-SA counterpart is in $(E_h(W), \Pr\{W \leq A\})$-space. In both cases, investors prefer higher μ and higher $E_h(W)$, but lower σ and lower $\Pr\{W \leq A\}$. Hence, the mean-variance frontier is obtained by maximizing μ for fixed σ, and the BPT-SA frontier is obtained by maximizing $E_h(W)$ for fixed $\Pr\{W \leq A\}$. In this section, we present three results. The first characterizes the BPT-SA efficient frontier, the second characterizes the mean-variance frontier, and the third establishes that, typically, a BPT-SA efficient portfolio is not mean-variance efficient.

Consider a market in contingent claims at date zero, where a state-i contingent claim pays one unit of consumption at date one if state i occurs at date one, and zero otherwise. Let the price of a state-i claim be v_i, and imagine that the states are ordered so that state prices per unit probability, v_i/p_i, are monotonically decreasing in i. Suppose an investor has wealth W_0 at date zero, and seeks to maximize date one expected wealth $E_h(W)$, subject to a safety-first constraint, by purchasing a bundle of date one contingent claims, $W_1, \ldots, W_n$, whose value $\Sigma v_i W_i$ does not exceed W_0.

Theorem 1. Any solution $W_1, \ldots, W_n$ to

$$\max : E_h(W) = \Sigma r_i W_i, \qquad \text{subject to} : \text{Prob}(W \leq A) \leq \alpha,$$

has the following form. There is a subset T of states, including the nth state s_n such that

$$\begin{aligned}
(3) \qquad W_i &= 0, \quad \text{for } i \notin T, \\
W_i &= A, \quad \text{for } i \in T \backslash \{s_n\}, \\
W_n &= (W_0 - \Sigma v_i W_i)/v_n \quad \text{which exceeds } A \text{ when } W_0 > v_n A,
\end{aligned}$$

where the summation in the previous equation is from one to $n-1$. Moreover, $\text{Prob}\{T\} \geq \alpha$, but no proper subset T' of T features $\text{Prob}\{T'\} \geq \alpha$. If all states are equiprobable, then there is a critical state i_c such that the optimal portfolio has the form,

$$\begin{aligned}
(4) \qquad W_i &= 0, \quad \text{for } i \leq i_c, \\
W_i &= A, \quad \text{for } i_c \leq i < n, \\
W_n &= (W_0 - \Sigma v_i W_i)/v_n \quad \text{which exceeds } A \text{ when } W_0 > v_n A,
\end{aligned}$$

where the summation in the previous equation is from one to $n-1$, and i_c is the lowest integer for which $\Sigma_{i>ic} p_i \geq \alpha$.

Proof. Note that $E_h(W)$ is a sum of probability wealth products $r_i W_i$. Consider the unconstrained maximization of $E_h(W)$. To maximize the sum of probability wealth products, focus on the state that features the lowest price, per unit probability, for purchasing contingent wealth. By construction, this will be state n. That is, $v_n/r_n = \min_i\{v_i/r_i\}$. An unconstrained optimum for the $E_h(W)$ maximization is the corner solution $W_n = W_0/v_n$ with $W_i = 0$ for all other i. In the special case when $p_n \geq \alpha$, the unconstrained maximum will also be a constrained maximum. But this is not generally so. To modify the unconstrained maximum, consider the least expensive way of satisfying the constraint. To this end, consider all sets $\{T''\}$ that include s_n and have the property that $\text{Prob}\{T''\} \geq \alpha$, but no proper subset T' of T'' features $\text{Prob}\{T'\} \geq \alpha$. To each such set, associate the sum $v_A(T'') = A\Sigma_{i\in T''} v_i$. From the finite collection of sets T'' so defined, choose one, T, that features a minimum value $v_A(T'')$. Now modify the unconstrained optimum by reallocating $v_A(T)$ in value from claims that only pay in state s_n to claims that pay exactly A units of consumption for the states in $T\backslash\{s_n\}$.

In the case of equiprobable states, $p_i = p$ for all i, the minimum number of states required to achieve the probability constraint is the first integer n_α larger than α/p. Since the ratio v_i/r_i declines with i by construction, the cheapest reallocation from the unconstrained optimum involves holding positive claims in the top n_α states, and zero claims in the bottom $n - n_\alpha$ states. As before, claims to exactly A units of wealth are held in the top n_α states, with the exception of state s_n. □

Theorem 1 characterizes an efficient BPT-SA solution. We note that, for sufficiently high values of either A or α, it will be impossible to satisfy the probability constraint and, therefore, no optimal solution will exist.

What is the role of equiprobabilities? If unequal, consider a three-state case where $p_1 = 0.6$, $p_2 = p_3 = 0.2$, and $\alpha = 0.55$. In this case, it is impossible to satisfy the probability constraint without featuring positive consumption in state s_1. But this means that it is not necessary to have positive consumption in state s_2.[3]

In the next two theorems, we demonstrate that the BPT-SA efficient portfolios described in Theorem 1 are typically not mean-variance efficient.

Theorem 2. In the discrete-state case, a mean-variance efficient portfolio has the form,

$$W_i = 1/b\left[1 - \left([\Sigma v_j - bW_0]/\Sigma v_j^2/p_j\right) v_i/p_i\right], \tag{5}$$

where b is a positive constant.

Proof. The mean-variance efficient portfolios can be obtained by maximizing expected quadratic utility,

$$\Sigma p_i \left(W_i - (b/2)W_i^2\right),$$

[3]The equiprobable framework provides the basis for moving from discrete states to continuous states in the limit. Hence, we focus on the special case (4) rather than the more general (3).

subject to the constraint $\Sigma v_i W_i \leq W_0$. To establish the theorem, use the Lagrangean technique to solve for the constrained maximum. This yields the first order condition,

$$W_i = 1/b[1 - \lambda v_i/p_i],$$

where λ is the Lagrange multiplier. Solving for λ from the budget constraint $\Sigma v_i W_i \leq W_0$, yields

$$\lambda = [\Sigma v_j - bW_0]/\Sigma v_j^2/p_j.$$

Substitution of λ into the first order condition completes the proof. □

A non-negativity condition has not been imposed in Theorem 2. Note that, for sufficiently low values of the parameter b, the solution (5) will turn negative. Imposing a non-non-negativity constraint on date one wealth leads to a situation where W_i is zero at some of the highest-priced states, and the summations in Theorem 2 are over states that feature positive consumption.

Non-negativity also interferes with two-fund separation. Note that, when expressed in vector form $[W_i]$, (5) implies the two-fund separation property. However, when a non-negativity constraint is imposed on W_i, and (5) is replaced by a constrained version, the separation property becomes local. The separation property holds only for portfolios that feature positive claims in the exact same states.

Theorem 3. If there are at least three states that feature positive consumption in a BPT-SA efficient portfolio, with distinct values for v_i/p_i, then that portfolio is not mean-variance efficient.

Proof. As noted in Theorem 2, a mean-variance solution has the form,

$$W_i = 1/b\left[1 - \left([\Sigma v_j - bW_0]/\Sigma v_j^2/p_j\right) v_i/p_i\right],$$

which is a strictly concave function of v_j/p_j, the probability normalized price. But the optimal BPT-SA portfolio is strictly increasing in i and features exactly three payoff levels, of which one is zero. □

V. The Character of Optimal BPT-SA Portfolios

BPT-SA investors choose optimal portfolios by maximizing $U(E_h(W), D(A))$ along the BPT-SA efficient frontier. Theorem 1 describes the return distribution of efficient BPT-SA portfolios.

Consider a numerical example. Table 1 contains the data for a simple eight-state equiprobable case. Figure 2 depicts the efficient BPT-SA frontier in $[E_h(W), D(A)]$-space, for the case $A = \$2$. Figure 2 displays the return distribution for a particular efficient portfolio, a \$1 investment at date zero, as a function of the realized state at date one. Note that returns have three possible values in this figure: zero, A, and a value W_n higher than A.

This payoff pattern can be described as the combination of payoffs from a portfolio consisting of a bond and a lottery ticket that pays off only in state 8. The zero payoffs in states 1 and 2 indicate that the bond is risky. However, another

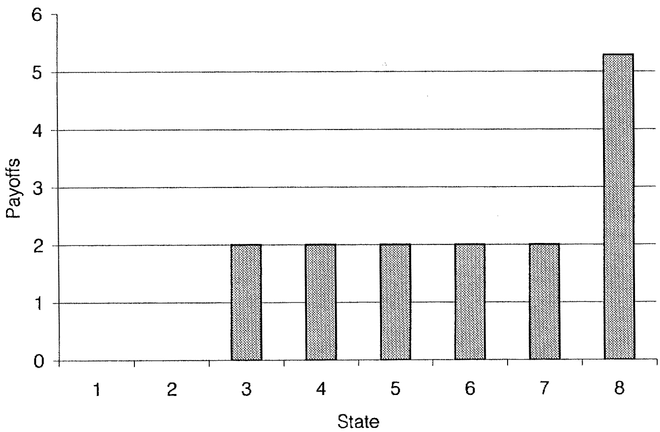
TABLE 1

Prices of State Securities and Per Capita Endowments of State Securities (Market Portfolio) (each state security pays $1 if the state occurs and zero otherwise)

State		Prices of State Securities (in $)	Per Capita Endowment of State Securities
Deep Recession	1	0.37	0.671
	2	0.19	1.050
	3	0.12	1.176
	4	0.09	1.239
	5	0.07	1.277
	6	0.06	1.302
	7	0.05	1.320
Explosive Boom	8	0.04	1.357

FIGURE 2

Payoffs for a BPT-SA Efficient Portfolio with a (Low) $2 Aspiration Level



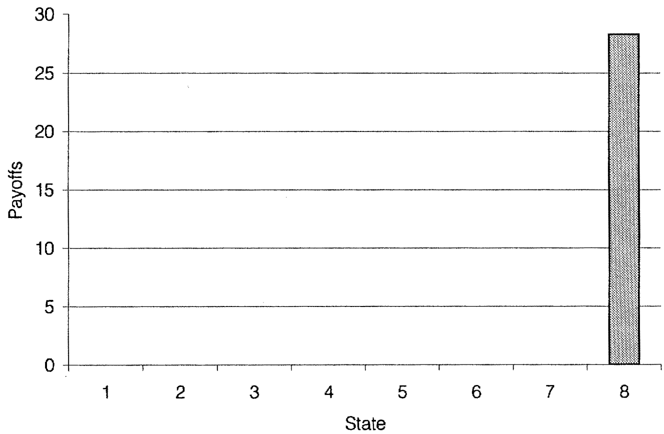
BPT-SA efficient portfolio within the parameters of this example, one for which $A = 0.9$, features strictly positive payoffs in all states. This portfolio return pattern corresponds to the return pattern of a combination risk-free bond that pays either A or zero, and a lottery ticket that pays $W_n - A$ in state s_n and zero otherwise.[4]

Figure 3 displays an efficient payoff when A = $12. In this case, there is only one efficient portfolio, and it pays off in state 8 alone. When the aspiration level A is sufficiently high, the only efficient portfolio resembles a lottery ticket.

[4]For those wishing to see an example illustrating Theorem 3, consider the following. Begin with the parameters as they appear above. That is, $n = 8$ with $p_i = 0.125$ for all i. The state prices v_i are, respectively: 0.37, 0.19, 0.12, 0.09, 0.07, 0.06, 0.05, 0.04. Let $q_s = q_p = 0$, $A = 0.9$, $W_0 = 1$, and $\alpha = 1$. The SP/A-efficient portfolio features $W_i = 0.9$ for $i < 8$, and $W_8 = 3.7$. The expected return to this portfolio is 25.37%, and the return variance is 87.56%. By setting the parameter b in Theorem 2 equal to 1.6155, we obtain a mean-variance efficient portfolio with the same expected return, but a return variance equal to 9.41%. Hence, the specific SP/A portfolio is not on the mean-variance frontier.

FIGURE 3

Payoffs for a BPT-SA Portfolio with a (High) \$12 Aspiration Level



Lopes (1987) illustrates the intuition underlying choice in the SP/A framework through the choice of crops by farmers. She notes that subsistence farmers often choose between two types of crops, food crops and cash crops. The prices of food crops are low but generally stable. In contrast, the prices of cash crops are volatile, but they offer the potential for higher wealth. Lopes notes that these farmers tend to plant food crops to the point where their subsistence needs are met. Farmers allocate the remainder of their land to cash crops. She suggests that farmers gamble on cash crops because they aspire to escape poverty.

Note that there are two implicit aspiration levels in the agricultural analogue. One is subsistence, and the other is some level above subsistence. In Lopes' framework, the *fear* of falling below subsistence motivates the allocation to food crops. This is the safety-first approach. The *aspiration* of escaping poverty motivates the allocation of the remainder to cash crops.

The farmer's problem has as its analogue Friedman and Savage's observation that people simultaneously purchase insurance and lottery tickets. As we have argued, such simultaneity is the hallmark of efficient BPT-SA portfolios.

A mean-variance investor's preferences can be expressed by the function $\mu - \sigma^2/d$ where d is the degree of risk tolerance. This is because attitude toward risk in a mean-variance framework is measured by a single parameter, d. In SP/A theory, risk is multidimensional, and is described with five parameters:

i) q_s measures the strength of fear (need for security);

ii) q_p measures the strength of hope (need for potential);

iii) A is the aspiration level;

iv) δ determines the strength of fear relative to hope; and

v) γ, discussed below, determines the strength of the desire to reach the aspiration level relative to fear and hope.

Just as increasing the parameter d alters a mean-variance investor's optimal mean-variance efficient portfolio, so altering any of the above parameters alters the BPT-SA investor's choice of portfolio. We discuss the impact of these parameters on the character of the selected portfolio later in the paper.

VI. What Tchebyshev's Inequality Does and Does Not Imply

Roy (1952) used Tchebyshev's inequality to extend his analysis from the case where portfolio returns are normally distributed. Elton and Gruber ((1995), p. 240) indicate that, as long as the probability distribution of returns is "sufficiently well behaved that the Tchebyshev inequality holds," Roy's argument implies that optimal safety-first portfolios lie along the mean-variance efficient frontier.

Tchebyshev's inequality is a general result, holding for all distributions with a finite second moment. Of course, there is no mean-variance frontier for distributions lacking a second moment. Theorem 3 makes the case that an optimal safety-first portfolio typically lies off the mean-variance frontier. If the argument in Elton-Gruber (1991) were correct, Theorem 3 would be false.

Theorem 3 is true, which leads to the next question: where is the flaw in the Elton-Gruber argument? In this paper's notation, Tchebyshev's inequality implies that $\text{Prob}\{(W - \mu_P)/\sigma_P < -K\} \leq 1/K^2$. Elton-Gruber suggest that a safety-first optimum can be identified by setting $K = (\mu_P - s)/\sigma_P$, and then choosing a portfolio P that maximizes K. There are several difficulties with this argument. First, the $1/K^2$ bound in Tchebyshev's inequality is a two-sided bound, and the safety-first probability constraint $\text{Prob}\{W < s\}$ is one-sided. Second, $1/K^2$ is an inequality bound. It need not hold with equality, and certainly cannot hold with equality when $K < 1$.

Moreover, it may even be undesirable that the bound hold with equality. There are cases where it is possible to find a portfolio such that $\text{Prob}\{W < s\} = 0$, yet maximizing K leads to inefficiency because it reduces expected wealth. This can be seen in the numerical example presented earlier, for $A = s = 0.9$. In this case, $\mu_P = 1.25$, $\sigma_P = 0.94$, and $\text{Prob}\{W < s\} = 0$. For the efficient portfolio, the corresponding value of K is 0.377, and the upper bound, $1/K^2$, on $\text{Prob}\{W < s\}$ is seven, not only well above zero, but well above unity. Consider what happens when we alter the portfolio by changing the state 1 payoff to 1.0614, and the state 8 payoff to 2.03. We note that this change leaves the value of $\text{Prob}\{W < s\}$ at zero, and causes the standard deviation to fall to 0.37. More importantly, the mean payoff also declines to 1.0614, yet the value of K increases to 0.4364. Hence, increasing K, the variable that Elton-Gruber suggest be maximized, has made the investor worse, not better, off.

VII. BPT-SA when Returns are Normally Distributed

It is not difficult to build a discrete-state model in which an investor can choose a portfolio whose return distribution is approximately normal. Nevertheless, Theorem 1 establishes that BPT-SA investors reject such opportunities, in favor of portfolios formed from a combination of a (possibly risky) bond and a lottery ticket. If we restrict all portfolio return distributions to be normal, this preferred return pattern is unavailable. Yet, there is still something interesting to say about the difference between the mean-variance frontier and the BPT-SA frontier.

Consider a world with two securities, X and Y, both of which feature normally distributed returns. X has an expected return of 16% with a standard deviation of 20%, while Y has an expected return of 10% with a standard deviation of 15%. The correlation between the returns of X and Y is zero.

Imagine an investor with \$1 of current wealth who chooses \$1 as *A*, his aspiration level. A portfolio consisting entirely of Y offers an expected wealth of \$1.10 along with a 25.2% probability of missing the aspiration level. Portfolio Y is not on the BPT-SA efficient frontier since it is dominated by portfolio Z. Portfolio Z, combining \$0.50 of X with \$0.50 of Y, has a lower probability of missing the aspiration level, 14.9%, along with higher expected wealth, \$1.13. Figure 4, A shows the BPT-SA efficient frontier extending from Z to X—call this the frontier for the low aspiration investor.

Now consider an investor with a high aspiration level, such as the Dubins and Savage (1976) investor, who is in a casino with \$1,000 and desperately needs \$10,000 by morning; anything less is worth nothing—call this the high aspiration investor.

Dubins and Savage concluded that the optimal portfolio for this high aspiration investor is concentrated in a single bet, a bet that offers a chance for a \$10,000 payoff. The optimality of a single bet portfolio for a high aspiration investor can also be illustrated in our two-security example.

Imagine a high aspiration investor with \$1 of current wealth who defines the aspiration level *A* as \$1.20, a high aspiration level relative to the expected 10% return of security Y and the expected 16% return of security X. The efficient BPT-SA frontier for this investor contains only one portfolio, consisting of a single bet on security X (Figure 4, B).

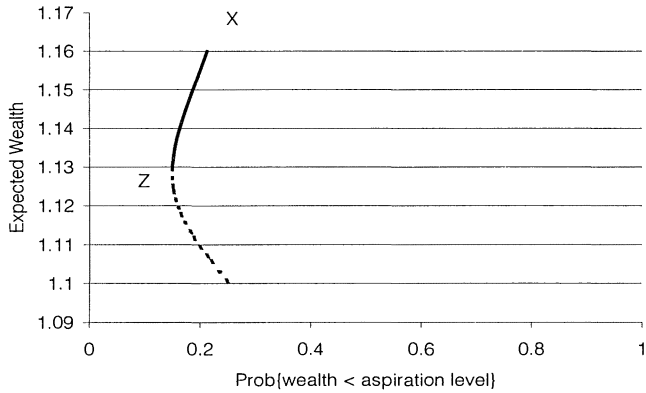
When returns are normally distributed and no short sales are allowed, some optimal BPT-SA portfolios are on the mean-variance frontier. For example, mean-variance investors with little aversion to variance, like investors with high aspiration levels, choose X as their portfolio. However, not all portfolios on the BPT-SA efficient frontier are identical to portfolios on the mean-variance efficient frontier. The distinction is evident most when high aspiration investors and mean-variance investors consider casino-type securities, securities with low expected returns and high variance.

Consider securities L and M. Security L has an expected return of 2% and a standard deviation of 90%, while security M has an expected return of 20% and a standard deviation of 30%. Security L is a casino-type security since its expected return is lower than M's while its variance is higher.[5] A portfolio composed entirely of L is not on the mean-variance efficient frontier; it is dominated by other portfolios, including the one composed entirely of M. Given the prohibition against short sales, a portfolio composed entirely of L is on the efficient frontier for a high aspiration investor with an aspiration level of \$1.30, since it provides the highest probability of reaching the aspiration level (see Figure 5). High

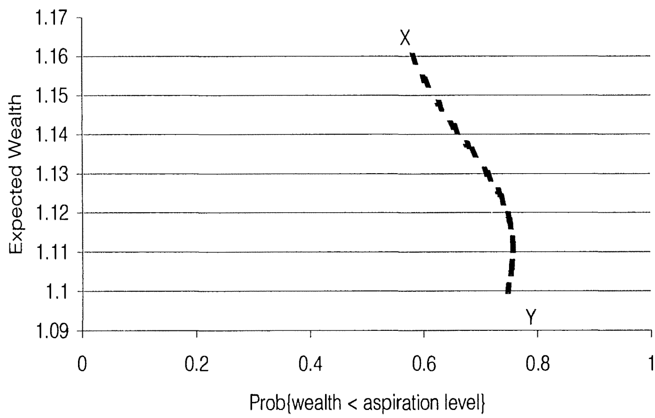
[5]The argument would change little if the expected return of L were actually negative, as is common in casinos.

FIGURE 4

4A. The BPT-SA Efficient Frontier for an Investor with a (Low) $1 Aspiration Level (the frontier extends from Z to X)



4B. The BPT-SA Efficient Frontier for an Investor with a (High) $1.20 Aspiration Level (the frontier consists only of X)



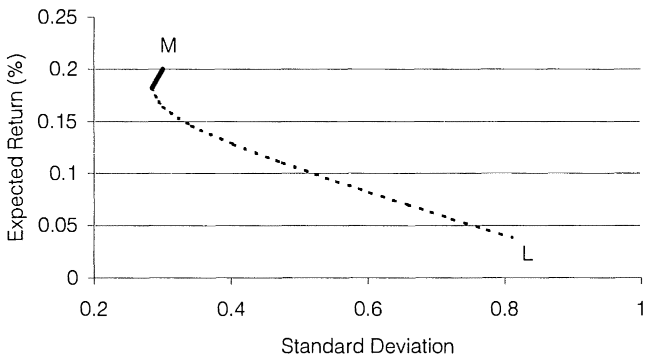
aspiration investors, like Dubins-Savage investors, are not risk seeking. They choose the high risk L because it provides the highest probability of reaching the aspiration level, not because they like risk.

The SP/A framework is similar to the value-at-risk (VaR) framework. In both, optimization involves tradeoffs between expected wealth and probabilities of falling short of an aspiration level. The usual aspiration level in VaR analyses is a poverty level; the goal is to combine a low probability of falling below a poverty level with the highest possible expected wealth. But we can think of a VaR framework that corresponds to the high aspiration framework, where the aspiration level pertains to riches, not poverty.

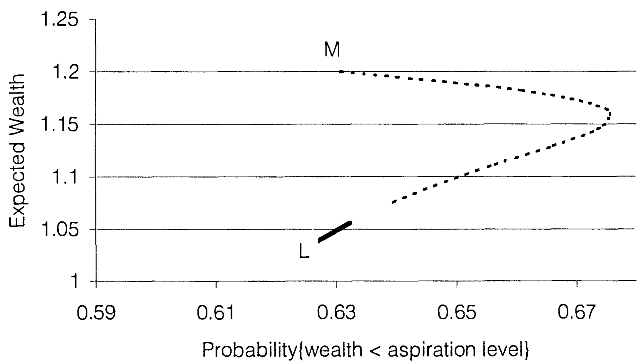
FIGURE 5

The Mean-Variance Efficient Frontier and the BPT-SA Efficient Frontier for an Investor with a $1.30 Aspiration Level

5A. Mean-Variance Frontier



5B. BPT-SA Frontier



VIII. BPT-MA: Behavioral Portfolio Theory with Multiple Accounts

Some investors have low aspirations and others have high aspirations, exclusively. Most investors combine the two; they want to avoid poverty, but they also want a shot at riches. Portfolios that combine low and high aspirations are often depicted as layered pyramids where investors divide their current wealth between a bottom layer, designed to avoid poverty, and a top layer, designed for a shot at riches.

Mental accounting is the feature that underlies the difference between BPT-SA, the single account version of BPT, and BPT-MA, the multiple account version. BPT-SA investors act as if they consider covariances—they integrate their portfolios into a whole. BPT-MA investors act as if they overlook covariances—they segregate their portfolios into distinct mental accounts. This feature is captured in the adage "people keep their money in separate pockets."

Mental accounting is a feature of prospect theory, a framework Kahneman and Tversky (1979) developed to model the tendency to make decisions based on gains and losses relative to a reference point. Shefrin and Statman (1985) suggest that original purchase price serves as a reference point in investment decisions, and use prospect theory to explain why investors sell winners too early and hold losers too long (the disposition effect). Although the disposition effect can be formally captured in this paper by setting $A = W_D$, our focus here is on the structure of portfolios rather than the timing of buy/sell decisions for individual securities.

Tversky and Kahneman (1986) present evidence of the difficulty that covariance and other properties of joint probability distributions impose on mental processes. People simplify choices by dividing joint probability distributions into mental accounts and in the layered pyramid structure of portfolios.

There is considerable evidence, from experiments and practice, that investors overlook covariances. Kroll, Levy, and Rapoport (1988) conducted experiments where three groups of subjects were given expected returns and the variance-covariance matrix of three securities, A, B, and C, and asked to form portfolios. The correlations between A and B and A and C were set at zero for all three groups, but the correlation between B and C was set at zero for the first group, at 0.8 for the second group, and at −0.8 for the third group. The differences between the covariances set for the three groups are such that if the subjects in the three groups considered covariances, the average proportion allocated to each of the stocks would have been different across the three groups. Yet Kroll, Levy, and Rapoport find no significant differences between the portfolios of the three groups. In essence, subjects ignored covariances as they constructed their portfolios.

Jorion (1994) presents similar evidence from the practice of institutional investors, who invest globally, to often put securities into one layer of the pyramid and currencies into another. They separate the management of securities from the management of currencies and assign the latter to "overlay" managers. As Jorion notes, the overlay structure is inherently suboptimal because it ignores covariances between securities and currencies. He calculates the annual efficiency loss that results from overlooking covariances as the equivalent of 40 basis points.

We present BPT-MA for the case of two mental accounts. To understand the mental accounting structure of portfolios in BPT-MA, imagine an investor who contains three entities, a principal we call the "planner" and two agents we call "doers." This follows the self-control framework developed by Thaler and Shefrin (1981) and applied by Shefrin and Statman (1984) in the context of dividends. The first doer has a low aspiration level, and the second doer has a high aspiration level. Each doer is associated with one mental account, and the planner balances the two doers to maximize overall utility as he divides current wealth W_0 between the two.

As we develop the BPT-MA model, we use the two-security example, X and Y, from the previous section to illustrate some of the key concepts. Imagine that utility of the low aspiration doer possesses a Cobb-Douglas function,

$$U_s = P_s^{1-\gamma} E_h(W_s)^{\gamma},$$

where P_s is the probability of falling short of the low aspiration level A_s (\$0.20), W_s is the terminal wealth of the low aspiration doer, and γ is a non-negative weighting parameter (0.1). Similarly, the utility of the high aspiration doer is

$$U_r = P_r^{1-\beta} E_h(W_r)^{\beta},$$

where P_r is the probability of falling short of the high aspiration level A_p (\$1.20), W_r is the terminal wealth of the high aspiration doer, and β is 0.1.

The utility function of the planner combines the utilities of the low and high aspiration doers, U_s and U_r, where the weight attached to the high aspiration doer is $K_r = 10{,}000$, much higher than the weight attached to the low aspiration doer, $K_s = 1$. We assume that the planner's utility takes the form,

$$U = \left[1 + K_{dr}\left(P_r^{1-\beta} E_h(W_r)^{\beta}\right)\right] K_{ds}\left[P_s^{1-\gamma} E_h(W_s)^{\gamma}\right]. \tag{6}$$

The planner divides initial wealth W_0 into two portions, $W_{s,0}$ for the low aspiration account and $W_{r,0}$ for the high aspiration account. Note that the planner's utility function (6) has a safety-first bent. The utility of the planner is zero when the utility of the low aspiration doer is zero, but it is not necessarily zero when the utility of the high aspiration doer is zero. This implies that the *first* dollar of wealth W_0 will be allocated to the low aspiration account. In other words, achieving low aspiration, or safety comes *first*.

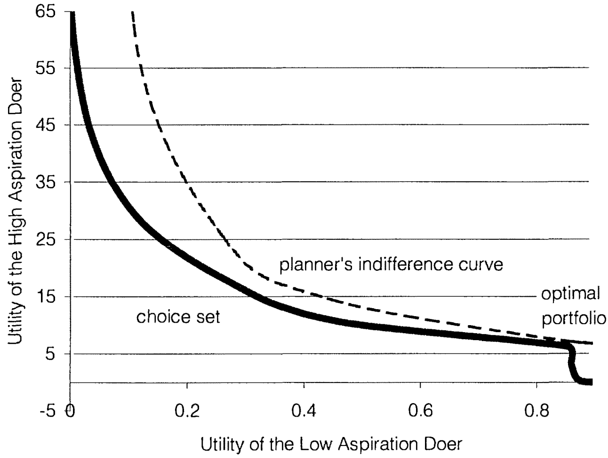
Consider a specific example. Imagine that the investor faces two securities D and E. D has an expected return of 3% with a standard deviation of 1%, while E has an expected return of 5% with a standard deviation of 10%. Figure 6 depicts the utilities of each of the two doers under each feasible division of the \$1 current wealth. It also depicts the optimal portfolio of each doer. The optimal portfolio for the planner is composed of an allocation of \$0.20 for the low aspiration doer and \$0.80 for the high aspiration doer. The low aspiration doer places 93% of his wealth in the low expected return security D, while the high aspiration doer places 100% of his wealth in the high expected return security E. The probability that the low aspiration doer will miss his aspiration level is 5.8%, while the probability that the high aspiration doer will miss his aspiration level is lottery-like, greater than 99.9%.

We have excluded short sales from our analysis so far, but if short sales are allowed, the high aspiration doer would take a short position in D to increase his holdings of E. As a result, the planner would end up with a short position in D in his high aspiration account, but a long position in D in his low aspiration account. This stems from the lack of integration between the two mental accounts; covariances are overlooked.

In summary, BPT-MA investors match mental accounts with goals. The two mental accounts are not integrated. As a result, BPT-MA investors may take offsetting positions, borrowing for leverage in their high aspiration accounts, while they lend in their low aspiration accounts.

FIGURE 6

The Choice Set of BPT-MA Portfolios Available to a Planner with Two Doers and the Optimal Portfolio of a Planner



IX. Structural Issues

Theorem 1 implies that both the low aspiration doer and the high aspiration doer will choose portfolios formed by combining (possibly risky) bonds and a lottery ticket. But the mental accounting maximum solution will push the accounts in the layers to the extremes. That is, the low aspiration account will look more like risk-free bond than the high aspiration account, which, in turn, looks more like a lottery ticket.

To see why, consider the nature of the Engel curve that describes the impact of a marginal dollar of wealth on the investor's demand for contingent claims. For the sake of exposition, focus on the equiprobable case. Consider how the low aspiration doer allocates successive increments of his allocation $W_{s,0}$. According to the argument used to establish Theorem 1, the low aspiration doer allocates the first dollar to claims in state s_n. Indeed, the low aspiration doer's initial allocation $W_{s,0}$ would have to be at least $A_s(v_n + v_{n-1})$ before it would be worthwhile for him to choose positive claims in two states. Choosing positive claims in two states would lower $E_h(W_s)$ without raising P_s. For $W_{s,0} \leq A_s v_n$, $U_s = 0$ because $P_s = 0$. For $A_s v_n \leq W_{s,0} \leq A_s(v_n + v_{n-1})$, U_s is a concave function, $U_{s,n} = p^{1-\gamma} E_h(W_s)^\gamma$, featuring diminishing marginal utility. A low aspiration doer with a higher allocation $W_{s,0}$ will find it worthwhile to hold positive claims that pay off in states s_{n-1} and s_n. In this case, utility is $U_{s,n-1} = (2p)^{1-\gamma} E_h(W_s)^\gamma$. Continuing this way, recursively define $U_{s,n-j}$ equal to $((j+1)p)^{1-\gamma} E_h(W_s)^\gamma$.

Because the low aspiration doer selects his optimal allocation by making a choice about the number of states in which to hold positive claims, the optimal value of U_s will be equal to $\max\{U_{s,n-j}\}$, for $j = 0$ to $n - 1$. This implies that the *indirect* utility function $U_s(W_{s,0})$ associated with optimal allocations will be not be concave everywhere. At crossover points where positive claims to new states are brought in, marginal utility increases. This is because the reallocation of wealth from state s_n to a lower state both raises P_s and lowers $E_h(W_s)$.

If state prices are low at the upper end of the state range and the aspiration point A_s is low, then increasing $W_{s,0}$ leads to "rapid" crossovers, since increasing the probability of reaching A_s by shifting the allocation from state s_n to a lower state comes at a low cost in terms of foregone $E_h(W_s)$. This means that the low aspiration doer responds to increased allocation by shifting into less risky bonds. However, further increases in $W_{s,0}$ lead the low aspiration doer to consider states with higher state prices. As this happens, the cost of increasing P_s in terms of foregone $E_h(W_s)$ mounts. Now larger absolute increments in $W_{s,0}$ than before are necessary to induce crossovers.

The Engel structure of the high aspiration subproblem is the same as above, except that $A_r > A_s$. Qualitatively, the higher aspiration point leads to a lower sensitivity of the crossover points with respect to $W_{r,0}$ than with respect to $W_{s,0}$.

The difference between Engel curves shows up in the character of the portfolios selected for the two layers. The low aspiration doer will tend to use additional allocations $W_{s,0}$ to form a less risky bond, thereby moving the lottery component to zero. On the other hand, the high aspiration doer will tend to use additional allocations $W_{r,0}$ for the lottery component, focusing on the maximization of the expected payoff $E_h(W_r)$. Moderate levels of wealth will lead to the Friedman-Savage insurance/lottery property when the aspiration level for the low aspiration doer is low, and the aspiration level for the high aspiration doer is high. Such investors choose riskless bonds in the low aspiration layer of their portfolios and a lottery ticket for their high aspiration layer.

Just as the risk tolerance parameter d affects the choice of mean-variance investors among portfolios on the mean-variance frontier, the BPT parameters determine the portfolios of BPT investors. Recall from Section V that in BPT-SA, risk tolerance is captured by a combination of parameters. Tolerance for risk in BPT, both the single account and multiple account versions, is determined by the degree of fear (which is increasing in q_s), the degree of hope (which is increasing in q_p), and the strength of fear relative to hope (which is increasing in δ). Increased fear accentuates the tendency of BPT investors to make the bond portion of their low aspiration mental accounts risk free. Increased hope accentuates the tendency of BPT investors to increase the maximum possible payoffs in their high aspiration mental accounts.

Lopes discusses a special case of her model she calls "cautious optimism." Here, fear is the predominant emotion throughout most of the domain, except for the upper end of the range. A cautiously optimistic individual overweights the probabilities attached to both the worst and best outcomes. That is, cautious optimism gives rise to a weighting scheme with thicker tails than the underlying distribution. A cautious optimist is inclined to extremes in the two layers of his portfolio, a risk-free bond for his low aspiration account and a high expected payoff for his high aspiration account.

Will the planner's portfolio, obtained by combining the low and high aspiration subportfolios, be mean-variance efficient? The following theorem provides the answer.

Theorem 4. If there are at least five states that feature positive consumption in a BPT-MA portfolio P_M, and distinct values for v_i/p_i, then P_M is neither mean-variance efficient nor BPT-SA efficient.

Proof. The proof is similar to the argument establishing Theorem 3. There can be at most four distinct positive consumption values in an SP/A portfolio: A_s, A_r, $A_s + A_r$, and an additional value for W_n in excess of A_r. Now any monotone increasing mean-variance portfolio with positive consumption in five or more states will feature at least five distinct values of consumption. Therefore, P_M is mean-variance inefficient. Similarly, Theorem 1 implies that P_M is BPT-SA inefficient.[6] □

X. Evolutionary Implications

As noted earlier, BPT optimal portfolios are typically off the mean-variance efficient frontier. Does this result have evolutionary implications? In particular, do BPT investors lose wealth to mean-variance investors over the long term? Do they vanish?

Blume and Easley (1992) develop a general framework for analyzing the evolutionary questions about trading. Their model implies that BPT investors can, but need not, lose wealth to mean-variance investors over the long run. To be sure, mean-variance portfolios are not the most "fit" from the perspective of long-term survival. This distinction belongs to log-utility portfolios, and a log-utility portfolio does not typically lie on the mean-variance frontier.

Consider a multiperiod model in which, at each date, state i occurs with probability p_i. A portfolio that is unfit loses wealth in the long run. Blume-Easley establish that the fitness of a portfolio with payoff return $(W_1, \ldots, W_n)$ is determined by its budget share profile $(v_1 W_1/W_0, \ldots, v_n W_n/W_0)$. The fittest portfolio is one whose budget share profile coincides with the probability vector $p = (p_i)$.

Consider the budget share per unit probability $(v_i W_i/p_i W_0)$. Blume-Easley show that portfolios can be ranked by fitness using the non-negative entropy function,

$$\Sigma p_i \ln(p_i W_0/v_i W_i).$$

Given two portfolios, the fitter portfolio is the one with the lower entropy. Note that minimum entropy is achieved when budget share per unit probability equals unity across all states.

[6]Note that a BPT-MA portfolio is not BPT-SA inefficient just because the former has two aspiration levels while the former has one. It is straightforward to define a version of BPT-SA with two aspiration levels. The major source of the inefficiency is the lack of integration between the two accounts. In an efficient BPT-SA portfolio, the associated payoff values are as follows: zero, the aspiration levels, and possibly an additional amount associated with the s_n. This is not what a BPT-MA investor chooses.

Consider the budget share of the BPT-SA efficient portfolio (4) and mean-variance efficient portfolio (5). The budget share per unit probability for the BPT-SA portfolio is

$$(7) \qquad \begin{array}{ll} \infty, & \text{for } i \leq i_c, \\ (p_i/v_i)(W_0/A), & \text{for } i_c \leq i < n, \\ p_n W_0/(W_0 - \Sigma v_i W_i). & \end{array}$$

Note that the probability to budget share ratio is infinite for the lowest states, and that it is an increasing function for $i_c \leq i < n$. Clearly, this ratio will not be the unity function.

The budget share per unit probability for a mean-variance efficient portfolio has the form,

$$(8) \qquad \left\{(v_i/bp_iW_0)\left[1 - \left([\Sigma v_j - bW_0]/\Sigma v_j^2/p_j\right) v_i/p_i\right]\right\}^{-1}.$$

For a non-negative solution, it is straightforward to establish that this function is monotone increasing in v_i/p_i. If the non-negative constraint is active, then the probability to budget share ratio will be infinite in the states with the highest prices per unit probability, and be monotone increasing in states that feature positive consumption.

Both BPT-SA efficient portfolios and mean-variance efficient portfolios feature similar patterns in their budget share per unit probability profiles. In particular, a zero budget share in a state with positive probability leads to infinite entropy, in which case, long-term wealth falls to zero with probability one. Clearly, this can occur with both efficient BPT-SA and mean-variance portfolios.

We note that a given BPT-SA portfolio can have a higher entropy than some portfolios along the mean-variance frontier, and a lower entropy than others.[7] The mean-variance frontier is effectively traced out through variation in parameter b. The case $b = 0$ corresponds to risk neutrality. Higher values of b correspond to higher risk aversion. Both very low and very high risk aversion lead to portfolios that are low in fitness. High levels of risk aversion lead to low expected returns, while levels of risk aversion close to zero (risk neutrality) lead to low long-run wealth because wealth declines precipitously during disasters.

Consider a multiperiod framework where one-period state prices v are constant over time, and all investors' portfolio budget shares likewise remain constant. In this case, the portfolio with the lower entropy value will dominate in the long run. If budget shares vary with time, the entropy condition becomes more complex, but the same general idea applies.

If the aspiration value A is constant over time, but wealth varies, then the budget shares for a BPT-SA efficient portfolio also vary. This follows from (4) and the property of the Engel function discussed earlier. Indeed, if wealth becomes low enough, the budget share for some state will fall to zero, thereby leading to infinite entropy.

[7] This can be seen in the eight-state numerical example discussed earlier. Consider the case when $A = 0.9$. When the mean-variance parameter $b = 1.9$, the mean-variance portfolio features higher entropy. When $b = 1.8$, the SP/A portfolio features higher entropy.

Finally, consider what happens when the aspiration level A is tied to wealth, and the wealth ratio A/W remains constant. In this case, the budget shares of BPT-SA investors do remain constant over time. Hence, the Blume-Easley entropy function serves to rank order BPT-SA efficient portfolios and mean-variance efficient portfolios by fitness. In general, BPT-SA portfolios are more fit than some mean-variance efficient portfolios, but less fit than others.

XI. Real World Portfolios and Securities

The optimal portfolios and securities we described were constructed to fit optimally the preferences of specific investors. Such optimal securities will be constructed in a cost-free world. As Allen and Gale (1987) noted, however, the construction costs of real world securities influence their design and limit their number. Shefrin and Statman (1993) describe real world securities as a design that meets the general preferences of many investors rather than the specific preferences of one.

Swedish lottery bonds come close to the optimal security for a low aspiration mental account (Green and Rydqvist (1999)). Holders of lottery bonds receive lottery tickets in place of interest coupons. All bondholders receive the bonds' face value at maturity, but lottery winners receive much more than a usual coupon payment while losers receive a zero coupon payment.

Lottery bonds with one coupon to maturity resemble the optimal security for a low aspiration account where the aspiration level is equal to the face value of the bond. Bondholders receive neither coupons nor face value in some low states where the Swedish government goes bankrupt. Beyond these are states where bondholders lose the lottery and receive only the face value of their bonds. Last, there is a high state where the face value of the bond is augmented by a lottery payoff.

Swedish lottery bonds are not the only lottery bonds. Britain's premium bonds are lottery bonds as well. Still, most bonds are regular bonds, offering interest coupons rather than lottery tickets. Investors do not necessarily need the government to design lottery bonds; many investors design lottery bonds on their own. Some of these investors buy both bonds and lottery tickets, as in the Friedman-Savage puzzle. Others are investors who combine money market funds with call options. McConnell and Schwartz (1993) describe the insight of Lee Cole, Merrill Lynch options marketing manager, who discovered that many investors who held money market funds used the interest payments to buy call options.

Call options are different from lotteries that offer single size prizes. Call options offer many prizes, low prizes when they are slightly in-the-money at expiration and high prizes when they are deep in-the-money. We can think of call options as securities designed to appeal to many investors with different aspiration levels. Call options do not match the precise aspiration level of any particular investor, but match approximately the aspiration levels of many investors.

Cole's observation led to the construction of LYONs, securities that combine the security of bonds with the potential of call options. The same observation led many brokerage firms and insurance companies to offer equity participation notes, securities that combine a secure floor, usually equal to the amount of the initial investment, with some potential linked to an index such as the S&P 500.

Treasury bills are right for investors with very low aspiration levels, while equity participation notes are right for investors with higher aspiration levels. Investors with even higher aspiration levels choose stocks and those with yet higher aspiration levels choose out-of-the-money call options and lottery tickets. Stocks, call options, and lottery tickets feature many states with zero payoffs, but they also features states with payoffs that meet high, even exceedingly high, aspiration levels.

Cash, bonds, and stocks are the most common elements of portfolios and they are the elements of the portfolio puzzle discussed by Canner, Mankiw, and Weil (CMW (1997)). CMW note that investment advisors recommend that investors increase the ratio of stocks to bonds if they want to increase the aggressiveness of their portfolios. This recommendation is puzzling within the CAPM since it violates two-fund separation. Two-fund separation states that all CAPM efficient portfolios share a common ratio of stocks to bonds and that attitudes toward risk are reflected only in the proportion allocated to the risk-free asset.

The portfolio advice of mutual fund companies illustrates the CMW puzzle. As Fisher and Statman (1997) note, mutual fund companies often recommend that investors construct portfolios as pyramids of assets: cash in the bottom layer, bonds in the middle layer, and stocks in the top layer. Investors increase the aggressiveness of their portfolios by increasing the proportion allocated to stocks without necessarily increasing the proportion allocated to bonds.

Pyramid portfolios that are puzzling within CAPM are consistent with BPT, where two-fund separation fails. Consider a BPT-MA framework with three mental accounts in the hands of three "doers" whose aspiration levels range from low to medium and to high. BPT-MA investors do not follow two-fund separation. Greater aggressiveness might manifest itself by an increased portfolio allocation to the high aspiration doer and by a corresponding increase in the allocation to stocks in the portfolio.

XII. Conclusion

We develop a positive behavioral portfolio theory (BPT) and explore its implications for portfolio construction and security design. We present our model in two versions, BPT-SA, where the portfolio is integrated into a single mental account and BPT-MA, where the portfolio is segregated into multiple mental accounts, such that covariances among mental accounts are overlooked. BPT investors, like the investors in the Friedman-Savage puzzle, are simultaneously risk averse and risk seeking; they buy both bonds and lottery tickets.

Portfolios within BPT-MA resemble layered pyramids where each layer (i.e., mental account) is associated with a particular aspiration level. We explore a simple two-layer model with a low aspiration layer designed to avoid poverty and a high aspiration layer designed for a shot at riches. Since BPT-MA investors overlook covariance between layers, they might combine a short position in a security in one layer with a long position in the same security in another layer.

We explore the links between BPT portfolios and mean-variance, CAPM and VaR portfolios.

Optimal securities for BPT investors resemble combinations of bonds and lottery tickets. The bonds for the low aspiration mental account resemble risk-free or investment grade bonds, while the bonds for the high aspiration mental account resemble speculative (junk) bonds. We explore the similarities between optimal BPT securities and real world securities such as bonds, stocks, and options.

We plan to extend BPT in several ways. One extension involves the design of securities by corporations, especially in connection with capital structure and dividend policy. Capital structure and dividend policy are usually approached from the supply side; dividends are regarded as information signals and capital structure is regarded as a solution to agency problems. We think that capital structure and dividend policy also need to be approached from the demand side; some corporate securities fit better than others into the layered pyramid structure of BPT portfolios. Analysis of dividend policy requires a multi-period model.

A multi-period BPT model is also useful for analyzing risk and its relationship to time diversification. Proponents of time diversification argue that the risk of stocks declines as the time horizon increases, while opponents argue that it does not (see Kritzman (1994)). A multi-period BPT model would link time to perceptions of risk and show, for example, how investors revise portfolios when their original aspirations are reached.

Lastly, the road from BPT will lead to an equilibrium asset pricing model, extending Shefrin and Statman (1994), just as the road from mean-variance portfolio theory led to the CAPM.

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