Wiener Processes and Itô's Lemma

Chapter 12

Stochastic Processes

- A stochastic process describes the way a variable evolves over time that is at least in part random. i.e., temperature and IBM stock price.

- A stochastic process is defined by a probability law for the evolution of a variable x_t over time t. For given times, we can calculate the probability that the corresponding values x_1,x_2, x_3,etc., lie in some specified range.

Categorization of Stochastic Processes

- Discrete time; discrete variable

  Random walk: $x_t = x_{t-1} + \varepsilon_t$

  if ε_t can only take on discrete values
- Discrete time; continuous variable

  $$x_t = a + bx_{t-1} + \varepsilon_t$$

  ε_t is a normally distributed random variable with zero mean.
- Continuous time; discrete variable
- Continuous time; continuous variable

Modeling Stock Prices

- We can use any of the four types of stochastic processes to model stock prices

- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives

Markov Processes (See pages 259-60)

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are.

- We assume that stock prices follow Markov processes.

Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is consistent with weak-form market efficiency

Example of a Discrete Time Continuous Variable Model

- A stock price is currently at $40
- At the end of 1 year it is considered that it will have a normal probability distribution of with mean $40 and standard deviation $10

Questions

- What is the probability distribution of the stock price at the end of 2 years?
  - ½ years?
  - ¼ years?
  - Δt years?

Taking limits we have defined a continuous variable, continuous time process

Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

Variances & Standard Deviations
(continued)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

A Wiener Process (See pages 261-63)

- We consider a variable z whose value changes continuously
- Define $\phi(\mu, v)$ as a normal distribution with mean μ and variance v
- The change in a small interval of time Δt is Δz
- The variable follows a Wiener process if
  - $\Delta z = \varepsilon\sqrt{\Delta t}$ where ε is $\varphi(0,1)$
  - The values of Δz for any 2 different (non-overlapping) periods of time are independent

Properties of a Wiener Process

$$z(T) - z(0) = \sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t}$$

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

Taking Limits . . .

$$dz = \varepsilon\sqrt{dt}$$

- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving Δz and Δt is true in the limit as Δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

Generalized Wiener Processes

(See page 263-65)

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

Generalized Wiener Processes
(continued)

The variable *x* follows a generalized Wiener process with a drift rate of *a* and a variance rate of b^2 if

$$dx = a\,dt + b\,dz$$

$$\text{or: } x(t) = x_0 + at + bz(t)$$

Generalized Wiener Processes
(continued)

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

- Mean change in x in time T is aT
- Variance of change in x in time T is b^2T
- Standard deviation of change in x in time T is $b\sqrt{T}$

The Example Revisited

- A stock price starts at 40 and has a probability distribution of $\phi(40,100)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by $8 on average during the year, so that the year-end distribution is f(48,100), the process would be

$$dS = 8dt + 10dz$$

Why $b\sqrt{\Delta t}$?(1)

- It's the only way to make the variance of $(x_T - x_0)$ depend on T and not on the number of steps.

1. Divide time up into n discrete periods of length $\triangle t$, $n = T/\triangle t$. In each period the variable x either moves up or down by an amount $\triangle h$ with the probabilities of p and q respectively.

Why $b\sqrt{\Delta t}$?(2)

2.The distribution for the future values of x:

$$E(\triangle x)=(p-q)\triangle h$$

$$E[(\triangle x)^2]= p(\triangle h)^2+q(-\triangle h)^2$$

So, the variance of $\triangle x$ is:

$$E[(\triangle x)^2]-[E(\triangle x)]^2=[1-(p-q)^2](\triangle h)^2$$

3. Since the successive steps of the random walk are independent, the cumulated change(x_T-x_0)is a binomial random walk with mean:

$$n(p-q)\triangle h=T(p-q)\triangle h/\triangle t$$

and variance:

$$n[1-(p-q)^2](\triangle h)^2= T[1-(p-q)^2](\triangle h)^2/\triangle t$$

Why $b\sqrt{\Delta t}$?(3)

- When let $\triangle$t go to zero, we would like the mean and variance of (x_T-x_0) to remain unchanged, and to be independent of the particular choice of p,q, $\triangle$h and $\triangle$t.
- The only way to get it is to set:

$$\Delta h = b\sqrt{\Delta t}$$

$$p = \tfrac{1}{2}[1 + \frac{a}{b}\sqrt{\Delta t}], q = \tfrac{1}{2}[1 - \frac{a}{b}\sqrt{\Delta t}]$$

$$p - q = \frac{a}{b}\sqrt{\Delta t} = \frac{a}{b^2}\Delta h$$

Why $b\sqrt{\Delta t}$?(4)

- When $\triangle$t goes to zero, the binomial distribution converges to a normal distribution, with mean

$$t\frac{a}{b^2}\Delta h\frac{\Delta h}{\Delta t} = at$$

and variance

$$t[1-(\frac{a}{b})^2\Delta t]\frac{b^2\Delta t}{\Delta t} \rightarrow b^2 t$$

Sample path($a=0.2$ per year, $b^2=1.0$ per year)

- Taking a time interval of one month, then calculating a trajectory for x_t using the equation:

$$x_t = x_{t-1} + 0.01667 + 0.2887\varepsilon_t$$

A trend of 0.2 per year implies a trend of 0.0167 per month. A variance of 1.0 per year implies a variance of 0.0833 per month, so that the standard deviation in monthly terms is 0.2887.

See *Investment under uncertainty*, p66

Forecast using generalized Brownian Motion

- Given the value of x(t)for Dec. 1974, X_{1974}, the forecasted value of x for a time T months beyond Dec. 1974 is given by:

$$\hat{x}_{1974+T} = x_{1974} + 0.01667T$$

See *Investment under uncertainty*, p67

- In the long run, the trend is the dominant determinant of Brownian Motion, whereas in the short run, the volatility of the process dominates.

Why a Generalized Wiener Process Is Not Appropriate for Stocks

- The price of a stock never fall below zero.
- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

Itô Process (See pages 265)

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x,t)dt + b(x,t)dz$$

$$x(t) = x_0 + \int_0^t a(x,t)\,ds + \int_0^t b(x,t)\,dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

is only true in the limit as Δt tends to zero

An Ito Process for Stock Prices

(See pages 269-71)

$$dS = \mu S\,dt + \sigma S\,dz$$

where μ is the expected return, σ is the volatility.

- The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

Monte Carlo Simulation

- We can sample random paths for the stock price by sampling values for ε

- Suppose $\mu = 0.15$, $\sigma = 0.30$, and $\Delta t = 1$ week (=1/52 years), then

$$\Delta S = 0.00288S + 0.0416S\varepsilon$$

Monte Carlo Simulation – One Path (See Table 12.1, page 268)

Week	Stock Price at Start of Period	Random Sample for □	Change in Stock Price, □S
0	100.00	0.52	2.45
1	102.45	1.44	6.43
2	108.88	-0.86	-3.58
3	105.30	1.46	6.70
4	112.00	-0.69	-2.89

Itô's Lemma (See pages 269-270)

- If we know the stochastic process followed by x, Itô's lemma tells us the stochastic process followed by some function G (x, t)
- Since a derivative is a function of the price of the underlying and time, Itô's lemma plays an important part in the analysis of derivative securities

Taylor Series Expansion

- A Taylor's series expansion of G(x, t) gives

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta x\ \Delta t + ?\ \frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

Ignoring Terms of Higher Order Than Δt

In ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$$

In stochastic calculus this becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ? \frac{\partial^2 G}{\partial x^2}\Delta x^2$$

because Δx has a component which is of order $\sqrt{\Delta t}$

Substituting for Δx

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\Delta x = a\Delta t + b\ \varepsilon\sqrt{\Delta t}$$

Then ignoring terms of higher order than Δt

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ?\ \frac{\partial^2 G}{\partial x^2} b^2 \varepsilon^2 \Delta t$$

The $\varepsilon^2 \Delta t$ Term

Since $\varepsilon \sim \varphi(0,1)$, $E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \Delta t) = \Delta t$

The variance of Δt is proportional to Δt^2 and can be ignored.

Hence,

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \Delta t$$

Taking Limits

Taking limits: $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ? \ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting: $$dx = a \ dt + b \ dz$$

We obtain: $$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ? \ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \ dz$$

This is Itô's Lemma

Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + ? \ \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S\, dz$$

Examples

1. The forward price of a stock for a contract maturing at time T

$$G = S\ e^{r(T-t)}$$

$$dG = (\mu - r)G\ dt + \sigma G\ dz$$

2. $G = \ln S$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\ dz$$

Ito's Lemma for several Ito processes

- Suppose $F = F(x_1, x_2, \ldots, x_m, t)$ is a function of time and of the m Ito process $x_1, x_2, \ldots, x_m$, where

$$dx_i = a_i(x_1, x_2, \ldots, x_m, t)dt + b_i(x_1, x_2, \ldots, x_m, t)dz_i, i = 1, 2, \ldots, m$$

with $E(dz_i dz_j) = \rho_{ij} dt$

- Then Ito's Lemma gives the differential dF as

$$dF = \frac{\partial F}{\partial t}dt + \sum_i \frac{\partial F}{\partial x_i}dx_i + \frac{1}{2}\sum_i \sum_j \frac{\partial^2 F}{\partial x_i \partial x_j}dx_i dx_j$$

Examples

- Suppose F(x,y)=xy, where x and y each follow geometric Brownian motions:

$$dx = a_x x dt + b_x x dz_x$$

$$dy = a_y y dt + b_y y dz_y$$

with $E\left(dz_i dz_j\right) = \rho_{ij} dt$.

- What's the process followed by F(x,y) and by G=logF?

$$dF = xdy + ydx + dxdy$$

$$= \left(a_x + a_y + \rho b_x b_y\right) F dt + \left(b_x dz_x + b_y dz_y\right) F$$

$$dG = \left(a_x + a_y - \frac{1}{2} b_x^2 - \frac{1}{2} b_y^2\right) dt + b_x dz_x + b_y dz_y$$