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Abstract

This paper presents a generalized version of the lattice approach to pricing options. It shows how the control variate technique can produce significant improvements in the efficiency of the approach. The control variate technique is illustrated using American puts on dividend and nondividend paying stocks.

I. Introduction

In spite of theoretical advances made in recent years, many option pricing problems lack straightforward closed form solutions.[1] For example, American put options on a stock, American currency options, American options on commodity futures, and options on a stock with a stochastic volatility all appear to present intractible pricing problems.

The various approaches that have been suggested for calculating option prices when there is no closed form solution include analytic approximation, compound option methods, series solutions, Monte Carlo simulation, numerical integration, binomial models, and finite difference methods. Johnson (1983) and Macmillan (1986) show how analytic approximation can be used to value an American put on a nondividend paying stock. Blomeyer (1986) and Omberg (1987a) use analytic approximation to value American puts on dividend-paying stocks. Barone-Adesi and Whaley (1987) apply analytic approximation to other American options. Geske and Johnson (1984) present an ingenious compound option approach to valuing an American put on a dividend or nondividend-paying stock. Hull and White (1988) provide a series solution for valuing a European option on a stock when the volatility is stochastic. Monte Carlo simulation was

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[1] As pointed out by Geske and Johnson ((1984), p. 1513), describing a solution as closed form or analytical rather than numerical is tricky in the option context. In the end, all methods require numerical procedures. For example, the use of the Black-Scholes (1973) formula requires the numerical evaluation of the cumulative normal distribution function. This paper is concerned with option pricing problems in which existing numerical procedures are nontrivial so that improvements in numerical efficiency are of interest.

suggested by Boyle (1977) and has been used by both Johnson and Shanno (1985) and Hull and White (1987) to value options when the volatility is stochastic. Numerical integration has been used for American puts by Parkinson (1977). Both binomial models and finite difference methods have been used in a wide variety of situations. Binomial models are discussed by Cox, Ross, and Rubinstein (1979), while finite difference methods are discussed by Schwartz (1977), Brennan and Schwartz (1979), and Courtadon (1982).

All these numerical methods have a dual objective of accuracy and speed of computation. For any given method, greater accuracy can normally be achieved only by increasing the computation time. Geske and Shastri (1985) provide a careful comparison of binomial models and finite difference methods. They conclude that "researchers computing a smaller number of option values may prefer binomial approximation, while practitioners in the business of computing a larger number of option values will generally find that finite difference methods are more efficient" (p. 70).

The binomial model is a particular case of a more general set of multivariate multinomial models. For example, Boyle (1986) shows how stock options can be valued using a trinomial model, while Boyle (1988) shows how a bivariate multinomial model can be used to value options in which there are two underlying state variables. We shall refer to all multivariate multinomial models as lattice approaches. The main purpose of this paper is to show how a method known as the control variate technique can in some circumstances be used to improve the efficiency of lattice approaches.

The rest of the paper is organized as follows. Section II presents a generalized version of the lattice approach to option pricing. Section III discusses how dividends can be incorporated in lattice approaches. Section IV describes the control variate technique and its application to lattice approaches. Section V applies the technique to the valuation of American puts. Section VI discusses the application of the technique in other situations. Conclusions are in Section VII.

II. Lattice Approaches

In this section, we present a generalized version of the lattice approach to option valuation. As already mentioned, this approach involves a multivariate multinomial extension of the Cox, Ross, and Rubinstein (CRR) (1979) binomial model. The lattice approach can be viewed as an application of dynamic programming.[2]

A lattice approach requires the use of a risk-neutral argument.[3] If the underlying state variables are traded securities, risk-neutral valuation agruments can be applied in a straightforward way. If one or more of the variables are nontraded,

[2] For a description of the dynamic programming technique, see Bellman and Dreyfus (1962). It should be emphasized that moving from a binomial model to, say, a trinomial model does not mean that a different underlying stochastic process is being assumed. Instead, it means that the same stochastic process is being approximated in a different way.

[3] The lattice approach calculates the option price as the discounted value of the expected option pay-off. In order to know the appropriate discount rate, risk-neutral valuation arguments are necessary.

the basic risk-neutral valuation argument must be extended, as in Cox, Ingersoll, and Ross (1985).

In the generalized lattice approach, we consider a derivative security whose price depends on l underlying state variables. The life of the security, T, is divided into n subintervals of length Δt. At time $i\Delta t$ a finite number, m_i, of different possible states of the world is defined. We will denote these states of the world by x_{ij} $(1 \leqslant j \leqslant m_i)$. Each x_{ij} is a $l \times 1$ vector of values of the state variables. When $i = 0$, the state of the world is known and $m_i = 1$. Generally as i increases, m_i also increases.

Transition probabilities q_{ijk} are defined as follows

$$q_{ijk} = \text{probability of moving from state } x_{ij} \text{ at time } i\Delta t \text{ to state } x_{i+1,k} \text{ at time } (i+1)\Delta t \,.$$

The states of the world x_{ij} together with the transition probabilities q_{ijk} constitute the lattice that is used to model the underlying state variables. The q_{ijk} must be chosen so that they satisfy,

$$\sum_k q_{ijk} = 1 \quad \text{for all } i \text{ and } j$$

$$0 \leqslant q_{ijk} \leqslant 1 \quad \text{for all } i, j, \text{ and } k \,.$$

Also, the transition probabilities and values of the state variables x_{ij} must be chosen so that the lattice accurately represents the actual state variables in a risk-neutral world. This is usually achieved by requiring that the lattice give the correct values for the means and standard deviations of, and the coefficients of correlation between, the changes in the state variables in each time interval Δt.

Once the lattice has been set up, the dynamic programming method can be used. The value of the derivative security at time T is known for all m_n states of the world possible at that time. Furthermore the value for all m_i states of the world at time $i\Delta t$ can be calculated using risk-neutral valuation if the value is known for all m_{i+1} states of the world at time $(i+1)\Delta t$. By moving backwards through the lattice, the value at time 0 can be obtained.

In the CRR binomial model, there is one state variable, the stock price S. This is assumed to follow the stochastic process

$$dS = \mu S dt + \sigma S dz \,, \tag{1}$$

where σ is the constant instantaneous standard deviation of dS/S, μ is the instantaneous drift of dS/S, and dz is a Wiener process. In a risk-neutral world, Equation (1) becomes

$$dS = r_f S dt + \sigma S dz \,, \tag{2}$$

where r_f is the risk-free interest rate.

If the initial stock price is S_0, the stock prices corresponding to the $m_i =$

$i+1$ states of the world, x_{ij}, $1 \leqslant j \leqslant i+1$, considered at time $i\Delta t$ in the CRR approach, are as follows,

$$\begin{aligned} i &= 0 : \quad S_0, \\ i &= 1 : \quad dS_0, uS_0, \\ i &= 2 : \quad d^2S_0, S_0, u^2S_0, \\ i &= 3 : \quad d^3S_0, dS_0, uS_0, u^3S_0, \\ &\text{etc.,} \end{aligned}$$

where

(3) $$u = \exp\left(\sigma\sqrt{\Delta t}\right),$$

(4) $$d = 1/u.$$

The transition probabilities, q_{ijk}, are defined as

$$\begin{aligned} q_{ijk} &= p \quad \text{when} \quad k = j+1, \\ q_{ijk} &= 1-p \quad \text{when} \quad k = j, \text{ and} \\ q_{ijk} &= 0 \quad \text{otherwise,} \end{aligned}$$

where

(5) $$p = \frac{r-d}{u-d}$$

and

(6) $$r = \exp\left(r_f \Delta t\right).$$

The definitions of p, u, and d in Equations (3), (4), and (5) provide a binomial approximation to the stochastic process in Equation (2).[4]

[4] Actually, the CRR values for p, u, and d only give the correct mean and standard deviation of stock price changes in the limit as $\Delta t \to 0$. It is slightly more accurate to solve

$$\begin{aligned} pu + (1-p)d &= m_1, \\ pu^2 + (1-p)d^2 &= m_2, \\ \text{and } u &= 1/d, \end{aligned}$$

where m_1 and m_2 are the first and second moments of the lognormal distribution of $S(t+\Delta t)/S(t)$. This gives

$$\begin{aligned} m_1 &= \exp\left(r_f \Delta t\right), \\ m_2 &= \exp\left(2r_f \Delta t + \sigma^2 \Delta t\right), \\ u &= \left[\left(m_2+1\right) + \sqrt{\left(m_2+1\right)^2 - 4m_1^2}\right] \Big/ 2m_1, \\ d &= \frac{1}{u}, \\ \text{and } p &= \frac{m_1 - d}{u-d}. \end{aligned}$$

For all the calculations in this paper, the CRR binomial lattice was modified so that these values of u, d, and p were used.

Boyle (1986) has suggested an alternative to the CRR binomial pricing model where $m_i = 2i + 1$ and there are three (rather than two) nonzero q_{ijk} for each i and j. He shows that a more efficient computational algorithm results. In another paper, Boyle (1987) has suggested using the lattice approach for the situation in which there are two independent variables. In this case, he finds it necessary to have five nonzero q_{ijk}'s for each i and j, and $m_i = 1 + 2i(i+1)$.

For the lattice approach to work efficiently, the total number of nodes (unique values of x_{ij}) considered at any given time should not be allowed to become unnecessarily large. This is because computation time is proportional to the total number of nodes in the lattice. An efficient lattice also requires the x_{ij} to be representative of the possible future states of the world.[5] Generally, a satisfactory lattice is one in which the transition probabilities are not regularly close to either 0.0 or 1.0.

III. Dividends

There is an interesting issue concerned with the use of a CRR-type lattice (and lattices, in general) when there are known payouts, such as dividends. Suppose that it is known that a dividend of D will be paid and that the stock will go ex-dividend at time τ ($0 \leqslant \tau \leqslant T$). We can define a variable $S^*(t)$ as follows,

$$S^*(t) = S(t) - De^{-r_f(\tau - t)} \quad \text{when } t \leqslant \tau ,$$
$$S^*(t) = S(t) \quad \text{when } t > \tau .$$

When $t \leqslant \tau$, the stock price can be viewed as having two components: a part S^* that is stochastic and a part $De^{-r_f(\tau - t)}$ that is nonstochastic and will be used to pay a dividend D at time τ.[6] Define σ^* as the instantaneous standard deviation of dS^*/S^*. It is logical to assume that σ^* rather than σ is constant. If we make this assumption, the parameters p, u, and d can be calculated by replacing σ by σ^* in Equations (3), (4), and (5). At times $i\Delta t$, the values of the stock prices considered in the lattice are then

$$S^*(0)u^j d^{i-j} + De^{-r_f(\tau - i\Delta t)}, \quad j = 0, 1, \ldots, i ,$$

when $i\Delta t < \tau$, and

$$S^*(0)u^j d^{i-j}, \quad j = 0, 1, \ldots, i ,$$

when $i\Delta t > \tau$. For a European option on a stock with known dividends, the assumption that has been made here corresponds to the common practice of using the Black-Scholes price with the stock price reduced by the present value of the dividends.[7]

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[5] For example, a lattice in which all values considered for a state variable are within one tenth of its standard deviation is not normally appropriate. Similarly, it is likely to be wasteful to consider values that are ten standard deviations from the mean.

[6] This corresponds to Rubinstein's (1983) displaced diffusion model.

[7] We have assumed that the stock price declines by D when it goes ex-dividend. This may not always be the case because of tax considerations. If the stock price is expected to decline by αD, the same conclusions apply with D replaced by αD.

An alternative valuation approach (that does not have quite as much theoretical appeal) is to define a variable S' by

$$S'(t) = S(t) , \qquad t \leq \tau ,$$
$$S'(t) = S(t) + De^{r_f(t-\tau)} , \qquad t > \tau ,$$

and to assume that the instantaneous standard deviation of dS'/S', say σ', is constant. (Since $S^*(t) = S'(t) - De^{-r_f(\tau - t)}$, this implies that the known dividend affects the volatility of the stochastic component of the stock price.) The parameters u, d, and p in Equations (3), (4), and (5) should then be calculated using σ' rather than σ. At time $i\Delta t$, the values considered in the lattice are then

$$S(0)u^j d^{i-j}, \quad j = 0, 1, \ldots, i ,$$

when $i\Delta t \leq \tau$, and

$$S(0)u^j d^{i-j} - De^{r_f(i\Delta t - \tau)}, \quad j = 0, 1, \ldots, i ,$$

when $i\Delta t > \tau$. For a European option on a stock with known dividends, this assumption corresponds to using the Black-Scholes formula with the exercise price increased by the value of the dividends compounded to time T at the risk-free rate.

The assumption that is frequently made for American options is that σ is constant. This implies that either the return from the known dividend is uncertain, or the risk of the stock price declines after the dividend is paid. This is also the assumption for which it is most difficult to construct a lattice (see Cox and Rubinstein (1984), p. 241). A straightforward extension of CRR in which u, d, and p are held constant leads to a great increase in the number of nodes. This is because the branches of the binomial tree do not recombine. Suppose that $i\Delta t \leq \tau < (i+1)\Delta t$. At time $(i+1)\Delta t$,the nodes on the lattice correspond to stock prices,

$$\left[S(0)u^j d^{i-j} - D\right]u \text{ and } \left[S(0)u^j d^{i-j} - D\right]d ,$$

for $j = 0, 1, \ldots, i$, so that there are, in general, $2(i+1)$ rather than $i+2$ nodes. At time $(i+q)\Delta t$, there are $(q+1)(i+1)$, rather than $i+q+1$ nodes. If more than one dividend is expected, the number of terminal nodes is liable to be very large. It is possible to design lattices in which the number of nodes at time $i\Delta t$ is always $i+1$. However the transition probabilities vary from node to node.

If the dividend yield rather than the dividend itself is assumed to be known, Cox and Rubinstein (1984) show that a straightforward extension of the CRR lattice is possible. At times $i\Delta t$, the values of the stock price considered in the lattice are

$$S(0)u^j d^{i-j},$$

when $i\Delta t \leq \tau$, and

$$S(0)(1-\delta)u^i d^{i-j},$$

when $i\Delta t > \tau$, where δ is the dividend yield (i.e., the proportion of the stock price paid out as a dividend at time τ).

Finally, we note the observation of Cox, Ross, and Rubinstein (1979) that a slight modification of their lattice can be used to value an option on a stock paying a continuous dividend when the instantaneous dividend yield, γ, is constant. It is necessary to replace r_f by $r_f - \gamma$ in the definition of r in Equation (6). Currencies and commodities are analogous to stocks paying continuous dividends. In the case of a currency, the analogue to the dividend is the foreign risk-free rate; in the case of a commodity, it is the convenience yield net of storage costs. Options on currencies and commodities therefore can be valued using a suitably modified CRR lattice.

IV. The Control Variate Technique

The control variate technique can be used to improve the efficiency of numerical valuation procedures. It is applicable when we wish to value an option, A, and we have an accurate valuation for a similar option, B. Boyle (1977) suggested the use of the control variate technique in conjunction with Monte Carlo simulation. Here, we show that it can also be applied in conjunction with lattice approaches.

The key element in the control variate technique is that the same numerical procedure is used to value both option A and option B. This may appear wasteful since an accurate value for B is already available. However, if the estimation errors from using the numerical procedure to value A and B are unbiased (or equally biased) and highly correlated, the technique enables a much better estimate of the value of A to be produced.

Define C_B = the accurate value of option B,

$\hat{C}_A$ = the estimated value of option A using the numerical procedure,

$\hat{C}_B$ = the estimated value of option B using the numerical procedure,

σ_A = the standard error of $\hat{C}_A$,

σ_B = the standard error of $\hat{C}_B$, and

ρ = the correlation between $\hat{C}_A$ and $\hat{C}_B$.

The control variate technique's estimate for the value of A, $\hat{C}_A^*$, is given by

$$\hat{C}_A^* = C_B + \left(\hat{C}_A - \hat{C}_B\right). \tag{7}$$

This has a standard error of

$$\left[\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B\right]^{1/2},$$

which is less than σ_A if

$$\rho > \frac{\sigma_B}{2\sigma_A}. \tag{8}$$

When the control variate technique is used in conjunction with Monte Carlo simulation, the same random normal deviates are used to calculate $\hat{C}_A$ and $\hat{C}_B$. When it is used in conjunction with the lattice approach, the same lattice is used to calculate $\hat{C}_A$ and $\hat{C}_B$. In both cases, providing an accurate value of B is available, Equation (7) can be used to calculate the control variate estimate $\hat{C}_A^*$.

The lattice approach and the Monte Carlo simulation approach have some similarities.[8] Monte Carlo simulation considers a finite number of randomly selected paths for the state variables. Lattice approaches consider a finite number of representative paths for the state variables. In both cases, the paths considered will inevitably lead to less than perfectly accurate price estimates. However, in both approaches, there is a tendency for the errors in the price estimates of the two similar options to be highly correlated when the same set of paths is considered.

The control variate technique can also be used in conjunction with the lattice approach when derivatives of option prices are required. Suppose that we wish to evaluate $\partial C_A/\partial\phi$ at $\phi = \phi_0$, where ϕ is an option parameter and ϕ_0 is its current value. We assume that an accurate value of $\partial C_B/\partial\phi$ at $\phi = \phi_0$ is available, and we denote this value by D_B. First, $\hat{C}_A(\phi_0)$ and $\hat{C}_B(\phi_0)$ are calculated using the lattice in the usual way. Then a small change $\Delta\phi_0$ in ϕ_0 is chosen, and $\hat{C}_A(\phi_0+\Delta\phi_0)$ and $\hat{C}_B(\phi_0+\Delta\phi_0)$ also are calculated using the lattice. Define

$$\hat{D}_A = \frac{\hat{C}_A\left(\phi_0+\Delta\phi_0\right) - \hat{C}_A\left(\phi_0\right)}{\Delta\phi_0},$$

$$\hat{D}_B = \frac{\hat{C}_B\left(\phi_0+\Delta\phi_0\right) - \hat{C}_B\left(\phi_0\right)}{\Delta\phi_0}.$$

The control variate estimate of $\partial C_A/\partial\phi$, $\hat{D}_A^*$ is given by[9]

$$\hat{D}_A^* = D_B + \left(\hat{D}_A - \hat{D}_B\right).$$

This has a standard error of

$$\frac{1}{\Delta\phi_0}\hat{\sigma}_A^*\sqrt{2-2\rho_\Delta\left(\phi_0,\Delta\phi_0\right)},$$

where $\hat{\sigma}_A^*$ is the standard error of $\hat{C}_A^*$, and $\rho_\Delta(\phi_0,\Delta\phi_0)$ is the correlation between $\hat{C}_A^*(\phi_0+\Delta\phi_0)$ and $\hat{C}_A^*(\phi_0)$. This is smaller than $\hat{\sigma}_A^*$ if

$$\rho_\Delta\left(\phi_0,\Delta\phi_0\right) > 1 - \frac{\Delta\phi_0^2}{2}.$$

[8] However, the two approaches are not direct substitutes for each other. Monte Carlo simulation can be used only for European options, whereas lattice approaches can be used for European or American options.

[9] For greater accuracy, it may be desirable to calculate $\hat{D}_A$ and $\hat{D}_B$ using symmetrical rather than one-sided first differences.

V. American Put Options

In this section, we illustrate the control variate technique by considering American put options on a stock. The natural option to use as option B in this case is a European put option on the same stock with the same exercise price and maturity. Consider first the case in which there are no dividends. Define

S_0 = initial stock price,
X = exercise price,
T = time to maturity in years,
σ = instantaneous proportional standard deviation per annum,
r_f = risk-free interest rate per annum,
Δt = length of time subinterval for lattice approach in years,
$n = T/\Delta t$ = total number of subintervals, and
P = the American put price.

The control variate estimate of P, $\hat{C}_A^*$, is calculated from Equation (7). C_B is the European put price. $\hat{C}_A$ is the value obtained using the CRR lattice approach, as described in Section II, and $\hat{C}_B$ is the value obtained for the European put using the same lattice (i.e., the same value of n).

In Table 1, the control variate technique is illustrated for the option defined by $r_f = 0.0488$, $X = 35.0$, $T = 0.5833$, $\sigma = 0.2$, and $S_0 = 40.0$. This is one of the options considered by Johnson (1983), Geske and Johnson (1984), Macmillan (1986), and Cox and Rubinstein (1984). The true value of the option, based on the use of the CRR lattice approach with $n = 500$, is 0.433. This is close to the value obtained by other authors. Table 1 shows that as n increases, $\hat{C}_A$ oscillates about 0.433. $\hat{C}_B$ tends to oscillate about the true value of the European put option, 0.417, in a corresponding way.[10] This provides the basis for the control variate technique. It will be noted that $\hat{C}_A^*$ converges to the true value far more quickly than $\hat{C}_A$.

The time taken to perform the calculations for the CRR lattice approach is approximately proportional to $n(n+1)$. For a given value of n, the time taken to perform calculations for the control variate approach is approximately twice that for the CRR lattice approach. The relative efficiency of the CRR lattice approach and the control variate approach (measured by the relative computing time required to achieve a specified level of accuracy) therefore can be estimated by comparing the value of n required to achieve a specified level of accuracy in pricing. If n_{CRR} and n_{CV} are the values of n required to achieve the desired accuracy using the CRR lattice and the control variate approach, respectively, the relative efficiency is given by

$$\frac{n_{CRR}(n_{CRR}+1)}{2n_{CV}(n_{CV}+1)}.$$

[10] The convergence of binomial and other models to the true option value is discussed by Omberg (1987b). Whereas the Geske-Johnson compound option model converges uniformly, convergence for the binomial model is oscillatory.

TABLE 1

Use of the Control Variate Technique for an American Put Option on a Stock when $r_f = 0.0488$, $X = 35.0$, $T = 0.5833$, $\sigma = 0.2$, and $S_0 = 40.0$[a]

n	C_B	$\hat{C}_B$	$\hat{C}_A$	$\hat{C}_A^*$
1	0.4170	0.3728	0.3728	0.4170
5	0.4170	0.4785	0.4884	0.4269
10	0.4170	0.4412	0.4530	0.4287
15	0.4170	0.4182	0.4319	0.4306
20	0.4170	0.4072	0.4276	0.4374
25	0.4170	0.4271	0.4437	0.4336
50	0.4170	0.4141	0.4302	0.4331
75	0.4170	0.4192	0.4348	0.4325
100	0.4170	0.4195	0.4351	0.4326

[a] C_B is the Black-Scholes price of the European put with the same parameters. $\hat{C}_B$ and $\hat{C}_A$ are the estimates of the European and American put prices using a binomial tree with n intervals. $\hat{C}_A^*$ is the Control Variate price calculated as $C_B + (\hat{C}_A - \hat{C}_B)$.

Table 2 shows how the control variate technique, with $n = 25$, performs on all the options considered by Geske and Johnson (1984). This set of options has been widely used to compare different numerical procedures. The results from using the CRR lattice approach with $n = 500$ are shown for the purposes of comparison. The table shows that the control variate approach is highly efficient when there is a relatively small chance of early exercise. This is because ρ, the correlation between $\hat{C}_A$ and $\hat{C}_B$, is close to $+1$. For the nine out-of-the-money options in Table 2, the control variate approach was found to be highly efficient. The lowest relative efficiency measure was 4.8; the highest was over 2000. As the probability of early exercise increases, the control variate approach becomes relatively less efficient. Eventually, a stage is reached in which the control variate approach is actually less efficient than the CRR approach.[11] For the 21 at-the-money options in Table 2, the control variate technique was found to be, on average, 3.3 times more efficient than the CRR approach and, for the nine in-the-money options in Table 2, the control variate approach was found to be 1.2 times as efficient as the CRR approach, on average.

The control variate method can be used for American puts on stocks with known dividends. A European option that pays the same dividends as the American option is used as the control option B, and the appropriate lattice (see Section III) is used to value both the American and the European option. Table 3 compares the results obtained from using the control variate technique with those of Geske and Johnson (1984) for options on stocks in which the dividend yield is assumed known. Again, the results from using the normal CRR approach with $n = 500$ are given for the purposes of comparison. The estimates $\hat{C}_A$ and $\hat{C}_B$ were calculated using the Cox and Rubinstein (1984) lattice for stocks that pay a known dividend yield (see Section III) and $n = 25$. The relative efficiency measures are generally higher than those in Table 2. This is because put options are less likely to be exercised early when there are known dividends. For eight of the

[11] This is because ρ is so low that the condition in Equation (9) is not satisfied. An example would be an option such as the nineteenth one in Table 2 that should be exercised immediately. The basic lattice approach recognizes this and assigns the option its correct value for any value of n. The control variate technique only estimates the correct value in the limit as $n \rightarrow \infty$.

TABLE 2

Comparison of Control Variate Technique ($n = 25$), Geske-Johnson Procedure, and CRR ($n = 500$) when Used to Calculate Values for American Put Options on Nondividend-Paying Stocks[a]

r_f	X	σ	T	S_0	P Control Variate ($n = 25$)	P Geske-Johnson	P CRR ($n = 500$)	Relative Efficiency
0.1250	1.0	0.5	1.0000	1.0	0.1475	0.1476	0.1480	2.2
0.0800	1.0	0.4	1.0000	1.0	0.1258	0.1258	0.1260	1.9
0.0450	1.0	0.3	1.0000	1.0	0.1004	0.1005	0.1005	2.5
0.0200	1.0	0.2	1.0000	1.0	0.0711	0.0712	0.0711	3.5
0.0050	1.0	0.1	1.0000	1.0	0.0377	0.0377	0.0377	7.3
0.0900	1.0	0.3	1.0000	1.0	0.0858	0.0859	0.0861	1.7
0.0400	1.0	0.2	1.0000	1.0	0.0639	0.0640	0.0640	1.3
0.0100	1.0	0.1	1.0000	1.0	0.0357	0.0357	0.0357	3.1
0.0800	1.0	0.2	1.0000	1.0	0.0525	0.0525	0.0527	1.2
0.0200	1.0	0.1	1.0000	1.0	0.0322	0.0322	0.0322	1.5
0.1200	1.0	0.2	1.0000	1.0	0.0439	0.0439	0.0442	0.4
0.0300	1.0	0.1	1.0000	1.0	0.0292	0.0292	0.0293	0.8
0.0488	35.0	0.2	0.0833	40.0	0.0062	0.0062	0.0062	>100.0
0.0488	35.0	0.2	0.3333	40.0	0.2000	0.1999	0.2002	8.5
0.0488	35.0	0.2	0.5833	40.0	0.4336	0.4321	0.4331	8.2
0.0488	40.0	0.2	0.0833	40.0	0.8527	0.8528	0.8519	3.9
0.0488	40.0	0.2	0.3333	40.0	1.5793	1.5807	1.5794	2.2
0.0488	40.0	0.2	0.5833	40.0	1.9878	1.9905	1.9901	1.9
0.0488	45.0	0.2	0.0833	40.0	5.0007	4.9985	5.0000	0.5
0.0488	45.0	0.2	0.3333	40.0	5.0979	5.0951	5.0886	0.1
0.0488	45.0	0.2	0.5833	40.0	5.2626	5.2719	5.2674	0.2
0.0488	35.0	0.3	0.0833	40.0	0.0774	0.0774	0.0773	>100.0
0.0488	35.0	0.3	0.3333	40.0	0.6976	0.6969	0.6983	38.3
0.0488	35.0	0.3	0.5833	40.0	1.2273	1.2194	1.2212	4.8
0.0488	40.0	0.3	0.0833	40.0	1.3109	1.3100	1.3094	7.2
0.0488	40.0	0.3	0.3333	40.0	2.4833	2.4817	2.4819	3.1
0.0488	40.0	0.3	0.5833	40.0	3.1678	3.1733	3.1689	2.6
0.0488	45.0	0.3	0.0833	40.0	5.0626	5.0599	5.0597	1.1
0.0488	45.0	0.3	0.3333	40.0	5.7133	5.7012	5.7066	1.8
0.0488	45.0	0.3	0.5833	40.0	6.2421	6.2365	6.2446	1.9
0.0488	35.0	0.4	0.0833	40.0	0.2467	0.2466	0.2462	>100.0
0.0488	35.0	0.4	0.3333	40.0	1.3517	1.3450	1.3475	61.3
0.0488	35.0	0.4	0.5833	40.0	2.1571	2.1568	2.1552	36.7
0.0488	40.0	0.4	0.0833	40.0	1.7693	1.7679	1.7675	11.7
0.0488	40.0	0.4	0.3333	40.0	3.3891	3.3632	3.3865	5.2
0.0488	40.0	0.4	0.5833	40.0	4.3537	4.3556	4.3517	4.5
0.0488	45.0	0.4	0.0833	40.0	5.2927	5.2855	5.2872	1.7
0.0488	45.0	0.4	0.3333	40.0	6.5103	6.5093	6.5111	1.9
0.0488	45.0	0.4	0.5833	40.0	7.3839	7.3831	7.3851	1.3

[a] The variables n, r_f, X, σ, T, S_0, and P are the number of subintervals used, the risk-free interest rate, exercise price, volatility p.a., time to maturity (years), initial stock price, and estimated option price, respectively. The relative efficiency measure shows the improvement in efficiency when the control variate method was used instead of CRR and 1-percent accuracy was required.

nine out-of-the-money options in Table III, the control variate method was over 100 times more efficient than CRR. For the nine at-the-money options, the control variate method was on the average 39.3 times more efficient than CRR. For the nine in-the-money options, it was, on average, 16.2 times more efficient.

TABLE 3

Comparisons of Control Variate Technique ($n = 25$), Geske-Johnson Procedure, and CRR ($n = 500$) for American Put Options on Dividend-Paying Stocks.[a]

X	σ	T	P Control Variate ($n = 25$)	P Geske-Johnson	P CRR ($n = 500$)	Relative Efficiency
35.0	0.2	0.0833	0.0116	0.0116	0.0116	>100.0
35.0	0.2	0.3333	0.3092	0.3071	0.3070	16.6
35.0	0.2	0.5833	0.6566	0.6580	0.6568	>100.0
40.0	0.2	0.0833	1.1113	1.1079	1.1087	15.8
40.0	0.2	0.3333	2.0224	2.0120	2.0135	2.9
40.0	0.2	0.5833	2.5781	2.5717	2.5762	3.4
45.0	0.2	0.0833	5.4119	5.4209	5.4139	0.5
45.0	0.2	0.3333	5.6736	5.6900	5.6710	3.8
45.0	0.2	0.5833	6.0203	6.0300	6.0208	1.3
35.0	0.3	0.0833	0.1078	0.1073	0.1074	>100.0
35.0	0.3	0.3333	0.8853	0.8837	0.8831	>100.0
35.0	0.3	0.5833	1.5456	1.5454	1.5457	>100.0
40.0	0.3	0.0833	1.5594	1.5590	1.5595	60.0
40.0	0.3	0.3333	2.9106	2.9072	2.9109	33.0
40.0	0.3	0.5833	3.7473	3.7435	3.7501	95.0
45.0	0.3	0.0833	5.4977	5.4996	5.4976	0.5
45.0	0.3	0.3333	6.3027	6.3089	6.2982	52.5
45.0	0.3	0.5833	6.9997	6.9977	6.9999	33.0
35.0	0.4	0.0833	0.3049	0.3049	0.3051	>100.0
35.0	0.4	0.3333	1.5834	1.5798	1.5794	>100.0
35.0	0.4	0.5833	2.5275	2.5277	2.5301	>100.0
40.0	0.4	0.0833	2.0127	2.0120	2.0127	52.5
40.0	0.4	0.3333	3.8087	3.8033	3.8072	52.5
40.0	0.4	0.5833	4.9187	4.9116	4.9229	39.0
45.0	0.4	0.0833	5.7033	5.7015	5.7018	7.5
45.0	0.4	0.3333	7.0834	7.0774	7.0700	33.0
45.0	0.4	0.5833	8.0973	8.0914	8.1014	14.0

[a] Dividends equal to 1.25 percent of the stock are paid in ½, 3½, and 6½ months. Thus, one-, four-, and seven-month options (T = 0.0833, 0.3333, 0.5833) have one, two, and three scheduled dividend payments, respectively. The variables n, X, σ, T, and P are the number of subintervals used, the exercise price, volatility p.a., time to maturity (years), and the estimated option price, respectively. The initial stock price, S_0 is 40, and the risk-free interest rate, r_f is 0.0488. The relative efficiency measure shows the improvement in efficiency when the control variate method was used instead of CRR, and 1-percent accuracy was required.

Geske and Johnson (1984) found their compound option approach to be ten times more efficient than standard numerical procedures. Our tests show that, overall, the control variate approach is also about ten times more efficient than the standard CRR approach when used to value American put options on stocks. Both the control variate technique and the compound option approach work best when there is a low chance of early exercise.[12] The control variate approach has the advantage that it is somewhat easier to implement.

[12] In the limit, when there is no chance of early exercise, both methods work perfectly since the compound option collapses to a simple option and $\hat{C}_A = \hat{C}_B$, leaving $\hat{C}_A^* = C_B$ in the control variate approach.

As an additional test of the control variate technique, the procedure described in Section IV was used to estimate the partial derivative of the option price with respect to the stock price for the options in Table 2 with $n = 25$.[13] The control variate technique produced an overall increase in efficiency. The absolute errors in the estimates relative to the values given by the normal CRR approach with $n = 500$ were, on average, about half those given by the Geske and Johnson (1984) approach using the same data. This provides additional evidence that the control variate method is an attractive alternative to the Geske-Johnson approach.

VI. Application to Other Options

The control variate approach can be used for American currency and commodity options. The natural option to use as option B is the European option with the same parameters. In the case of currency options, we find that the control variate technique generally works very efficiently for calls when the foreign interest rate is below the domestic interest rate, for puts when the foreign interest rate is above the domestic interest rate, and for all out-of-the-money options.

Consider next the pricing of an American put option on a nondividend-paying stock when the volatility is stochastic. This presents an interesting problem in lattice design. Assume that

$$dS = \mu S dt + \sqrt{V} S dz_1 ,$$
$$dV = \xi V dz_2 ,$$

where V $(= \sigma^2)$ is the instantaneous variance of dS/S, ξ^2 is the instantaneous variance of dV/V, and dz_1 and dz_2 are uncorrelated Wiener processes. This model is discussed by Hull and White (1987) who produce a series solution for the value of a European call. The value of the corresponding European put can be obtained using put-call parity. This option can be used as option B in the control variate approach.

Since there are two state variables, a two-dimensional lattice is necessary. The transition probabilities q_{ijk} must give correct values for the mean and standard deviation of changes in S, the mean and standard deviation of changes in V, and a zero coefficient of correlation between S and V. Experimentation shows that a tree with six non-zero q_{ijk} for each i and j works well. If, at time $i\Delta t$, the state of the world is $\{S_i, V_i\}$ (i.e., the stock price is S_i and the volatility is V_i), the six possibilities at time $(i+1)\Delta t$ are

$$\{S_i u_1, V_i u_2\}; \qquad \{S_i u_1, V_i d_2\};$$
$$\{S_i, V_i u_2\}; \qquad \{S_i, V_i d_2\};$$
$$\{S_i d_1, V_i u_2\}; \text{ and } \{S_i d_1, V_i d_2\},$$

[13] The hedge ratio was calculated by slightly extending the lattice so that three values of the stock price, S_0, $S_0 u^2$, and $S_0 d^2$, were considered at time zero. For both the European and American options, the difference between the option prices at the $S_0 u^2$ and $S_0 d^2$ nodes was divided by $S_0(u^2 - d^2)$ to obtain the hedge ratio.

where $u_1 = 1/d_1$ and $u_2 = 1/d_2$.

This problem is more complicated than the two state variable problem considered by Boyle (1988) because the variance of one of the state variables, the stock price, is different in different parts of the lattice. If u_1 and d_1 are constant throughout the lattice, they must be large enough to accommodate the maximum value of V in the lattice. The result is that the probability of a zero stock price change is liable to become close to 1.0 in other parts of the lattice where V is low, and the lattice may not be a good representation of possible states of the world at all times. Clearly, it is necessary to vary u_1 and d_1 according to the value of V. This must be done carefully so that the total number of nodes does not increase too fast. One solution that the authors have used successfully involves putting $u_1 = \alpha^m$ and $d_1 = 1/\alpha^m$, where α is a constant and m is an integer whose value depends on V.

VII. Conclusions

This paper has considered the use of the lattice approach to option pricing. It has shown that, in principle, lattices can be constructed to deal with any number of state variables. It also has examined how a lattice can be used to deal efficiently with the situation in which there are known dividends (as opposed to known dividend yields) during the life of the option.

The main contribution of the paper has been to show how the control variate approach can be used in conjunction with the lattice approach. If the option under consideration is similar to another option for which an analytic solution is available, the same lattice can be used to evaluate both options, and a considerable improvement in numerical efficiency often results. The control variate approach also may be used to improve the efficiency of other numerical procedures such as the finite difference method.[14]

[14] Examples of its use in conjunction with the finite difference method are provided by Hull (1988).

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