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Two-State Option Pricing

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I. Introduction

IN THIS PAPER WE present an elemental two-state option pricing model (TSOPM) which is mathematically simple, yet can be used to solve many complex option pricing problems.¹ In contrast to widely accepted option pricing models which require solutions to stochastic differential equations, our model is derived algebraically. First we present the mathematics of the model and illustrate its application to the simplest type of option pricing problem. Next, we discuss the statistical properties of the model and show how the parameters of the model can be estimated to solve practical option pricing problems. Finally, we apply the model to the pricing of European and American put and call options on both non-dividend and dividend paying stocks. Elsewhere, we have applied the model to the valuation of the debt and equity of a firm with coupon paying debt in its capital structure [9], the valuation of options on debt securities [7], and the pricing of fixed rate bank loan commitments [1, 2]. In the Appendix we derive the Black-Scholes [3] model using the two-state approach.

II. The Two-State Option Pricing Model

Consider a stock whose price can either advance or decline during the next period. Let H_t^+ and H_t^- represent the returns per dollar invested in the stock if the price rises (the + state) or falls (the - state), respectively, from time $t - 1$ to time t , and V_t^+ and V_t^- the corresponding end-of-period values of the option. With the assumption that the prices of the stock and its option follow a two-state process, it is possible to form a riskless portfolio with the two securities. [See Black and Scholes [3] for the continuous time analog of riskless hedging.] Since the end-of-period value of the portfolio is certain, the option should be priced so that the portfolio will yield the riskless interest rate.

The riskless portfolio is formed by investing one dollar in the stock and

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¹ Since the original writing of this paper, the authors have learned that a similar procedure has been suggested by Rubinstein [10], Sharpe [11], and Cox, Ross, and Rubinstein [5].

purchasing α units of the option at a price of P_{t-1} . The value of α is chosen so that the portfolio payoffs are the same in both states, or

$$H_t^+ + \alpha V_t^+ = H_t^- + \alpha V_t^- \quad (1)$$

Solving for α we obtain the number of units of the option to be held in the portfolio per \$1 invested in the stock.

$$\alpha = \frac{H_t^- - H_t^+}{V_t^+ - V_t^-} \quad (2)$$

A negative value of α implies that the option is sold short (written) with the proceeds being used to partially fund the purchase of the stock.

The time $t - 1$ value of portfolio is $1 + \alpha P_{t-1}$. The end-of-period value is given by either side of (1). Discounting the left-hand side by the riskless interest rate, R , and setting the discounted value equal to the present value of the portfolio, a pricing equation for the option is obtained.

$$1 + \alpha P_{t-1} = \frac{H_t^+ + \alpha V_t^+}{1 + R} \quad (3)$$

Substituting the value of α from (2) into (3), the price of the option can be solved in terms of its end-of-period values.

$$P_{t-1} = \frac{V_t^+(1 + R - H_t^-) + V_t^-(H_t^+ - 1 - R)}{(H_t^+ - H_t^-)(1 + R)} \quad (4)$$

Equation 4 is a recursive relationship that can be applied at any time $t - 1$ to determine the price of the option as a function of its value at time t .

Note that in equation (4) we make a notational distinction between an option's value (V) and its price (P). Assuming that an investor will exercise an option when it is in his best interest to do so,

$$V_t = \text{MAX}[P_t, \text{VEXER}_t], \quad (5)$$

where VEXER_t is the value of exercising the option at time t .

The distinguishing feature among American and European puts and calls is in the definition of their exercisable values. American options can be exercised at any time whereas European options can only be exercised at maturity. Calls are options to buy stock at a set price whereas puts are options to sell. Letting S_t represent the time t price of the stock, X the option's exercise price, and T the maturity date of the option, we obtain

American:

$$\begin{array}{ll} \text{Call} & \text{VEXER}_t = S_t - X \quad \text{for all } t, \\ \text{Put} & \text{VEXER}_t = X - S_t \quad \text{for all } t, \end{array}$$

European:

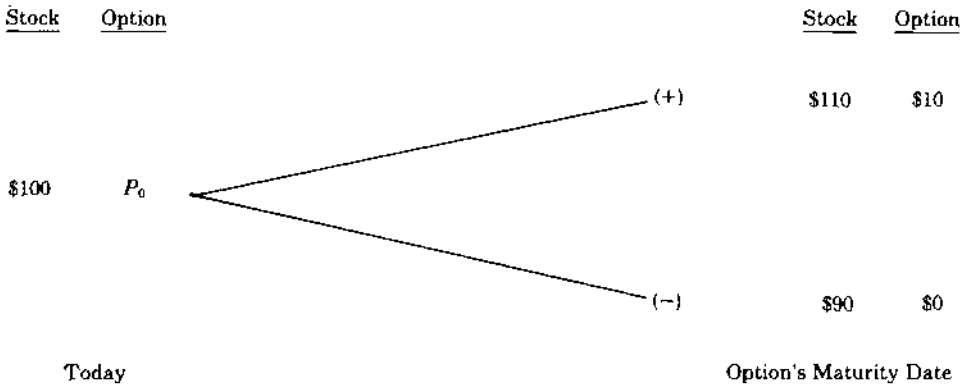
$$\begin{array}{ll} \text{Call} & \text{VEXER}_t = S_t - X \quad \text{for } t = T \\ & \text{VEXER}_t = 0 \quad \text{for } t < T, \\ \text{Put} & \text{VEXER}_t = X - S_t \quad \text{for } t = T \\ & \text{VEXER}_t = 0 \quad \text{for } t < T. \end{array}$$

Recognizing that for both American and European puts and calls

$$P_T = 0, \quad (7)$$

since there is no value associated with maintaining an option position beyond maturity, (4-7) represent the formal specification of the two-state model. Through repeated application of (4), subject to (5-7), one can begin at an option's maturity date and recursively solve for its current price.

To illustrate the model, consider a call option on a stock with an exercise price of \$100. The current price of the stock is \$100 and the possible prices of the stock on the option's maturity date are \$110 and \$90, implying $H^+ = 1.10$ and $H^- = .90$. Assuming that the option is exercised if the stock price rises to \$110 and is allowed to expire worthless if the stock price falls to \$90, the present prices and the end-of-period payoffs of the stock and option can be represented by the following two-branched tree diagram.



If an investor purchases the stock and writes two call options, the end-of-period portfolio value will be \$90 in both states. Equivalently, for every \$1 invested in the stock, a riskless hedge requires that $\alpha = (.90 - 1.10)/(10 - 0) = -.02$, or that .02 options are written. Assuming a risk free interest rate of 5%, the present value of the riskless portfolio should be $\$90/1.05$ or \$85.71 to ensure no riskless arbitrage opportunities between the stock-option portfolio and a riskless security. Since the riskless portfolio involves a \$100 investment in the stock which is partially offset by the two short options, an option price of \$7.14 is required to obtain an \$85.71 portfolio value. The option price can also be obtained directly from (4):

$$\begin{aligned}
 P_0 &= \frac{\$10(1 + .05 - .90) + \$0(1.10 - 1.05)}{(1.10 - .90)(1 + .05)} \\
 &= \frac{\$10(.15)}{.2(1.05)} = \$7.14.
 \end{aligned}$$

Although this example is unrealistic, it nevertheless illustrates two of the most important features of the TSOPM. We can observe that the option price does not depend upon the probabilities of the up (+) or down (-) states occurring or the risk preferences of the investor. Two investors who agreed that the stock price is in equilibrium, but had different probability beliefs and preferences, would both view \$7.14 as the equilibrium option price. As long as they agreed on the magnitudes of the underlying stock's holding period returns (H^+ and H^-), they would agree on the price of the option.

The example can be extended to a multiperiod framework in which the price of the underlying stock can take on only one of two values at any time t given the price of the stock at $t - 1$. Consider the case in which a non-dividend paying stock's holding period return is 1.175 in all up states and .85 in all down states. Given an initial stock price of \$100, these return parameters imply the four-period price pattern shown in Figure 1.

Assume that we wish to value a call option which matures at the end of period 4 and has an exercise price of \$100. Given a riskless interest rate of 1.25% per period (5% per year, assuming a one-year maturity), the sequence of option values corresponding to the stock prices in Figure 1 is given in Figure 2.

In Figure 2 the prices \$90.61 and \$37.89 are the values of the call obtainable by exercising at maturity. For those states at maturity where the price of the stock falls below the exercise price of \$100, the option expires worthless. Each of the time 3 option prices is obtained from (4). Similarly, the prices at time 2, 1, and 0 are obtained by recursive application of (4) resulting in a current call option price of \$14.41.

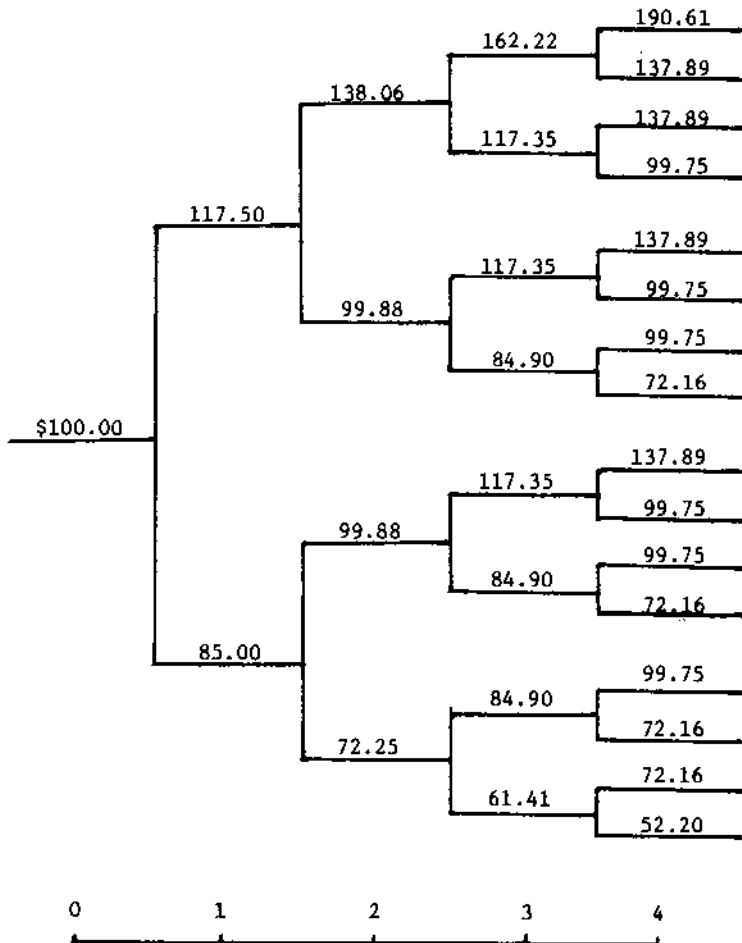


Figure 1. Price Path of Underlying Stock

Although the above example considers only four periods of time, one can always choose an interval of time to recognize price changes that more realistically captures expected stock price behavior. In Section IV we demonstrate the sensitivity of option prices to the choice of the time differencing interval under the assumption that H^+ and H^- are chosen to hold the mean and variance of the distribution of stock price changes constant over the life of the option. In the Appendix, a generalized formula for the multiperiod case is derived for the situation where R , H^+ , and H^- are constant. This formula is extended under the assumption that the two-state process evolves over an infinitesimally small interval of time.

III. Operationalizing the TSOPM

In the TSOPM, the only parameters describing the probability distribution of returns of the underlying stock are the magnitudes of the holding period returns, H^+ and H^- . Although our examples assume that H^+ and H^- remain constant

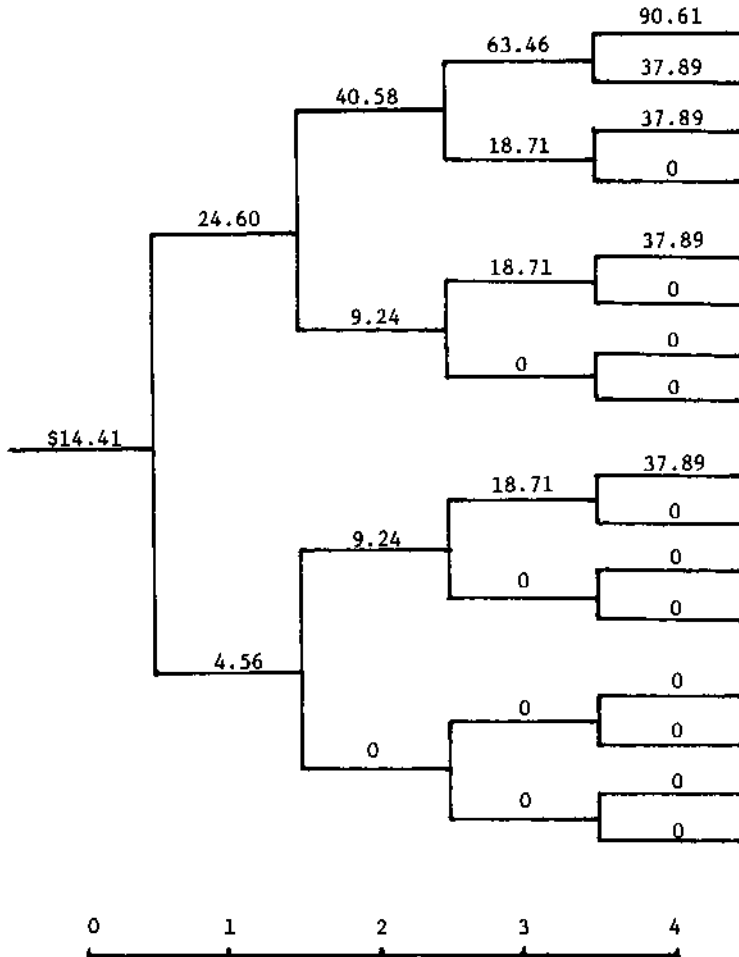


Figure 2. Price Path of European Call Option

through time, this is not a necessary assumption for the implementation of the model. Thus, if one can simply specify the pattern of H^+ and H^- through time, it is possible to value the option.

The TSOPM can be used as a method for obtaining exact values of options when the magnitudes of H^+ and H^- are known in advance. As a practical matter, the values of H^+ and H^- will not be known, but must be estimated. For example, if the probabilities associated with the occurrence of the + and - states remain stable over time along with the magnitudes of H^+ and H^- , then the two-state model implies a binomial distribution for the returns of the stock. It is well known that both the Normal and Poisson distributions can be viewed as limiting cases of the Binomial. Thus, the Binomial distribution can be employed as an approximation procedure for deriving option prices when the actual distribution of returns is assumed to be either Normal or Poisson. We will illustrate how the values of H^+ and H^- can be determined when the binomial distribution is used as an approximation to the lognormal distribution.

If the magnitudes of the relative price changes in our model and their associated probabilities remain stable from one period to the next, then the distribution of returns which is generated after T time periods will follow a log-binomial distribution with a mean

$$\mu = T[h^+ \theta + h^-(1 - \theta)] = T[(h^+ - h^-)\theta + h^-], \quad (8)$$

and variance

$$\sigma^2 = T(h^+ - h^-)^2\theta(1 - \theta), \quad (9)$$

where:

θ = the probability that the price of the stock will rise
in any period,

$$h^+ = \ln(H^+),$$

$$h^- = \ln(H^-).$$

In the last four-period example where $H^+ = 1.175$ and $H^- = .85$, the value of σ and μ for the entire four periods would be .324 and -.003, respectively, if a value of θ equal to .5 is assumed.

It is also possible to determine the values of H^+ and H^- that are implied by the values of μ , σ , θ , and T . By solving (8) and (9) in terms of these parameters and recognizing that $H = \exp(h)$, we obtain the following implied values of H^+ and H^- .

$$H^+ = \exp\left(\mu/T + (\sigma/\sqrt{T})\sqrt{\frac{(1-\theta)}{\theta}}\right) \quad (10)$$

$$H^- = \exp\left(\mu/T - (\sigma/\sqrt{T})\sqrt{\frac{\theta}{(1-\theta)}}\right) \quad (11)$$

As T becomes large, the log-binomial distribution will approximate a lognormal distribution with the same mean and variance.

IV. Applications of the Model

European Puts and Calls on Non-Dividend Paying Stocks

In this section we price European put and call options on non-dividend paying stocks using the two-state model as an approximation procedure for the case in which stock prices are assumed to follow a lognormal distribution.² Given the assumptions of no dividends and lognormal returns, the Black-Scholes [3] model provides the exact values for both types of options, thereby serving as a benchmark to assess the accuracy of the two-state model as a numerical procedure.

In Table 1, we present prices of one-year European put and call options with exercise prices of \$75, \$100, and \$125 assuming a current stock price of \$100. The riskless interest rate is assumed to be 5% per year. To conform with the Black-Scholes model, continuous compounding of interest is assumed. Thus, $R = e^{.05/N} - 1$, where N is the number of time intervals per year employed in the analysis. The values of H^+ and H^- are chosen so that the annual standard deviation of the logarithmic return is .324 as in the previous four-period example. The expected value of the logarithmic return is assumed to take on values of .5, .1, 0, $-.1$ and $-.5$ per year, and a value of θ equal to .5 is assumed. Finally, option prices are calculated by partitioning the year into 12, 52, and 100 time periods.

Consider the panel of Table 1 in which the stock's growth rate (μ) is assumed to be 0%. When the year is divided into 100 time intervals, the two-state prices of all the put and call options are quite close to their corresponding Black-Scholes prices. With these two parameters ($\mu = 0$, $T = 100$), the greatest absolute percentage difference between the Black-Scholes and two-state prices is .6%. Even if only 12 time differencing intervals are assumed, the two-state and Black-Scholes prices are remarkably close.

For growth rates of 10% and -10% , the two-state prices do not appear to be significantly different from those obtained when a zero growth rate is assumed. Thus, within this range of growth rates, the option price does not appear to be significantly affected by the growth rate.

If extreme growth rates are assumed ($\mu = .5$ and $\mu = -.5$), the two-state model does not appear to provide an accurate approximation to the Black-Scholes price for low T values. However, for 100 time intervals, the two-state and Black-Scholes prices are reasonably close.

The entries in Table 1 reveal that the option price is slightly dependent upon the stock's growth rate. In addition, if θ were varied, we would also discover a slight dependence on investor probability beliefs. These findings seem to contradict the earlier observation that two-state prices are independent of both investor preferences (which would be revealed through μ) and probability beliefs.

This dependence results from the fact that H^+ and H^- are chosen in the two-state model to conform with a given continuous distribution. Since the two-state model is only an approximation, the values of μ and θ implicit in the continuous distribution may be reflected in the two-state solution. In the limit as $T \rightarrow \infty$, the two distributions will be identical, and therefore, preferences and probabilities

² See Brennan and Schwartz [4] and Parkinson [7] for descriptions of alternative numerical procedures for solving option pricing equations.

Table 1
Comparison Between TSOPM and Black-Scholes Option Prices^a
(European)

Option Parameters			TSOPM		Black-Scholes Model		Percent Difference ^b	
μ	X	T	Call	Put	Call	Put	Call	Put
.5	75	12	29.94	1.28	30.74	2.08	- 2.6	-38.7
		52	30.57	1.91			- .5	- 8.2
		100	30.65	1.99			- .3	- 4.6
	100	12	14.23	9.35	15.14	10.26	- 6.0	- 8.9
		52	14.95	10.07			- 1.3	- 1.9
		100	15.03	10.16			- .7	- 1.1
	125	12	6.05	24.96	6.58	25.48	- 7.9	- 2.1
		52	6.38	25.28			- 3.0	- .8
		100	6.50	25.41			- 1.1	- .3
.1	75	12	30.58	1.93	30.74	2.08	- .5	- 7.6
		52	30.74	2.09			0	0
		100	30.72	2.06			- .1	- 1.1
	100	12	15.37	10.50	15.14	10.26	+ 1.5	+ 2.3
		52	15.12	10.25			- .1	- .2
		100	15.17	10.29			+ .2	+ .3
	125	12	6.78	25.69	6.58	25.48	+ 3.1	+ .8
		52	6.62	25.53			+ .7	+ .2
		100	6.57	25.47			- .1	0
0	75	12	30.85	2.19	30.74	2.08	- .4	+ 5.2
		52	30.74	2.09			0	0
		100	30.75	2.10			0	+ .6
	100	12	14.88	10.00	15.14	10.26	- 1.7	- 2.5
		52	15.08	10.20			- .4	- .6
		100	15.11	10.23			- .2	- .3
	125	12	6.55	25.46	6.58	25.48	- .3	- .1
		52	6.62	25.52			+ .7	+ .2
		100	6.60	25.50			+ .3	+ .1
-.1	75	12	30.60	1.94	30.74	2.08	- .5	- 7.0
		52	30.73	2.07			- .1	- .8
		100	30.74	2.08			0	- .3
	100	12	15.31	10.43	15.14	10.26	+ 1.1	+ 1.6
		52	15.11	10.23			- .2	- .3
		100	15.16	10.28			+ .1	+ .2
	125	12	6.55	25.43	6.58	25.48	- .8	- .2
		52	6.62	25.48			- .1	0
		100	6.60	25.43			- .8	- .2
-.5	75	12	30.25	1.59	30.74	2.08	- 1.6	-23.6
		52	30.65	2.00			- .3	- 4.2
		100	30.69	2.04			- .2	- 2.4
	100	12	13.85	8.97	15.14	10.26	- 8.6	-12.6
		52	14.87	10.00			- 1.8	- 2.6
		100	14.99	10.12			- 1.0	- 1.5
	125	12	4.65	23.55	6.58	25.48	-29.4	- 7.6
		52	6.12	25.02			- 7.0	- 1.8
		100	6.36	25.27			- 3.4	- .8

^a $S_0 = 100$, $R = e^{.05/N} - 1$, $\sigma = .324$, $\theta = .5$. In this table, $N = T$ in all cases.

^b Percent difference between the TSOPM and Black-Scholes (BS) prices is computed according to: $(TSOPM-BS)/BS$, rounded to the nearest one-tenth of one percent.

should not be reflected in option prices. In the Appendix, we derive the Black-Scholes equation using the two-state model. As expected, neither the growth rate nor probabilities enter the final solution. For practical applications, the two-state model appears to provide an accurate approximation to the Black-Scholes model if 100 or more time differencing intervals are assumed along with any reasonable growth rate. As we show below, however, it is possible to select a growth rate that will closely approximate the value of μ that minimizes the error in the two-state approximation.

Finding the Best Approximation

According to equation (A.11) in the Appendix, the price of a call option in the two-state model can be stated in terms of two binomial pseudo probability distributions. In each distribution, ψ and ϕ are the pseudo probabilities that the price of the underlying stock will rise. These pseudo probabilities are not necessarily equal to the true probability, θ , but nevertheless, the mathematics of probability theory are still applicable.

According to the Laplace-DeMoivre Limit Theorem, it can be shown that the best fit between the binomial and normal distributions occurs when the binomial probability (or pseudo probability in this case) is $\frac{1}{2}$. As a general rule, ψ and ϕ will not be identical. Therefore, it will usually be impossible to simultaneously set both pseudo probabilities to $\frac{1}{2}$. However, since $\psi = \phi(H^+/(1+R))$, and the term in parenthesis will generally be close to unity, the parameters of the underlying distribution that sets ϕ to $\frac{1}{2}$ will set ψ to approximately $\frac{1}{2}$.

By expanding ϕ in Taylor's series, we find that ϕ is approximately $\frac{1}{2}$ when

$$\mu = \frac{r - \frac{1}{2}\sigma\sqrt{T} \left[\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}} \right] - \frac{1}{4}\sigma^2 \left[\frac{1-\theta}{\theta} + \frac{\theta}{1-\theta} \right]}{\left[1 + \left(\sqrt{\frac{1-\theta}{\theta}} - \sqrt{\frac{\theta}{1-\theta}} \right) / \sqrt{T} \right]} \quad (12)$$

If the true probability, θ , is $\frac{1}{2}$, this expression simplifies to³

$$\mu = r - \frac{1}{2}\sigma^2. \quad (13)$$

For the parameters underlying Table 1, we find that (approximately) the best two-state approximation occurs when $\mu = -.002488$. The reasonableness of this result is confirmed by the $\mu = 0$ panel of Table 1.⁴

³ We wish to acknowledge the referee for suggesting that the best approximation would occur if $\mu = r - \frac{1}{2}\sigma^2$.

⁴ We repeated the analysis of Table 1 by setting μ to $-.002488$. Although the prices were almost identical to those obtained by setting μ to zero, they were slightly more accurate.

Pricing American Puts and Non-Dividend Paying Stocks

Table 2 shows the prices of American put options along with the value of the premature exercise privilege for the same parameters underlying Table 1, except that only a zero growth rate is assumed. Prices of American call options are not shown since, with no dividends, there will be no added value associated with the ability to exercise the call prior to maturity (see Merton [6]). Prices are shown for put options with exercise prices of 75, 100, and 125 under the assumption that the time differencing interval is 12, 52, 100, and 500 times per year. The differencing interval of 500 times is used as a proxy for the continuous case.

The prices in Table 2 suggest that the two-state model provides a fairly accurate approximation to the value of the premature exercise privilege, even for T values as low as 12. For all practical purposes, 100 time periods appears to provide sufficient accuracy for determining actual American put prices using the two-state model. For the options in Table 2, the prices obtained when $T = 100$ are within \$.01 of the $T = 500$ prices.

American and European Puts and Calls on Dividend Paying Stocks

If a stock pays a dividend, it may sometimes pay to prematurely exercise a call option on the stock before the dividend is paid rather than hold the option when the stock is almost certain to decline in value. Thus, one would expect American call options on dividend paying stocks to be worth more than their European counterparts. On the other hand, if a stock is expected to pay a dividend, it is less likely that an American put option will be exercised prematurely since the stock is likely to decrease in value when the dividend is paid due to the ex-dividend effect. The Black-Scholes model has been extended by Merton [6] to price

Table 2
American Put Prices Using the TSOPM^a

Option Parameters			American Put	Dollar Value of Premature Exercise ^b
μ	X	T		
.0	75	12	2.26	\$.05
		52	2.15	.06
		100	2.16	.06
		500	2.15	.07
	100	12	10.65	.65
		52	10.75	.55
		100	10.77	.54
		500	10.78	.52
	125	12	27.42	1.96
		52	27.40	1.88
		100	27.38	1.88
		500	27.37	1.89

^a $S_0 = 100$, $R = e^{.06/T} - 1$, $\sigma = .324$, $\theta = .5$.

^b Dollar value of premature exercise is computed as:

$$\text{TSOPM [American] Put Price} - \text{TSOPM [European] Put Price}, \text{ assuming } \mu = .0 \text{ in both cases.}$$

European puts and calls when dividends are paid continuously at a constant rate. This model is used in Table 3 as a benchmark for determining the accuracy of the two-state model for pricing European puts and calls.

In Table 3 the prices of European and American puts and calls are shown under the assumption that the underlying stock is expected to pay a quarterly dividend at an annual rate of 4%. The assumptions underlying Tables 1 and 2 are maintained.

The two-state prices of European puts and calls are all within \$.03 of the corresponding dividend-adjusted Black-Scholes-Merton prices when the life of the options are partitioned into 100 time intervals. Therefore, with dividends, the two-state model appears to provide an accurate approximation to the lognormal model.

As one would expect, the ability to exercise an American call option on a dividend paying stock prior to maturity can carry a significant premium. For example, for $X = 75$ and $T = 100$, the difference between the prices of American and European calls is \$1.03. This premium declines as the option's exercise price increases.

In contrast to the call option, the payment of a 4% dividend significantly lowers the value associated with the ability to exercise the put option prematurely. For example, for $T = 100$ and $X = 100$, the premature exercise premium is \$.54 for a non-dividend paying stock but only \$.16 if the stock pays a quarterly dividend at an annual rate of 4%.

V. Conclusions

This paper develops a simple two-state option pricing model and demonstrates the application of the model to several complex option pricing problems. Although the mathematics of the model are quite simple, especially when compared to the more conventional continuous time approach, the economics of both approaches to option pricing are essentially the same. Thus, the two-state approach opens the door to the understanding of modern option pricing theory without the added complications associated with the solutions to stochastic differential equations.

In addition to its pedagogic features, the two-state approach can be used as a numerical procedure for solving continuous time option pricing problems for which closed form solutions are unattainable. Moreover, the Black-Scholes equation can be derived from the two-state model as a special case. Admittedly, the mathematics of this derivation are as difficult as stochastic calculus itself, yet one need not carry the two-state model to its continuous limit to derive many interesting insights into both theoretical and practical applications of modern option pricing theory.

Appendix

Derivation of the Continuous Time Version of the TSOPM

In this Appendix we determine the value of a European call option, w_0 , using the TSOPM, assuming that the interval of time over which price changes in the underlying stock are recognized is infinitesimally small. This is equivalent to

Table 3
Comparison Between TSOPM and Black-Scholes-Merton Option Prices Including Dividends^a

μ	Option Parameters		TSOPM European Prices		TSOPM American Prices		Dollar Value of Premature Exercise ^d		Black-Scholes-Merton Prices ^c		
	X	T	INTVL ^b	Call	Put	Call	Put	Call	Put	Call	Put
.0	75	12	3	27.37	2.66	28.15	2.67	.78	\$.01	27.30	2.58
		52	13	27.32	2.59	28.29	2.61	.97	.02		
		100	25	27.30	2.59	28.33	2.60	1.03	.01		
	100	12	3	12.85	11.92	13.14	12.10	.29	.18	12.78	11.84
		52	13	12.83	11.89	13.33	12.05	.50	.16		
		100	25	12.80	11.86	13.35	12.02	.55	.16		
	125	12	3	5.40	28.24	5.52	28.93	.12	.69	5.27	28.11
		52	13	5.22	28.07	5.50	28.74	.28	.67		
		100	25	5.24	28.09	5.53	28.75	.29	.66		

^a $S = 100, R = e^{.06/T} - 1, \sigma = .324, \theta = .5$ and annual dividend yield = .04 for all contracts.

^b For $T = 12$, a dividend interval (INTVL) of 3 means that a dividend payment is made every 3rd period. If the entire time horizon is one year, then each (T, INTVL) pair implies a typical quarterly dividend payment.

^c Dividend adjusted Black-Scholes prices are computed by substituting $\$100(1 - .04/4)^t$ for the stock price in the Black-Scholes model.

^d Dollar value of premature exercise is computed as: [TSOPM American price] - [TSOPM European price].

allowing the number of time differencing intervals to become infinite over the fixed life of the option. Before deriving the continuous time version of the model, we will develop a valuation equation for the discrete time case under the assumptions that the distribution of returns of the stock is stationary over time and the stock pays no dividends.

The Discrete Time Model

When the option matures, there will be a one-to-one correspondence between the value of the option and the value of its underlying stock. The value of a call option at maturity, w_T , is $\max(0, S_T - X)$, where S_T is the value of the underlying stock at the maturity date, T , and X is the exercise price of the option. At the period prior to the option's maturity date, the value of the option is given by

$$w_{T-1} = \frac{w_T^+(1 + R - H^-) + w_T^-(H^+ - (1 + R))}{(H^+ - H^-)(1 + R)}. \quad (\text{A.1})$$

Similarly, the value of the option two periods prior to maturity is

$$w_{T-2} = \frac{w_{T-1}^+(1 + R - H^-) + w_{T-1}^-(H^+ - (1 + R))}{(H^+ - H^-)(1 + R)}. \quad (\text{A.2})$$

By substituting equation (A.1) into (A.2) and noting that the term w_T^+ is the option value at maturity, given that the price of the underlying stock advances in period $T - 1$ and falls in period T , the value of the option at period $T - 2$ becomes:

$$w_{T-2} = \frac{(w_T^+(1 + R - H^-) + w_T^-(H^+ - (1 + R)))(1 + R - H^-)}{(H^+ - H^-)^2(1 + R)^2} + \frac{(w_T^+(1 + R - H^-) + w_T^-(H^+ - (1 + R)))(H^+ - (1 + R))}{(H^+ - H^-)^2(1 + R)^2}. \quad (\text{A.3})$$

Equation (A.3) can be simplified by noting that $w_T^+ = w_T^-$, since the value of the underlying stock at maturity will be the same whether or not it advances first and then declines, or declines first and then advances. With this substitution, equation (A.3) can be restated as:

w_{T-2}

$$= \frac{w_T^+(1 + R - H^-)^2 + 2w_T^-(H^+ - (1 + R))(1 + R - H^-) + w_T^-(H^+ - (1 + R))^2}{(H^+ - H^-)^2(1 + R)^2}. \quad (\text{A.4})$$

If this same type of procedure is repeated for a total of T periods, there will always be $T + 1$ terms in the numerator of the option valuation equation. After T periods, there are exactly $\binom{T}{0}$ ways that a sequence of T pluses can occur, there are $\binom{T}{1}$ ways that $T - 1$ pluses can occur along with one minus, there are $\binom{T}{2}$ ways for $T - 2$ pluses and 2 minuses, and so on. In addition, the power to

which a term $(H^+ - (1 + R))$ associated with a particular w_T is raised is equal to the number of minus signs associated with the w_T . Therefore, if the valuation procedure is carried back to the present, the value of the option becomes:

$$\begin{aligned} w_0 = & \left[\binom{T}{0} w_T^+ \cdots^+ (1 + R - H^-)^T (H^+ - (1 + R))^0 \right. \\ & + \binom{T}{1} w_T^+ \cdots^+ (1 + R - H^-)^{T-1} (H^+ - (1 + R))^1 + \dots \\ & \dots + \binom{T}{T-1} w_T^+ \cdots^+ (1 + R - H^-)^1 (H^+ - (1 + R))^{T-1} \\ & \left. + \binom{T}{T} w_T^- \cdots^- (1 + R - H^-)^0 (H^+ - (1 + R))^T \right] / [(H^+ - H^-)(1 + R)]^T. \end{aligned} \quad (\text{A.5})$$

Next, we must determine the value of the option at maturity. If the stock advances i times and declines $(T - i)$ times, the price of the stock will be $S_0 H^+{}^i H^{-\ (T-i)}$ on the expiration date. The option will be exercised if

$$S_0 H^+{}^i H^{-\ (T-i)} > X, \quad (\text{A.6})$$

in which case, the maturity value of the option will be

$$w_T = S_0 H^+{}^i H^{-\ (T-i)} - X.$$

Otherwise, the option will expire worthless.

Let the symbol a denote the minimum integer value of i in (A.6) for which the inequality is satisfied. This value is given by:

$$a = 1 + \text{INT} \left[\frac{\ln(X/S_0) - T \cdot \ln(H^-)}{\ln(H^+) - \ln(H^-)} \right], \quad (\text{A.7})$$

where $\text{INT}[\cdot]$ denotes the integer operator. Thus, the maturity value of the option is given by

$$\begin{aligned} w_T &= S_0 H^+{}^i H^{-\ (T-i)} - X & \text{if } i \geq a \\ w_T &= 0 & \text{if } i < a. \end{aligned} \quad (\text{A.8})$$

By substituting (A.8) into (A.5), one obtains a generalized option pricing equation for the discrete time case.

$$w_0 = \frac{\sum_{i=a}^T \binom{T}{i} (S_0 H^+{}^i H^{-\ (T-i)} - X) (1 + R - H^-)^i (H^+ - (1 + R))^{T-i}}{(H^+ - H^-)^T (1 + R)^T}. \quad (\text{A.9})$$

The Continuous Time Model

In the derivation of the continuous time model, we will determine the option price when $T \rightarrow \infty$ assuming that the mean and variance of logarithmic returns of the stock are held constant over the life of the option.

Note that (A.9) can be rewritten as

$$w_0 = S_0 \sum_{i=a}^T \binom{T}{i} \left[\frac{(1+R-H^-)H^+}{(1+R)(H^+ - H^-)} \right]^i \left[\frac{(H^+ - 1 - R)H^-}{(H^+ - H^-)(1+R)} \right]^{T-i} - \frac{X}{(1+R)^T} \sum_{i=a}^T \binom{T}{i} \left[\frac{1+R-H^-}{H^+ - H^-} \right]^i \left[\frac{H^+ - 1 - R}{H^+ - H^-} \right]^{T-i} \quad (A.10)$$

The two bracketed terms in each term in (A.10) sum to unity and therefore can be interpreted as "pseudo probabilities." Although these pseudo probabilities do not represent the true probabilities that the price of the stock will either advance or decline, we can still apply the mathematics of probability theory to the solution of the problem. Let these pseudo probabilities be represented by

$$\psi = \frac{(1+R-H^-)H^+}{(1+R)(H^+ - H^-)}, \quad \text{and}$$

$$\phi = \frac{1+R-H^-}{H^+ - H^-}.$$

The option price can now be stated as:

$$w_0 = S_0 B(a, T, \psi) - \frac{X}{(1+R)^T} B(a, T, \phi), \quad (A.11)$$

where $B(a, T, (\cdot))$ is the cumulative binomial probability that the number of successes will fall between a and T after T trials, where (\cdot) is the probability associated with a success after one trial.

As T becomes large, the cumulative binomial density function can be approximated by the cumulative normal density function. The approximation will be exact in the limit as $T \rightarrow \infty$. Therefore,

$$w_0 \sim S_0 N(Z_1, Z'_1) - \frac{X}{(1+R)^T} N(Z_2, Z'_2), \quad (A.12)$$

where $N(Z, Z')$ is the probability that a normally distributed random variable with zero mean and unit variance will take on values between a lower limit of Z and an upper limit of Z' , and

$$Z_1 = \frac{a - T\psi}{\sqrt{T\psi(1-\psi)}}, \quad Z'_1 = \frac{T - T\psi}{\sqrt{T\psi(1-\psi)}}$$

$$Z_2 = \frac{a - T\phi}{\sqrt{T\phi(1-\phi)}}, \quad Z'_2 = \frac{T - T\phi}{\sqrt{T\phi(1-\phi)}}$$

Thus, the price of the option that will obtain when the two-state process evolves continuously is given by:

$$w_0 = S_0(\text{Lim}_{T \rightarrow \infty} Z_1, \text{Lim}_{T \rightarrow \infty} Z'_1) - \frac{X}{\text{Lim}_{T \rightarrow \infty} (1+R)^T} N(\text{Lim}_{T \rightarrow \infty} Z_2, \text{Lim}_{T \rightarrow \infty} Z'_2). \quad (A.13)$$

Let $1 + R = e^{r/T}$ to reflect the continuous compounding of interest. Then,

$$\lim_{T \rightarrow \infty} (1 + R)^T = e^r.$$

We will state without proof that

$$\lim_{T \rightarrow \infty} Z_1 = \lim_{T \rightarrow \infty} Z_2 = \infty.$$

Thus, all that remains in the derivation of the continuous time version of the two-state model is to determine $\lim_{T \rightarrow \infty} Z_1$ and $\lim_{T \rightarrow \infty} Z_2$.

In determining both limits, we will assume that both H^+ and H^- are chosen to hold the logarithmic mean and variance of returns of the stock constant over the option's life. Therefore, we make the following substitutions derived earlier in the text.

$$H^+ = e^{\mu/T + (\sigma/\sqrt{T})\sqrt{\theta(1-\theta)}/\theta}$$

$$H^- = e^{\mu/T - (\sigma/\sqrt{T})\sqrt{\theta(1-\theta)}/(1-\theta)}$$

Substituting H^+ and H^- into α ,

$$Z_1 = \frac{1 + \text{INT} \left[\frac{\ln(X/S_0) - \mu + \sigma\sqrt{T} \sqrt{\theta(1-\theta)}}{\sigma/\sqrt{T}\theta(1-\theta)} \right] - T\psi}{\sqrt{T}\psi(1-\psi)}.$$

In the limit, the term $1 + \text{INT}[\cdot]$ will simplify to the term in brackets. To simplify the exposition, we will replace $1 + \text{INT}[\cdot]$ with $[\cdot]$ at this point. With this substitution Z_1 can be restated as

$$Z_1 \sim \frac{\ln(X/S_0) - \mu}{\sigma\sqrt{\frac{\psi(1-\psi)}{\theta(1-\theta)}}} + \frac{\sqrt{T}(\theta - \psi)}{\sqrt{\psi(1-\psi)}}. \tag{A.14}$$

Substituting H^+ , H^- , and $1 + R = e^{r/T}$ in the expression for ψ and expanding in Taylor's series in T , we obtain

$$\begin{aligned} \psi \sim & \frac{1}{\sqrt{T}} \frac{r - \mu - \frac{1}{2}\sigma^2\left(\frac{\theta}{1-\theta}\right) + \sigma^2}{\sigma\left(\sqrt{\frac{1-\theta}{\theta}} + \sqrt{\frac{\theta}{1-\theta}}\right) + \frac{1}{2}(\sigma^2/\sqrt{T})\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{T}\right)} \\ & + \frac{\sqrt{\frac{\theta}{1-\theta}}}{\sqrt{\frac{1-\theta}{\theta}} + \sqrt{\frac{\theta}{1-\theta}} + \frac{1}{2}\left(\frac{\sigma}{\sqrt{T}}\right)\left(\frac{1-\theta}{\theta} - \frac{\theta}{1-\theta}\right) + o\left(\frac{1}{T}\right)} + o\left(\frac{1}{T}\right), \end{aligned}$$

where $o\left(\frac{1}{T}\right)$ denotes a function tending to zero more rapidly than $\frac{1}{T}$. [The derivation of the Taylor's series expansions of ψ and ϕ as well as the derivation of

the limits below will be made available by the authors upon request.] It can be shown that

$$\lim_{T \rightarrow \infty} \psi = \theta \quad \text{and}$$

$$\lim_{T \rightarrow \infty} \sqrt{T} (\theta - \psi) = -\frac{\sqrt{\theta(1-\theta)}}{\sigma} \left(r - \mu + \frac{1}{2} \sigma^2 \right)$$

Substituting $\lim_{T \rightarrow \infty} \psi$ for ψ and $\lim_{T \rightarrow \infty} \sqrt{T} (\theta - \psi)$ for $\sqrt{T} (\theta - \psi)$ into (A.14), we obtain:

$$\begin{aligned} \lim_{T \rightarrow \infty} Z_1 &= \frac{\ln(X/S_0) - \mu}{\sigma \sqrt{\frac{\theta(1-\theta)}{\theta(1-\theta)}}} - \frac{\sqrt{\theta(1-\theta)} \left(r - \mu + \frac{1}{2} \sigma^2 \right)}{\sigma \sqrt{\theta(1-\theta)}} \\ &= \frac{\ln(X/S_0) - r - \frac{1}{2} \sigma^2}{\sigma} \end{aligned}$$

The derivation of $\lim_{T \rightarrow \infty} Z_2$ closely parallels the corresponding derivation for Z_1 . It can be shown that

$$\lim_{T \rightarrow \infty} Z_2 = \frac{\ln(X/S_0) - r + \frac{1}{2} \sigma^2}{\sigma}$$

Recognizing that $N(Z, \infty) = N(-\infty, -Z)$, and letting

$$D_1 = -\lim_{T \rightarrow \infty} Z_1$$

$$D_2 = -\lim_{T \rightarrow \infty} Z_2,$$

the continuous time version of the two-state model is obtained:

$$w_0 = S_0 N(-\infty, D_1) - X e^{-r} N(-\infty, D_2)$$

$$D_1 = \frac{\ln(S_0/X) + r + \frac{1}{2} \sigma^2}{\sigma}$$

$$D_2 = D_1 - \sigma.$$

The above equation is identical to the Black-Scholes model.

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