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# Pricing Warrants: An Empirical Study of the Black-Scholes Model and Its Alternatives

BENI LAUTERBACH and PAUL SCHULTZ*

**ABSTRACT**

This paper uses a sample of over 25,000 daily warrant prices to empirically investigate potential problems with the commonly used warrant pricing model proposed by Black and Scholes as an extension of their call option model. One problem seems to be especially important: the constant variance assumption of the dilution adjusted Black-Scholes model appears to cause biases in model prices for almost all warrants and over the entire sample period. We show that more accurate price forecasts are obtained with a specific form of the constant elasticity of variance model.

SINCE THE DERIVATION OF the Black-Scholes Model, financial economists have speculated on its usefulness for warrant valuation. In their pathbreaking paper, Black and Scholes (1973) claim that in many cases their model "can be used as an approximation to give an estimate of the warrant value." However, Black and Scholes (1973) warn that

> The life of a warrant is typically measured in years, rather than months. Over a period of years, the variance rate of return on the stock may be expected to change substantially.

In similar fashion, the Black-Scholes assumption of a constant riskless interest rate may be especially troublesome given the long life of a warrant. Merton (1973) shows that the Black-Scholes model can be altered to accommodate stochastic interest rates. Merton's model is identical to the Black-Scholes model except that the yield to maturity for a default free bond that matures at the option's expiration date is used for the interest rate, and the variance of a portfolio of the stock and the riskless bond is used in place of the stock variance. (However, Merton's version of the option pricing model may be inappropriate for warrants as it assumes that the variance of the default free bond is constant. Over the long life of a warrant, the variance of bond prices may change.)

Other potential problems with using the Black-Scholes model for warrants arise because the Black-Scholes model assumes that warrants are European and expire at a given expiration date. In reality, warrant holders, like option holders, may elect to exercise early if the underlying stock pays sufficiently large divi-

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dends. However, an additional problem in warrant pricing that does not arise in pricing calls is that companies tend, for tax purposes, to extend the expiration date of warrants if an "out-of-the-money" expiration is imminent.

The applicability of the Black-Scholes model to warrant pricing is an empirical issue. To the extent that the Black-Scholes model performs poorly, a second empirical issue is which alternative models perform better. To date, these questions remain unanswered. There are only a handful of empirical studies of warrants, and these studies focus on demonstrating techniques for pricing contingent claims rather than testing warrant pricing models.

Chen (1975) uses dynamic programming to price five warrants. He concludes that this technique accurately prices warrants if reasonable estimates of the expected return and volatility of the underlying stock can be obtained. Noreen and Wolfson (1981) use a total of 52 observations of warrant prices to test a Black-Scholes warrant model that assumes the underlying stock follows a log-normal diffusion process and a model that assumes stock prices follow a constant elasticity of variance (CEV) diffusion process. Their purpose in examining warrant prices is to see if warrant pricing models can be used to price executive stock options. They claim that the CEV and Black-Scholes models work equally well. Schwartz (1977) uses a finite difference approach to approximate solutions to a partial differential equation that describes warrant values. He examines only 17 observations of AT&T warrant prices as the focus of his paper is demonstrating the finite difference technique for pricing contingent claims.

In this paper we examine over 25,000 observations of daily prices of listed warrants. We use a dilution-adjusted version of the Black-Scholes model to estimate implied equity standard deviations (henceforth ISDs). Systematic changes in ISDs across time are then used to determine if other models could price warrants more accurately than the Black-Scholes model. The observed behavior of the ISDs is consistent with predictions of models that allow for an inverse relation between equity values and variances. A direct test confirms that the constant elasticity of variance (CEV) model provides better predictions of warrant prices than the Black-Scholes model. In contrast, we find little evidence that other modifications of the Black-Scholes model are useful in practice.

The remainder of the paper is organized as follows. Section I discusses differences between warrants and call options and shows how option pricing models may be modified to price warrants. Section II describes how the Black-Scholes model is tested against alternatives. Section III describes the data employed in this study and provides details of how the data are used in the model. Section IV presents results of the empirical tests. Finally, a summary of the paper and suggestions for further research are given in Section V.

## I. Using the Black-Scholes Model to Price Warrants

When the Black-Scholes call option model is adjusted for the dilution that occurs when warrants are exercised, model warrant prices are given by

$$W = \left(\frac{N}{N/\gamma + M}\right)\left[\left(S - \sum_i e^{-rt_i}D_i + \frac{M}{N}\,W\right)N(d_1) - e^{-rT}xN(d_2)\right], \qquad (1)$$

where

$$d_1 = \frac{\ln\left(\frac{S - \sum_i e^{-rt_i} D_i + (M/N)W}{x}\right) + rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2},$$

$d_2 = d_1 - \sigma\sqrt{T}$,
$W$ = the warrant price,
$S$ = the stock price,
$x$ = the exercise price,
$N$ = the number of outstanding shares of stock,
$M$ = the number of warrants,
$\gamma$ = the number of shares that can be purchased with each warrant,
$r$ = the risk-free interest rate,
$T$ = the time until expiration,
$\sigma$ = the standard deviation of the return of $S + (M/N)W$ per unit time,
$N(d)$ = the cumulative normal distribution function evaluated at $d$,
$t_i$ = the time until the $i$th dividend is paid, and
$D_i$ = the dollar amount (per share) of the $i$th dividend.

The need for modifications to the Black-Scholes call option model arises because warrants are not written by other investors; they are supplied by the firm. When warrants are exercised, the firm receives the exercise price and the size of the corporate pie increases. The firm then issues additional shares with the result that the corporate pie is cut into more pieces.

Work by Galai and Schneller (1978) and by Black and Scholes (1973) shows that the Black-Scholes and other models for pricing European Options can be altered to incorporate dilution and thus price warrants if three modifications are made in the models. First, $S + (M/N)\ W$ must be substituted for the stock price. This sum can be thought of as the equity per share of stock. A second modification is that the standard deviation used in the formula is now the standard deviation of equity, that is $S + (M/N)\ W$, rather than the stock volatility. Finally, the entire formula is multiplied by $N/((N/\gamma) + M)$, where $\gamma$ is the number of shares that can be purchased with each warrant. These effects of dilution are discussed further in the appendix.[1]

Note that in equation (1) we adjust for dividends by subtracting the present value of dividends from the equity value. This dividend adjustment technique is commonly used in option pricing. However, it has the disadvantage of ignoring early exercise. This is not a problem if the dividends are small, as they generally are in this sample, and if the warrants are broadly held.[2]

[1] In a recent paper, Schulz and Trautman (1989) discuss how failure to account for dilution may affect estimated warrant prices.

[2] If large dividends are paid, complications arise in pricing warrants that do not appear with options. Optimal exercise policy depends on how the firm uses the proceeds from warrant exercise. If the firm invests this money in risky projects, it will be optimal for only a portion of the warrant holders to exercise, but the value of the warrants will be the same as if all were exercised simultaneously. On the other hand, if proceeds from the exercise of warrants are paid out in dividends, a warrant holder's optimal exercise policy depends on the actions of other warrant holders. In this case there are a number of possible equilibrium prices for the warrants. See Constantinides (1987) for a discussion of these issues.

The dilution-adjusted Black-Scholes model cannot be used to price all warrants. Many have unique or complicated exercise provisions. One common provision allows warrant holders to pay the warrant's exercise price by redeeming bonds or preferred stock at par. Other unorthodox warrants may have changing exercise prices, be callable, or be perpetuities. Warrants with complex provisions are excluded from the empirical analysis since the Black-Scholes model is not applicable to these securities.

## II. Testing the Black-Scholes Model Against Its Alternatives

To test the Black-Scholes warrant pricing model, the following regression is run for each warrant for each quarter over 1971–80:

$$\mathrm{ISD}_t = \alpha_0 + \alpha_1\left(\frac{S - \sum_i D_i e^{-rt_i} - e^{-rT}x}{e^{-rT}x}\right)_{t-1} + \alpha_2 r_{t-1} + \varepsilon_t. \tag{2}$$

Note that in equation (2), the ISD on day $t$ is regressed against the percentage that the warrant is in or out of the money on the previous day and the default free interest rate of the previous day. Use of contemporaneous independent variables would result in biased coefficients. To see this, consider what happens when the warrant trades less frequently than the underlying stock. The observed warrant price is determined when the market price of the underlying stock differs from its observed closing price. If the stock price increases (decreases) after the last warrant trade, $\mathrm{ISD}_t$ will tend to be downward (upward) biased. At the same time, the in-the-money variable will tend to be larger (smaller) than average when the stock price increases (decreases) after the last trade. Thus, the nonsynchronous trading will induce a spuriously negative estimate of the $\alpha_1$ coefficient.[3] Use of lagged independent variables eliminates this potential bias by making the measurement error of the ISD independent of that of the "out-of-the-money" variable.

Our estimates of equation (2) tell us if ISDs of individual warrants vary systematically with interest rates and the value of the underlying equity. Rubinstein (1985) conducts a similar study of the ISDs of call options. Rubinstein (1985) looks for systematic variation of ISDs across options with different exercise prices and times to expiration as a test of the Black-Scholes model and to infer which models might work better. We are not able to study ISDs of warrants cross-sectionally as companies do not issue several warrants at the same time. However, our estimates of equation (2) provide an analogous time series examination of systematic variation in ISDs.

Table I describes some of the potential problems with the Black-Scholes warrant model, how these problems would show up in estimates of $\alpha_1$ and $\alpha_2$, and alternative models that would not suffer from the same deficiencies. Several of the potential problems arise from the Black-Scholes model's assumptions

[3] Any measurement error (including the bid-ask spread) introduces a similar bias. Suppose $S = S_{true} + \varepsilon$ where $\varepsilon$ is measurement error. A positive (negative) $\varepsilon$ decreases (increases) the ISD and increases (decreases) the "out-of-the-money" variable. Consequently, a spuriously negative $\alpha_1$ coefficient is induced.

**Table I**

**Potential Shortcomings of the Black-Scholes Warrant Pricing Model and the Implications of Model Misspecification for Coefficients of the Regression**

$$\mathrm{ISD}_t = \alpha_0 + \alpha_1 \left( \frac{S - \sum_i e^{-rt_i} D_i - e^{-rT} X}{e^{-rT} X} \right)_{t-1} + \alpha_2 r_{t-1} + \varepsilon_t$$

$\mathrm{ISD}_t$ is the implied standard deviation estimated for a warrant for day $t$ using the Black-Scholes model. $S$ is the stock price, $\sum_i e^{-rt_i} D_i$ is the present value of dividends to be paid over the life of the warrant, $T$ is the time until the warrant's expiration, $X$ is the warrant's exercise price, and $r_{t-1}$ is the day $t-1$ return on a portfolio of Treasury Bonds with an average maturity equal to the warrant's expiration date.

| Deficiency in the Black-Scholes Model | Appropriate Alternative Models | Implications for $\alpha_1$ | Implications for $\alpha_2$ |
|---|---|---|---|
| 1. None | — | $\alpha_1 = 0$ | $\alpha_2 = 0$ |
| 2. Interest rate is stochastic with constant variance | Merton (1973) Variable Interest Rate | $\alpha_1 = 0$ | $\alpha_2 = 0$ |
| 3. Equity variances are inversely related to equity values | i) Constant Elasticity of Variance (Cox (1975)) ii) Compound Option Model (Geske (1979)) iii) Stochastic volatility that is inversely correlated with equity values (Wiggins (1987)) | $\alpha_1 < 0$ | |
| 4. Equity variance is stochastic but uncorrelated with equity values | Wiggins (1987) Stochastic Variance Models | $\alpha_1 < 0$ (Out-of-the-Money Warrants) $\alpha_1 > 0$ (In-the-Money Warrants) | |
| 5. Interest rate is stochastic and the variance of the interest rate is correlated with the interest rate | | | $\alpha_2 \lesseqgtr 0$ |
| 6. Early exercise is possible | American Warrant Model (Numerical Pricing) | $\alpha_1 > 0$ | $\alpha_2 < 0$ |
| 7. Warrants may be extended | Longstaff (1990) Model | $\alpha_1 < 0$ | $\alpha_2 > 0$ |

about the stochastic processes followed by the underlying variables. These problems might be more acute for warrants than options because of warrants' long lives.

For example, the Black-Scholes warrant model assumes a constant equity variance. There is empirical evidence (e.g., Christie (1982)) that stock volatilities decrease as stock prices rise. In this case, ISDs would fall as the stock price increases and $\alpha_1$ would be negative. Several alternative models explicitly allow for an inverse relation between stock price and volatility. These models include constant elasticity of variance models, compound option models, and models that assume a stochastic variance that is negatively correlated with the stock price.

Alternatively, the stock variance could be stochastic but unrelated to the equity value. In this case, as Wiggins (1987) notes, stochastic volatility affects option values in much the same way as a jump component. Both can significantly increase the chance that the out-of-the-money options will finish in the money and both can enhance the insurance value of deep-in-the-money options by increasing the probability of out-of-the-money expiration. Thus, as Wiggins (1987) shows, the Black-Scholes model will under-price both in- and out-of-the-money options when volatility is stochastic but uncorrelated with stock prices. In terms of our tests, this implies that $\alpha_1$ will be negative for out-of-the-money warrants and positive for in-the-money warrants. Other specifications of a stochastic volatility model have different implications. Wiggins (1987) shows that $\alpha_1$ will be negative if volatility is stochastic and is inversely correlated with the stock price.

Another possible deficiency in the Black-Scholes model arises from its assumption of a constant default free interest rate. Merton (1973) has generalized the model to allow for stochastic interest rates. However, if the variance of the interest rate is correlated with the interest rate level or the stock price, $\alpha_2$ could differ from zero.

A different set of potential problems with the Black-Scholes model arise from institutional factors that the model ignores. One is the possible early exercise of warrants on dividend paying stocks. The right to exercise early is of little value when warrants are out of the money and Black-Scholes and market warrant prices should be similar. But, as equity values rise, the right to exercise early becomes valuable and market warrant prices rise faster than Black-Scholes prices. The ISDs (inferred from market prices of the warrant and stock using the Black-Scholes model) therefore rise with equity values. Hence, $\alpha_1$ will be positive. At the same time, $\alpha_2$ will be negative. As interest rates increase, investors may choose to earn interest on the warrant's exercise price rather than exercise early, the market and Black-Scholes prices will converge, and ISDs will fall. Hence, $\alpha_2$ will be negative.

Another institutional factor not incorporated in the Black-Scholes model is the potential for extension.[4] On April 24, 1972, the Internal Revenue Service ruled that if a warrant expires worthless, the initial issue price of the warrant is taxable income for the firm. As a result, some firms have extended warrants to avoid taxation. The possibility of extension can be thought to place a floor on the value of a warrant. So, actual market prices of deep out-of-the-money warrants will be higher than the Black-Scholes price, and ISDs inferred from

[4] In a recent working paper, Longstaff (1990) discusses the problem of pricing extendable calls. He derives a closed form solution for a call option that can be extended for a finite time period. This model has the interesting property that under certain circumstances call prices may actually decrease as stock prices rise. This is because of the decreased probability of extension that occurs as the stock price increases.

market prices with the Black-Scholes model will be high for low equity values. Hence $\alpha_1$ will be less than zero. At the same time, $\alpha_2$ will be positive. The possibility of extension means the expected life of the warrant is greater than assumed by the Black-Scholes model. This extra time on the warrant will be especially valuable when the interest rate is high. Thus ISDs will be large for high interest rates and $\alpha_2$ will be positive.

Extension is probably a less important factor during our sample period than in succeeding years. Most warrants that we examine were issued prior to the original Internal Revenue Service ruling and do not have explicit extension provisions. This makes it impossible to tell when, for how long, and under what circumstances a warrant will be extended. But, more important, this enables a firm to minimize warrant values while preventing an out of the money expiration. Firms may do this by extending warrants numerous times for short periods, by offering to exchange a token number of shares for the warrants, or by lowering the exercise price to just below the current stock price.

## III. Data

The warrant prices used here are taken from a tape owned by the Center for Research in Securities Prices (CRSP). This tape contains daily warrant prices for all New York and American Stock Exchange listed warrants for January 1971 through December 1980. Although the tape is owned by CRSP, it is not part of the data that is sold to CRSP subscribers.[5] Thus CRSP has not updated the data set since 1980 nor do they guarantee the accuracy of the prices. We employ a filter to examine warrant prices for coding errors, and find only 17 obviously erroneous warrant prices. These observations, which are eliminated from the sample, include warrant prices that appear to have misplaced decimal points or omitted digits.

Data on the exercise provisions of the warrants are obtained from several sources, including company 10k reports, Value Line Options and Convertibles, the R.H.M. Survey of Warrants, Options & Low Price Stocks and Moody's Industrial Manual. The information gleaned from these sources includes the warrant's exercise price, its expiration date, and the number of shares that can be purchased with each warrant. We also use these sources to determine whether warrants had special exercise provisions that would prevent them from being included in our final sample.

Daily prices of stocks underlying the warrants are obtained from the CRSP Daily Master Tape. This tape is also the source of information about dividends paid by the underlying securities. Pricing of warrants using equation (1) requires that the sum of the discounted values of all dividends to be paid over the life of the warrant be subtracted from the stock price. We assume that investors employ a naive forecast of dividends for pricing warrants. Investors are assumed to know the exact amount of dividends to be paid within the calendar quarter. If this

[5] CRSP obtained the tape with the stipulation that the data were not to be sold or released to subscribers. The authors of this paper were given access to the data set while graduate students at the University of Chicago. This tape is also used by Stickel (1986) in his study of preferred stock.

known dividend is announced as a quarterly dividend, dividends of the same dollar amount are assumed to be paid every quarter year over the remaining life of the warrant. Special dividends to be paid within a quarter are included in the total dividends subtracted from stock prices, but they are not extrapolated into future quarters. Semiannual and annual dividends are extrapolated into the future and assumed to be paid at 6 and 12 months intervals.

Data on number of shares outstanding and number of warrants outstanding are taken from the CRSP Daily Master Tape, company 10k reports, and Moody's Industrial Manuals. Stock split information is also obtained from the CRSP Daily Master Tape. Warrants outstanding are adjusted to reflect splits as of the date of the split.

Interest rate data used in this paper are obtained from a sample of daily dealer quotes assembled by the Federal Reserve Board.[6] Daily rates for 12 month T-bills and 3, 5, and 7 year Treasury notes are utilized. Warrants with more than 7 years to expiration are priced with interest rates from 7 year notes, while 12 month T-bill rates are used in the model for warrants with less than 1 year to expiration. Interest rates for warrants with maturities between 1 and 7 years are obtained by interpolating rates of notes with longer and shorter maturities. For example, if a warrant expires in 4½ years, the interest rate used to price the warrant is obtained by adding ¼ times the rate on 3 year notes and ¾ times the rate on 5 year notes.

Less than half of the observations on the warrant tape are utilized here. Most warrants are not included in the tests because they have exercise provisions that make the Black-Scholes model clearly inappropriate. Table II details why warrants are dropped from the sample.

Examination of Table II reveals that warrants are excluded from the study for three reasons other than special exercise provisions. Because of the uncertain treatment of warrants in the event of a merger, warrants are discarded if the firm is referred to as a takeover candidate by the Wall Street Journal in the year prior to the first warrant observation on the tape. Warrants are also deleted if different sources provide contradictory information on exercise provisions, or if available information is incomplete. Finally, warrants with few or no observations at market prices of $1 or more are deleted from the sample. Individual daily observations are not used when the warrant price is below $1 as the bid-ask spread will typically be large relative to the price of these warrants. In this case, most of the fluctuations in ISDs are spurious.

Individual daily observations are discarded for reasons other than prices under $1. Observations are also rejected if the warrant prices violate arbitrage boundary conditions. Third, if warrants are extended, the announcement date for the extension is obtained from the *Wall Street Journal* and all observations in the three months prior to the extension are omitted. Finally, if there are fewer than 20 usable observations during a quarter, no prices from that quarter are used.

In total 25,171 daily price observations from 39 warrants meet our inclusion criteria. Table III describes this sample using the inclusion dates, number of

[6] We are grateful to Ken French for allowing us to use this data.

**Table II**

**Reasons for Excluding Warrants from the Study**

| | |
|---|---|
| Total Number of Warrants Listed on NYSE and AMEX 1971–1980 | 100 |
| Excluded Because Other Securities Can be Used to Pay Exercise Price | −21 |
| Excluded Due to Insufficient Number of Usable Observations | −15 |
| Excluded Because Warrants are Callable | −6 |
| Excluded Because of Scheduled Exercise Price Increase | −5 |
| Excluded Because Warrants are Perpetuities | −4 |
| Excluded Because of Insufficient or Contradictory Information on Exercise Provisions | −10 |
| Excluded For Miscellaneous Reasons (Merger Candidate, Unique Exercise Provisions) | −5 |
| Number of Warrants in Two Exclusion Categories | +5 |
| Total Number of Warrants Included in Sample | 39 |

observations, and means of parameter values for each warrant. Examination of Table III reveals that the warrant observations range from deep-out-of-the-money to deep-in, and cover a wide range of dilution factors and times to expiration.

Of special interest is the implied standard deviations listed in Table III. When compared to ISDs computed from listed options, the ISDs in Table III seem large. To gain perspective on the ISD estimates, we calculate an average (weighted by the derivative of the warrant price with respect to the standard deviation) of the daily ISDs for each warrant during each quarter. The average ISDs are then compared with standard deviations realized the following quarter. There are 339 warrant quarter observations where an ISD one quarter can be compared with a standard deviation estimated from at least 40 returns in the subsequent quarter. The average ISD over the 339 warrant quarters is .556, while the average equity (defined as $S + (M/N)W$) standard deviation realized the following quarter is a similar .594. On a cautionary note, the average realized stock standard deviation in the subsequent quarter is .415. This serves to emphasize that equity volatilities, not stock volatilities, should be used to price warrants. It also illustrates why the ISDs in Table III may appear large relative to ISDs from stock options.

## IV. Results

### *A. Regression Tests of the Black-Scholes Warrant Pricing Model*

Table IV presents a summary of the estimates of the regression

$$\text{ISD}_t = \alpha_0 + \alpha_1\left(\frac{S - \sum_i D_i e^{-rt_i} - e^{-rT}x}{e^{-rT}x}\right)_{t-1} + \alpha_2 r_{t-1} + \varepsilon_t. \tag{2}$$

**Table III**

## Descriptive Statistics for the Warrants Included in the Sample

The dilution factor is the number of shares that can be purchased by warrants divided by the sum of the number of existing shares and the number of shares that can be purchased with warrants. The in-the-money factor is calculated as $(S - \sum_i e^{-r_i t_i} D_i - e^{-rT}X)/e^{-rT}X$ where $S$ = the stock price, $e^{-rT}X$ is the discounted value of the exercise price, and $\sum_i e^{-r_i t_i} D_i$ is the present value of dividends to be paid over the life of the warrant. The conversion ratio is the number of shares that can be purchased with a warrant.

| Company | CUSIP | Date In Sample | Number of Valid Observations | Mean Warrant Price $ | Mean Dilution Factor | Mean Time to Expiration (Yr) | Mean In The Money Factor (%) | Mean Stock Price $ | Mean Discounted Dividends $ | Mean Implied Standard Deviation | Exercise Price | Conversion Ratio |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| American Broadcasting | 2473511 | 761005-801215 | 308 | 18.37 | .0377 | 3.5932 | 94.55 | 35.90 | 3.83 | .323 | 24.00, 16.00[a] | 1.00, 1.50 |
| American Century Mtge | 2507811 | 711027-731109 | 500 | 3.42 | .2724 | 2.6556 | −3.55 | 23.51 | 4.66 | .331 | 23.00 | 1.00 |
| American Realty Trust | 2917712 | 720104-731203 | 414 | 1.98 | .3440 | 3.8851 | 4.25 | 10.19 | 2.30 | .356 | 9.625 | 1.00 |
| American Telephone and Telegraph | 3017712 | 710105-750331 | 1000 | 6.28 | .0537 | 2.3738 | −7.71 | 47.31 | 5.72 | .309 | 52.00 | 1.00 |
| Beneficial Std | 8178811 | 710701-740820 | 578 | 3.98 | .4110 | 2.4141 | −6.74 | 21.36 | 5.43 | .563 | 20.00 | 1.00 |
| Bluebird Inc. | 9608611 | 710118-721222 | 462 | 2.47 | .0784 | 2.3060 | −42.20 | 5.46 | 0.00 | 1.076 | 10.75 | 1.00 |
| Carrier Corp | 14446511 | 720103-750331 | 723 | 6.60 | .0445 | 3.0768 | −10.32 | 24.11 | 1.37 | .484 | 41.00, 27.33 | 1.00 |
| Chrysler Corp | 17119611 | 710701-750923 | 1068 | 8.80 | .0327 | 2.7567 | −25.86 | 23.01 | 2.47 | .856 | 34.00 | 1.00 |
| Citizens Mtge Investors Trust | 17476512 | 711027-740501 | 466 | 2.23 | .3326 | 2.3073 | −7.95 | 14.28 | 2.75 | .420 | 15.00 | 1.00 |
| Diversified Ind | 25526411 | 711108-730612 | 351 | 2.01 | .0996 | 3.9325 | −41.64 | 4.41 | 0.16 | .862 | 9.25 | 1.00 |
| Fed Mart Corp | 31308112 | 800401-800630 | 22 | 1.03 | .2215 | 1.2427 | −22.22 | 9.74 | 0.00 | .481 | 14.00 | 1.00 |
| Fibreboard | 31571111 | 710118-770331 | 1551 | 5.98 | .1854 | 4.7943 | −9.95 | 16.76 | 2.49 | .577 | 22.50 | 1.00 |
| BF Goodrich | 38238811 | 730102-801215 | 1890 | 3.79 | .0806 | 4.7936 | −18.44 | 21.60 | 4.44 | .370 | 30.00 | 1.00 |
| Gould Inc. | 38349211 | 710118-760225 | 1146 | 6.57 | .0234 | 2.9459 | −22.53 | 29.61 | 3.17 | .538 | 55.00, 36.67 | 1.00 |
| Greyhound Corp | 39802812 | 720417-810215 | 1737 | 2.71 | .1252 | 4.8606 | −34.87 | 14.81 | 4.18 | .480 | 23.50 | 1.00 |
| Gulf South Mtge | 40252311 | 730702-731214 | 112 | 1.68 | .3955 | 3.4206 | −30.17 | 17.91 | 7.02 | .443 | 20.00 | 1.00 |
| Kane Miller | 48409811 | 750721-790330 | 747 | 3.43 | .1965 | 2.4423 | 83.06 | 9.19 | 0.57 | .563 | 24.25, 19.40, 12.93 | 1.25, 1.875 |

**Table III**—*Continued*

| Company | CUSIP | Date In Sample | Number of Valid Obser-vations | Mean Warrant Price $ | Mean Dilution Factor | Mean Time to Expiration (Yr) | Mean In The Money Factor (%) | Mean Stock Price $ | Mean Discounted Dividends $ | Mean Implied Standard Deviation | Exercise Price | Conversion Ratio |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Kaufman & Broad | 48617011 | 730515-740208 | 169 | 9.72 | .0382 | .4312 | 83.06 | 19.18 | 0.06 | 1.045 | 10.84 | 1.00 |
| LCA Corp | 50180011 | 720427-740627 | 541 | 5.79 | .0845 | 3.9122 | −39.46 | 22.35 | 0.80 | .580 | 46.75 | 1.00 |
| Lerner Stores | 52676811 | 710118-730328 | 266 | 30.23 | .1753 | 10.2660 | 366.22 | 43.96 | 6.59 | .389 | 15.00 | 1.00 |
| Mattel | 57708111 | 780809-800730 | 471 | 5.40 | .2400 | 6.6562 | 230.64 | 8.43 | 1.47 | .603 | 4.00 | 1.00 |
| McCrory Corp | 57986511 | 710118-730522 | 525 | 9.01 | .0680 | 4.0215 | 35.05 | 25.60 | 4.38 | .421 | 20.00 | 1.00 |
| Molycorp | 60874411 | 730102-761231 | 929 | 9.93 | .2045 | 2.3930 | 54.54 | 20.24 | 0.50 | .674 | 15.00 | 1.00 |
| North Amer Mtge | 65703912 | 730102-750327 | 364 | 2.68 | .1391 | 5.4173 | −35.86 | 24.07 | 10.49 | .404 | 31.125 | 1.00 |
| Patrick Petroleum | 70334711 | 800118-801215 | 230 | 6.64 | .1109 | .7704 | 14.71 | 20.46 | 0.00 | .845 | 20.00, 13.33 | 1.00 |
| Plaza Realty | 72890011 | 730525-731205 | 134 | 1.28 | .5000 | 4.2297 | −31.69 | 14.30 | 4.95 | .411 | 18.50 | 1.00 |
| Republic Air | 76027411 | 730523-780630 | 1069 | 1.36 | .1773 | 3.8511 | −12.94 | 3.85 | 0.19 | .624 | 5.50 | 1.00 |
| Rossmoor | 77846611 | 720222-740925 | 593 | 2.99 | .1387 | 3.1948 | −21.73 | 7.86 | 0.00 | .686 | 12.50 | 1.00 |
| South Atlantic Financial Corp | 83636511 | 710118-750507 | 547 | 4.32 | .3029 | 4.6578 | 5.84 | 13.32 | 1.41 | .437 | 15.00 | 1.00 |
| Sutro Mtge | 86937612 | 771124-780929 | 100 | 1.20 | .2316 | 4.0394 | −37.61 | 9.92 | 0.00 | .367 | 22.00 | 1.00 |
| Telex | 87957311 | 720403-741231 | 524 | 2.66 | .0843 | 3.4160 | −38.60 | 5.41 | 0.08 | 1.048 | 11.00 | 1.00 |
| Tesoro Petroleum | 88160911 | 711027-760630 | 1178 | 15.00 | .1287 | 2.4872 | 54.42 | 27.79 | 0.00 | .699 | 27.625, 13.80 | 1.00 |
| United Brands | 90966102 | 710119-760524 | 875 | 2.73 | .4827 | 6.2293 | −63.17 | 11.51 | 0.34 | .626 | 46.00 | 1.00 |
| United Telecom | 91302511 | 710205-760930 | 1026 | 4.31 | .0389 | 2.4373 | 17.75 | 17.42 | 0.00 | .280 | 17.50, 16.93 | 1.00, 1.03 |
| Uris Buildings | 91727011 | 720502-730320 | 162 | 19.17 | .0968 | 2.5311 | 143.28 | 14.61 | 2.28 | .975 | 5.89 | 1.00 |
| Ward Foods | 93405111 | 710119-760329 | 1003 | 2.21 | .1425 | 5.4831 | −79.75 | 9.57 | 1.18 | .732 | 60.00 | 1.00 |
| Western Pacific | 95909011 | 720207-750904 | 795 | 3.36 | .2855 | 3.3183 | −33.85 | 10.70 | 0.00 | .677 | 20.50 | 1.00 |
| Williams Cos. | 96945711 | 710112-721108 | 127 | 24.72 | .2255 | 4.0462 | 146.16 | 39.30 | 0.00 | .444 | 20.00 | 1.00 |

[a] If a warrant has more than one exercise price, it is because the underlying stock split. If a warrant has contractually increasing or step-up exercise prices, it is only included in our sample after the last scheduled exercise price increase.

This regression is run separately for each warrant each quarter. In adjusting for autocorrelation, residuals of each regression are assumed to follow an AR(1) process.[7] A maximum likelihood estimation procedure is used.

The first three columns of Table IV give the mean of the coefficients across different regressions. The following two columns provide $t$-statistics for the average $\alpha_1$ and $\alpha_2$ coefficients. The table also shows the percentage of $\alpha_1$ and $\alpha_2$ estimates that are less than zero and $t$-statistics of a nonparametric test to determine if there is an equal proportion of positive and negative coefficients.

The first row in Table IV shows averages of the coefficients across all 451 regressions. $t$-statistics shown in this row are based on the assumption that the coefficients of the 451 regressions are independent observations. Even so, we cannot reject a null hypothesis that $\alpha_2$ is zero. However, the $t$-statistic of the $\alpha_1$ coefficient is highly significant.

The independence of $\alpha_1$ estimates for different warrants during the same quarter is suspect. So, for the second row of Table IV, an average of each coefficient is calculated for each quarter of the sample period. Grand means of the quarterly averages of $\alpha_0$, $\alpha_1$, and $\alpha_2$ are then calculated. $t$-statistics test whether the grand means are significantly different from zero. In effect, these $t$-statistics are based on the assumption that regressions are independent across quarters but perfectly correlated within quarters. Again, $\alpha_1$ is significantly less than zero, indicating that ISDs decrease with equity value.

Finally, average coefficients in the last row of Table IV are calculated by first averaging coefficients across quarters for the same warrant. A grand mean of the warrant average coefficients is then calculated. In this case $t$-statistics are based on the assumption that coefficients are independent across warrants but that quarterly regressions for the same warrant are perfectly correlated. Again, the grand mean of average $\alpha_1$'s is found to be significantly less than zero. Thus the main conclusion to be drawn from Table IV is that ISDs of warrants are inversely related to the value of the underlying equity.

Table V presents a summary of estimates of equation (2) broken down by quarters. Within each quarter, the median of each coefficient across regressions for different warrants is shown. Next, the median $t$-statistics (after adjustment for autocorrelation) for the $\alpha_1$ and $\alpha_2$ coefficients are presented. In succeeding columns the table shows the percentage of individual warrant regressions with negative $\alpha_1$ estimates, the percentage of regressions with negative $\alpha_2$ estimates, the percentage of $t$-statistics for $\alpha_1$ that are less than $-2$, and the percentage of $t$-statistics for $\alpha_1$ that exceed 2.

Table V documents that $\alpha_1$ is consistently less than zero over the 1971–80 period. In only two of the forty quarters, are median estimates of $\alpha_1$ greater than zero. In one of these quarters only 4 warrants trade, while only 3 are included in the sample in the other quarter. The consistent inverse relation between ISDs generated by the Black-Scholes model and underlying stock prices contrasts with the transitory biases reported by Rubinstein (12) in his study of options.

In Table V, as in Table IV, we find little evidence of a relation between model pricing errors and interest rates. Exactly half of the medium $\alpha_2$'s are negative.

[7] Higher order autoregressive parameters are significantly different from zero in fewer than 10% of the regressions.

**Table IV**

## Summary Statistics for the Coefficients of the Regression

$$\text{ISD}_t = \alpha_0 + \alpha_1 \left( \frac{S_t - \sum_i e^{-rt_i} D_i - e^{-rT} x}{e^{-rT} x} \right)_{t-1} + \alpha_2 r_{t-1} + \varepsilon_t$$

$\text{ISD}_t$ is the implied standard deviation estimated for a warrant for day $t$ using the Black-Scholes model. $S_t$ is the stock price, $\sum_i e^{-rt_i} D_i$ is the present value of dividends to be paid over the life of the warrant, $T$ is the time until the warrant's expiration, $X$ is the warrant's exercise price, and $r_{t-1}$ is the day $t-1$ return on a portfolio of Treasury Bonds with an average maturity equal to the warrant's expiration date. Simple average coefficients are calculated as the mean of the 451 regression coefficients. It is implicitly assumed that the regressions are independent when calculating $t$-statistics. Average "by quarter" regression coefficients are calculated in two steps. First, average, $\alpha_1$ and $\alpha_2$ coefficients are calculated across all warrants within a quarter. Then the average of these 40 quarterly means are calculated for $\alpha_1$ and $\alpha_2$ and presented in the table. Average "by warrant" coefficients are calculated in an analogous way. Coefficients are first calculated across quarters for each warrant, and these "by warrant" means are then averaged and presented in the table. The $t$-statistics that test whether $\alpha_1$ coefficients are equally likely to be positive or negative are calculated as $t(\%\ \alpha_1) = (\pi_1 - .5)/\sqrt{.5 \cdot .5/N}$ where $\pi_1$ is the proportion of positive $\alpha_1$ estimates and $N$ is the total number of observations. The statistic $t(\%\ \alpha_2)$ is calculated in an analogous way.

| Averaging Method | $\bar{\alpha}_0$ | $\bar{\alpha}_1$ | $\bar{\alpha}_2$ | $t(\bar{\alpha}_1)$ | $t(\bar{\alpha}_2)$ | % $\alpha_1 < 0$ | % $\alpha_2 < 0$ | $t(\%\alpha_1)$ | $t(\%\alpha_2)$ | Number of Observations |
|---|---|---|---|---|---|---|---|---|---|---|
| Simple | .55 | −.20 | −.73 | −11.23 | −1.65 | 71.0 | 52.3 | −8.92 | −.98 | 451 |
| By Quarter | .55 | −.20 | −.99 | −7.19 | −1.59 | 90.0 | 55.0 | −5.06 | −.63 | 40 |
| By Warrant | .56 | −.22 | −.69 | −6.56 | −1.23 | 82.1 | 56.4 | −4.01 | −.80 | 39 |

**Table V**

## Quarterly Summary Statistics for the Coefficients of the Regression

$$\text{ISD}_t = \alpha_0 + \alpha_1 \left( \frac{S - \sum_i e^{-rt_i} D_i - e^{-rT} x}{e^{-rT} x} \right)_{t-1} + \alpha_2 r_{t-1} + \epsilon_t$$

$\text{ISD}_t$ is the implied standard deviation estimated for a warrant for day $t$ using the Black-Scholes model. $S$ is the stock price, $\sum_i e^{-rt_i} D_i$ is the present value of dividends to be paid over the life of the warrant, $T$ is the time until the warrant's expiration, $X$ is the warrant's exercise price, and $r_{t-1}$ is the day $t - 1$ return on a portfolio of Treasury Bonds with an average maturity equal to the warrant's expiration date.

| QTR | Median $\alpha_1$ | Median $\alpha_1$ | Median $\alpha_2$ | Median $t(\alpha_1)$ | Median $t(\alpha_2)$ | % $\alpha_1 < 0$ | % $\alpha_2 < 0$ | % $t(\alpha_1) < -2$ | % $t(\alpha_1) > 2$ | Number of Warrants |
|---|---|---|---|---|---|---|---|---|---|---|
| 1/71 | .5545 | −.0668 | −3.9012 | −.43 | −2.18 | 63.6 | 63.6 | 36.4 | 9.1 | 11 |
| 2/71 | .3967 | −.0395 | 1.0560 | −.37 | .41 | 55.6 | 22.2 | 11.1 | 0.0 | 9 |
| 3/71 | .5593 | −.1192 | −1.0813 | −1.58 | −1.02 | 75.0 | 75.0 | 41.7 | 0.0 | 12 |
| 4/71 | .5634 | −.0126 | .1163 | −.12 | .02 | 52.9 | 47.1 | 29.4 | 5.9 | 17 |
| 1/72 | .7033 | −.2217 | −3.1438 | −2.10 | −.75 | 70.0 | 70.0 | 50.0 | 5.0 | 20 |
| 2/72 | .5581 | −.0412 | −1.0813 | −.52 | −.10 | 60.9 | 52.2 | 26.1 | 4.3 | 23 |
| 3/72 | .6886 | −.1335 | −3.9703 | −1.23 | −.85 | 79.2 | 66.7 | 33.3 | 0.0 | 24 |
| 4/72 | .2999 | −.0139 | 4.7662 | −.16 | .61 | 61.9 | 38.1 | 23.8 | 9.6 | 21 |
| 1/73 | .3052 | −.1828 | −.0425 | −1.74 | −.02 | 72.0 | 52.0 | 40.0 | 4.0 | 25 |
| 2/73 | .3549 | −.1538 | 2.0294 | −2.00 | .33 | 74.1 | 40.7 | 48.1 | 7.4 | 27 |
| 3/73 | .6554 | −.1152 | .0225 | −1.05 | −.05 | 66.7 | 50.0 | 33.3 | 4.2 | 24 |
| 4/73 | .3687 | −.2900 | .1493 | −4.55 | .05 | 84.0 | 48.0 | 72.0 | 0.0 | 25 |
| 1/74 | .8211 | −.1146 | −1.8109 | −.66 | −.56 | 68.2 | 68.2 | 22.7 | 4.5 | 22 |
| 2/74 | .2938 | −.2999 | 1.2553 | −1.57 | .47 | 80.0 | 30.0 | 50.0 | 0.0 | 20 |
| 3/74 | .5327 | −.2298 | .3122 | −1.65 | .10 | 75.0 | 43.8 | 43.8 | 0.0 | 16 |
| 4/74 | .9662 | −.3434 | −4.7527 | −2.15 | −.99 | 84.6 | 69.2 | 53.8 | 0.0 | 13 |
| 1/75 | .4560 | −.2977 | −1.8527 | −1.54 | −.79 | 86.7 | 66.7 | 46.7 | 0.0 | 15 |
| 2/75 | .5117 | −.2473 | .1241 | −2.57 | .10 | 90.0 | 50.0 | 70.0 | 0.0 | 10 |
| 3/75 | .2214 | −.2815 | 3.6229 | −.66 | .61 | 63.6 | 27.3 | 45.5 | 9.1 | 11 |
| 4/75 | .3895 | −.1122 | 1.0701 | −.81 | .44 | 66.7 | 33.3 | 22.2 | 0.0 | 9 |
| 1/76 | .4140 | −.0096 | .4823 | −.30 | .17 | 55.6 | 44.4 | 22.2 | 0.0 | 9 |
| 2/76 | .7418 | −.2889 | −5.3515 | −2.35 | −2.44 | 75.0 | 87.5 | 50.0 | 0.0 | 8 |
| 3/76 | .8945 | −.1429 | −3.4791 | −1.35 | −.48 | 57.1 | 57.1 | 28.6 | 0.0 | 7 |

**Table V**—*Continued*

| QTR | Median $\alpha_1$ | Median $\alpha_1$ | Median $\alpha_2$ | Median $t(\alpha_1)$ | Median $t(\alpha_2)$ | % $\alpha_1 < 0$ | % $\alpha_2 < 0$ | % $t(\alpha_1) < -2$ | % $t(\alpha_1) > 2$ | Number of Warrants |
|---|---|---|---|---|---|---|---|---|---|---|
| 4/76 | .3157 | −.2556 | 1.8380 | −4.40 | .91 | 57.1 | 28.6 | 57.1 | 0.0 | 7 |
| 1/77 | .5975 | −.1081 | −2.9182 | −.90 | −1.14 | 66.7 | 66.7 | 33.3 | 0.0 | 6 |
| 2/77 | .8811 | .2432 | −2.1003 | 1.74 | −.74 | 25.0 | 75.0 | 25.0 | 25.0 | 4 |
| 3/77 | .7900 | −.2625 | −3.8864 | −1.82 | −.97 | 100.0 | 80.0 | 40.0 | 0.0 | 5 |
| 4/77 | .9250 | −.4490 | −5.1203 | −5.14 | −.70 | 80.0 | 80.0 | 60.0 | 0.0 | 5 |
| 1/78 | .6574 | −.0915 | .5506 | −.58 | .07 | 75.0 | 25.0 | 25.0 | 25.0 | 4 |
| 2/78 | .4852 | −.2662 | −.5169 | −2.10 | −.10 | 80.0 | 60.0 | 60.0 | 0.0 | 5 |
| 3/78 | .8113 | −.0902 | −4.2593 | −1.32 | −1.01 | 75.0 | 100.0 | 25.0 | 0.0 | 4 |
| 4/78 | .4471 | −.1892 | 1.2351 | −2.05 | .70 | 100.0 | 33.3 | 66.7 | 0.0 | 3 |
| 1/79 | −.0964 | −.0015 | 6.2179 | −.02 | .73 | 66.7 | 33.3 | 0.0 | 33.3 | 3 |
| 2/79 | 1.7772 | −.0669 | −13.9809 | −.58 | −.59 | 100.0 | 50.0 | 0.0 | 0.0 | 2 |
| 3/79 | −.2643 | .0895 | 6.9261 | .68 | 1.79 | 0.0 | 33.3 | 0.0 | 33.3 | 3 |
| 4/79 | .0403 | −.2249 | 2.4008 | −1.77 | .91 | 66.7 | 0.0 | 33.3 | 33.3 | 3 |
| 1/80 | .3754 | −.0426 | 1.5926 | −.45 | 1.28 | 75.0 | 0.0 | 25.0 | 0.0 | 4 |
| 2/80 | .4702 | −.1794 | −.2302 | −2.49 | −1.01 | 66.7 | 83.3 | 66.7 | 0.0 | 6 |
| 3/80 | .5892 | −.0934 | −.9231 | −2.11 | −1.11 | 80.0 | 60.0 | 60.0 | 0.0 | 5 |
| 4/80 | .3358 | −.4757 | .2612 | −3.84 | .26 | 75.0 | 50.0 | 75.0 | 0.0 | 4 |

### *B. The Economic Significance of the Black-Scholes Model Bias*

The economic significance of the bias in the Black-Scholes model can be demonstrated by comparing model and market prices of warrants. Our finding that ISDs are inversely related to equity values is equivalent to noting that warrant prices are less sensitive to underlying equity values than the Black-Scholes model predicts.

We use AT&T warrants to examine the dollar magnitude of the biases in the Black-Scholes model. AT&T warrants are used because these warrants trade everyday. Also, they are generally close to being at-the-money and the Black-Scholes model is known to work best for at-the-money calls. The 1972 time period used here is representative of the entire available time series of AT&T warrant prices. We use a weighted average of ISDs within a quarter to compute model prices for AT&T warrants during the same quarter.[8] Because the model prices and ISDs are estimated during the same quarter, average differences between model and market prices are artificially constrained to being close to zero.

The graphs in Figure 1 demonstrate in dramatic fashion that market prices of warrants (denoted "O") are less sensitive to underlying stock prices than model prices (denoted "/"). It is common for model warrant prices to exceed market prices by 50¢ to $1 for the highest equity values during a quarter. Thus even under the ideal conditions of an in-sample estimation of ISDs, the Black-Scholes model prices are significantly different from actual warrant prices. It is also interesting that in some quarters actual AT&T warrant prices remain almost unchanged over equity value ranges of several dollars. This implies that our results will not be sensitive to minor model changes.

### *C. Evaluating Alternative Explanations for the Bias*

The negative $\alpha_1$ estimates indicate that models that posit an inverse relation between equity variance and stock prices may be useful for pricing warrants. We have also explored four alternative explanations for the observed bias.

1. Equity volatility is stochastic but uncorrelated with equity values. Wiggins (1987) shows that in this case $\alpha_1$ estimates will be less than zero for out-of-the-money warrants and greater than zero for in-the-money warrants. So, if equity

[8] The weight assigned to the ISD obtained from the warrant price on day $t$ is given by

$$\text{Weight}_t = \left|\frac{\partial W}{\partial \sigma}\right|_t \Bigg/ \sum_{t=1}^{N} \left(\frac{\partial W}{\partial \sigma}\right)_t,$$

where

$$\frac{\partial W_t}{\partial \sigma_t} = \left|\frac{N}{(N/\gamma) + M}\right| \left|\left(S + \frac{M}{N} W\right)_t N'(d1)_t \sqrt{T}\right|.$$

This weighting scheme was first used by Latane and Rendlemen (1976) as a way of ensuring that deep in or out of the money options (that are relatively insensitive to volatility) do not exert undue influence in the determination of ISDs. The same principle is at work here, although the difference in weights across warrant observations within a quarter is small relative to the difference in weights across options with different exercise prices.

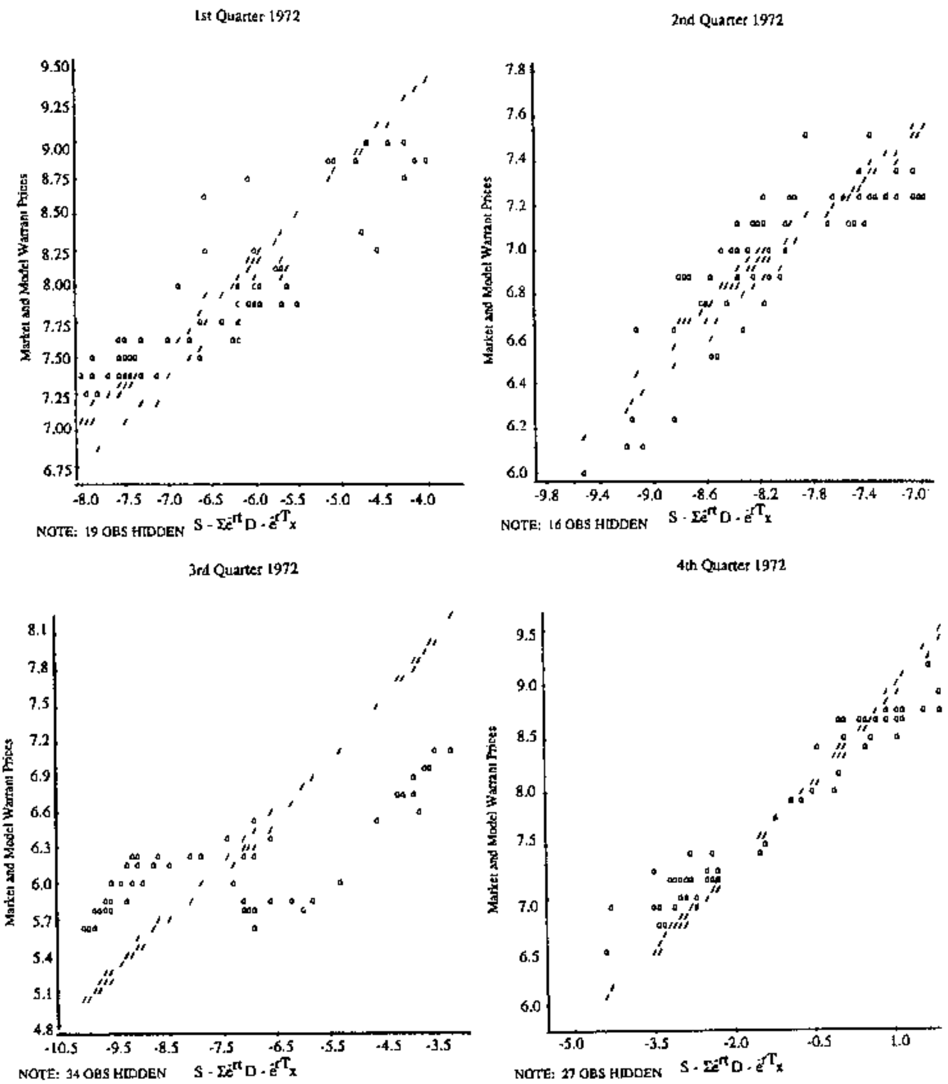


**Figure 1. Model and market prices of AT&T warrants.** Model prices are shown as "/" and market prices are given by "O". Model prices are generated by first computing an average implied standard deviation for the quarter using the Black-Scholes model. The average implied standard deviation is then inserted into the Black-Scholes model to generate model prices during the same quarter.

variances are stochastic, our results could arise from a sample of predominantly out of the money warrants. To test this, the sample of 451 regressions is split into observations of warrants that were, on average, in the money during a quarter and warrants that were, on average, out of the money during a quarter. Average $\alpha_1$ estimates for the two groups are shown in Panel A of Table VI. In contrast to the implications of stochastic equity volatility, $\alpha_1$ estimates are significantly negative for both in- and out-of-the-money warrants. However, it

**Table VI**

## The Effects of Various Factors on $\alpha_1$ Estimates from the Regression

$$\mathrm{ISD}_t = \alpha_0 + \alpha_1 \left( \frac{S - \sum_i e^{-rt_i} D_i - e^{-rT} x}{e^{-rT} x} \right)_{t-1} + \alpha_2 r_{t-1} + \epsilon_t$$

$\mathrm{ISD}_t$ is the implied standard deviation estimated for a warrant for day $t$ using the Black-Scholes model. $S$ is the stock price, $\sum_i e^{-rt_i} D_i$ is the present value of dividends to be paid over the life of the warrant, $T$ is the time until the warrant's expiration, $X$ is the warrant's exercise price, and $r_{t-1}$ is the day $t - 1$ return on a portfolio of Treasury Bonds with an average maturity equal to the warrant's expiration date. In all panels, $t(\%\ \alpha_1)$ is calculated as $(\pi_1 - .5)/(.5/N)$ where $\pi$ is the proportion of $\alpha_1$ coefficients that are positive and $N$ is the number of observations. In Panel B, expected dividends on the first day of the quarter are assumed to be a naive extrapolation of past dividends. For succeeding days in the quarter, expected dividends are assumed to be the same proportion of the stock price as the first day of the quarter.

Panel A: Comparison of $\alpha_1$ Estimates for In and Out of Money Warrants

| | In the Money $(S - \sum_i e^{-rt_i} D_i - e^{-rT} x > 0)$ | Out of the Money $(S - \sum_i e^{-rt_i} D_i - e^{-rT} x < 0)$ |
|---|---|---|
| $\bar{\alpha}_1$ | −.056 | −.268 |
| $t(\bar{\alpha}_1)$ | −2.23 | −10.84 |
| $\%\alpha_1 < 0$ | 54.8 | 74.9 |
| $t(\%\alpha_1)$ | −.96 | −9.26 |
| Number of Observations | 104 | 346 |
| $F$-test for equality of $\bar{\alpha}_1$ | 11.86 | |
| $P$-value | (.0006) | |

Panel B: Estimates of $\alpha_1$ When Expectations of Future Dividends are Assumed Proportional to Stock Prices

| | All Warrants |
|---|---|
| $\bar{\alpha}_1$ | −.165 |
| $t(\bar{\alpha}_1)$ | −7.52 |
| $t(\%\alpha_1)$ | −8.62 |
| Number of Observations | 451 |

Panel C: Comparison of $\alpha_1$ Estimates for Warrants of Profitable and Unprofitable Firms

| | Warrants of Profitable Firms | Warrants of Unprofitable Firms |
|---|---|---|
| $\bar{\alpha}_1$ | −.185 | −.338 |
| $t(\bar{\alpha}_1)$ | −10.34 | −4.58 |
| $\%\alpha_1 < 0$ | 70.6 | 73.5 |
| $t(\%\alpha_1)$ | −8.26 | −3.29 |
| Number of Observations | 402 | 49 |
| $F$-test | 7.25 | |
| $P$-value | (.0075) | |

should be noted that $\alpha_1$ estimates tend to be more negative for out-of-the-money warrants. Thus the results are at least somewhat consistent with a stochastic equity variance model.

2. Improper dividend adjustment. We project constant dollar dividends over a warrant's life. It is reasonable to assume that expected dividends increase as the stock price rises and decrease with declines in the stock price. Our assumption of constant expected dividends would thus lead to underestimated ISDs for high equity values and overestimated ISDs for low equity values. To test this we reestimate ISDs using a different dividend forecast. We again assume that investors know the dollar amount of dividends to be paid within the quarter. Projected future dividends as of the first day of the quarter are again obtained by assuming the same dividends will be paid in every future quarter. However, we now adjust dividend expectations for the other days of the quarter by assuming that a given percentage change in the stock price since the first day of a quarter is matched by an equal percentage change in expected future dividends. We discard observations of warrants that violate boundary conditions given by the new dividends. Regression results using these alternative model prices are shown in Panel B of Table VI. A comparison of these results with those shown in Table IV indicates that $\alpha_1$ estimates are slightly closer to zero with this dividend adjustment. However, they remain consistently and significantly negative.

We also compare the $\alpha_1$ coefficients of non-dividend paying firms with the $\alpha_1$ coefficients of firms that are paying dividends (not shown). The $\alpha_1$ coefficients are virtually identical for the two classes of firms. This is additional evidence that negative $\alpha_1$ coefficients are not an artifact of the dividend forecasting technique.

3. The possibility of extension is ignored. On April 24, 1972, the Internal Revenue Service issued the ruling that gave firms tax incentives to extend.[9] As Table V shows, we find $\alpha_1$ to be consistently negative for the period prior to that ruling. This indicates that extension is not sufficient to explain the inverse relation between ISDs and equity values. However, as a further indirect test of this hypothesis, we use information from Moody's Industrial Manual and Moody's Banking and Finance Manual to split the sample into regressions using warrants of firms that were profitable during a quarter and regressions using warrants of unprofitable firms. By examining warrants issued by unprofitable firms separately, we isolate a sample of warrants that may be less likely to be extended for tax reasons. However, as seen in Panel C of Table VI, $\alpha_1$ estimates are more negative for firms that are losing money.

4. Infrequent trading. It was noted earlier that non-synchronous trading could lead to spuriously negative $\alpha_1$ coefficients. To get around this problem, implied standard deviations are regressed on lagged values of the difference between stock and exercise prices and of the risk free interest rate. This adjustment could be inadequate if some warrants do not trade every day. As a crude test of this

[9] The ruling, Revenue Ruling 72-198, was thought to apply retroactively since changes in taxation are automaticially retroactive unless otherwise prescribed by the Secretary of the Treasury. The IRS was directed to reverse the automatic retroactive effect of the ruling by Treasury Secretary William Simon on January 19, 1977—his last day in office. (See Austen (1978), p. 215.)

explanation for the model bias, we note that $\alpha_1$ coefficients for AT&T warrants are negative in 14 of 17 quarters and are significantly less than zero in 9 quarters. AT&T warrants traded every day of the 17 quarters. The lowest daily trading volume during that time was 5600 warrants.

To summarize, our findings indicate that alternatives to the standard Black-Scholes warrant model that allow for early exercise, stochastic equity variance (that is uncorrelated with equity values), or stochastic interest rates do not hold the promise of superior warrant pricing abilities. Models that allow an inverse relation between equity value and equity volatility such as the CEV model, some versions of a stochastic variance model, or the Longstaff (1990) extendable warrant model are promising alternatives to the Black-Scholes model. Because we find similar biases in Black-Scholes prices before and after the IRS ruling that created tax incentives for extension, we believe that the CEV model is a particularly promising alternative.

### *D. A Comparison of Black-Scholes and CEV Model Forecasts*

In deriving his constant elasticity of variance (CEV) model, Cox (1975) assumes that the standard deviation of the stock's return when the stock's price is $S$ is given by

$$\sigma_S = \sigma_1 S^{\psi - 1}, \tag{3}$$

where

$\sigma_S$ = the standard deviation of the stock return when the stock price is $S$,
$\sigma_1$ = the standard deviation of the stock return when the price of the stock is \$1,
$S$ = stock price, and
$\psi$ = a constant between 0 and 1.

Under these circumstances, the elasticity of the stock return variance with respect to the stock price is given by the constant $2\psi - 2$. When $\psi = 1$, return variances are constant and the CEV model is the same as the Black-Scholes model.

Because the Black-Scholes model is a special case of the CEV model, the CEV model always prices securities more accurately if $\sigma_1$ and $\psi$ are known. However, the CEV model may not work well in practice when noisy estimates of the two parameters are used. So, the relevant empirical question is whether the CEV model provides sufficient improvement in price forecasts to overcome the noise associated with estimating an additional parameter.

To answer this question, we just assume that $\psi$ is ½, the midpoint of the range of potential parameter values discussed in Beckers (1980). This assumption also leads to a tractable approximation to the CEV model that lends itself well to our methodology of estimating ISDs and using the estimates to obtain future warrant prices. After applying the dilution adjustment used for the Black-Scholes warrant model to the model given in Beckers (1980), we obtain

$$W = \frac{N}{N/\gamma + M}\left[\left(S - \sum_i e^{-rt_i} D_i + \frac{M}{N} W\right) N(q(4)) - e^{-rT} x N(q(0))\right], \tag{4}$$

where, for $V = 0$ or 4,

$$q(V) = \frac{1 + h(h-1)p - h(h-1)(2-h)(1-3h)\tfrac{1}{2}p^2 - (z/(v+y))^h}{\{2h^2p(1-(1-h)(1-3h)p)\}}$$

$$h(V) = 1 - \frac{2(v+y)(v+3y)}{3(v+2y)^2}$$

$$p = \frac{v+2y}{(v+y)^2}$$

$$y = 4r\left(S - \sum_i e^{-rt_i}D_i + \frac{M}{N}W\right) \Big/ \sigma^2(1-e^{rT})$$

$$z = 4rx/\sigma^2(e^{rT}-1)$$

$N(q)$ = the cumulative normal distribution evaluated at $q$.

This model is often referred to as the square root CEV (henceforth SRCEV) model because it assumes return standard deviations are inversely related to the square root of the equity value.

To compare the Black-Scholes and SRCEV warrant pricing models we estimate ISDs for each warrant each day using both models. Daily observations are then weighted by the derivative of the warrant price with respect to the standard deviation and averaged over a quarter to get ISDs for each warrant each quarter. SRCEV and Black-Scholes model ISDs are then used to price the warrants in the subsequent quarter.

Results are presented in Table VII. The second column of the table shows the mean absolute dollar differences between Black-Scholes and market warrant prices for each quarter from the second quarter of 1971 through the fourth quarter of 1980. For example, the number .820 (the first observation of that column) indicates that during the second quarter of 1971 Black-Scholes prices differed from market prices of warrants by 82¢, on average. The corresponding number for the SRCEV model, .790, is given in the third column of the table, and the fourth column presents the difference between the average absolute errors of the Black-Scholes and SRCEV models. The value for the second quarter of 1971, .030, indicates that the SRCEV model came, on average, 3¢ closer to the market value of the warrant than the Black-Scholes model.

Examination of the first four columns of Table VII indicates that the SRCEV model is a consistently more accurate predictor of market prices than the Black-Scholes model. During the third quarter of 1971, the average SRCEV model error is 11.7¢ less than the corresponding Black-Scholes error. During the first quarter of 1976, the SRCEV model comes closer to market prices than the Black-Scholes model by 18.7¢. A simple average of the 39 quarterly mean differences is 6¢ with a $t$-statistic of 7.81 and a Wilcoxon signed rank test $Z$-statistic of 4.09.

The next three columns of Table VII provide mean absolute percentage differences between Black-Scholes model prices and market prices, mean absolute percentage differences between SRCEV model prices and market prices, and the difference between the two percentage pricing errors. Examination of the per-

## Table VII

## Quarterly Mean Absolute Values of the Pricing Errors of the Black-Scholes and Square Root Constant Elasticity of Variance Models

Mean absolute values of pricing errors for each quarter are obtained by averaging the absolute values across the pooled sample of all warrants and all days of the quarter. $W_{BS}$ is the Black-Scholes model price, $W_{MKT}$ is the market price, and $W_{SRCEV}$ is the square root constant elasticity of variance model price. Model prices are obtained by estimating implied standard deviations with the model during one quarter, and using the implied standard deviations in the model to price warrants during the following quarter. $t$-statistics are calculated as the mean of 39 quarterly means divided by 1/38 times the standard deviation of the quarterly means. To compute Wilcoxon $Z$-statistics, absolute values of the differences are ranked. $W$ is defined as the sum of the ranks of the positive differences. Under the null hypothesis that differences are equally likely to be positive or negative for all ranks,

$$E(W) = \frac{39(39+1)}{4}, \qquad \sigma_w^2 = \frac{39(39+1)(78+1)}{24}.$$

A large sample test of significance is $Z = \dfrac{W - E(W)}{\sigma_W}$.

| | | Average Absolute Value of Dollar Difference Between Model & Market Price | | | Average Absolute Value of Percentage Differences Between Model & Market Prices | | |
|---|---|---|---|---|---|---|---|
| Quarter | Number of Pooled Observations | Mean $\lvert W_{BS} - W_{MKT} \rvert$ | Mean $\lvert W_{SRCEV} - W_{MKT} \rvert$ | Difference | $\left\lvert \frac{W_{BS} - W_{MKT}}{W_{MKT}} \right\rvert$ | $\left\lvert \frac{W_{SRCEV} - W_{MKT}}{W_{MKT}} \right\rvert$ | Difference |
| 2/71 | 558 | .820 | .790 | .030 | .109 | .093 | .016 |
| 3/71 | 549 | .629 | .512 | .117 | .085 | .070 | .015 |
| 4/71 | 733 | .444 | .393 | .051 | .064 | .051 | .013 |
| 1/72 | 900 | .757 | .643 | .114 | .089 | .077 | .012 |
| 2/72 | 1110 | .557 | .487 | .070 | .098 | .070 | .028 |
| 3/72 | 1278 | .529 | .454 | .075 | .098 | .082 | .016 |
| 4/72 | 1197 | .550 | .455 | .095 | .091 | .073 | .018 |
| 1/73 | 1145 | .577 | .500 | .077 | .118 | .100 | .018 |
| 2/73 | 1332 | .715 | .568 | .147 | .193 | .148 | .045 |
| 3/73 | 1341 | .458 | .411 | .047 | .189 | .169 | .020 |
| 4/73 | 1331 | .631 | .507 | .124 | .198 | .163 | .035 |
| 1/74 | 1184 | .642 | .565 | .077 | .179 | .153 | .026 |
| 2/74 | 1155 | .466 | .444 | .022 | .190 | .162 | .028 |
| 3/74 | 839 | .508 | .443 | .065 | .225 | .187 | .038 |
| 4/74 | 690 | .560 | .426 | .134 | .247 | .179 | .068 |

**Table VII**—*Continued*

| | | Average Absolute Value of Dollar Difference Between Model & Market Price | | | Average Absolute Value of Percentage Differences Between Model & Market Prices | | |
|---|---|---|---|---|---|---|---|
| Quarter | Number of Pooled Observations | Mean $\lvert W_{BS} - W_{MKT}\rvert$ | Mean $\lvert W_{SRCEV} - W_{MKT}\rvert$ | Difference | $\left\lvert\frac{W_{BS} - W_{MKT}}{W_{MKT}}\right\rvert$ | $\left\lvert\frac{W_{SRCEV} - W_{MKT}}{W_{MKT}}\right\rvert$ | Difference |
| 1/75 | 622 | .295 | .240 | .055 | .148 | .114 | .034 |
| 2/75 | 587 | .422 | .313 | .109 | .161 | .110 | .051 |
| 3/75 | 507 | .302 | .259 | .043 | .111 | .093 | .018 |
| 4/75 | 537 | .476 | .459 | .017 | .210 | .195 | .015 |
| 1/76 | 458 | .509 | .322 | .187 | .145 | .100 | .045 |
| 2/76 | 420 | .482 | .456 | .026 | .132 | .125 | .012 |
| 3/76 | 348 | .406 | .368 | .038 | .154 | .141 | .013 |
| 4/76 | 350 | .442 | .359 | .083 | .116 | .089 | .027 |
| 1/77 | 339 | .286 | .239 | .047 | .088 | .075 | .013 |
| 2/77 | 252 | .258 | .252 | .006 | .067 | .070 | −.003 |
| 3/77 | 252 | .241 | .173 | .068 | .096 | .073 | .023 |
| 4/77 | 225 | .419 | .345 | .074 | .143 | .112 | .031 |
| 1/78 | 243 | .213 | .196 | .017 | .068 | .059 | .009 |
| 2/78 | 191 | .414 | .312 | .102 | .177 | .135 | .042 |
| 3/78 | 151 | .263 | .228 | .035 | .140 | .121 | .019 |
| 4/78 | 139 | .164 | .152 | .012 | .091 | .075 | .016 |
| 1/79 | 134 | .218 | .199 | .019 | .164 | .146 | .018 |
| 2/79 | 117 | .207 | .175 | .032 | .128 | .098 | .030 |
| 3/79 | 126 | .172 | .187 | −.015 | .059 | .061 | −.002 |
| 4/79 | 187 | .287 | .250 | .037 | .167 | .143 | .024 |
| 1/80 | 154 | .225 | .255 | −.030 | .104 | .110 | −.006 |
| 2/80 | 250 | .561 | .501 | .060 | .261 | .243 | .018 |
| 3/80 | 240 | .285 | .297 | −.012 | .088 | .074 | .014 |
| 4/80 | 183 | .286 | .281 | .005 | .064 | .065 | −.001 |
| Mean | | .430 | .370 | .060 | .135 | .113 | .022 |
| *t*-statistic | | | | (7.81) | | | (8.91) |
| Wilcoxon *Z*-statistic | | | | (4.09) | | | (5.30) |

Table VIII

## Comparison of Pricing Errors of Black Scholes and Square Root Constant Elasticity of Variance Models at Different Times During the Quarter and Across Warrants with Different Times to Expiration

$W_{BS}$ is the Black-Scholes model price for the warrant, $W_{MKT}$ is the warrant's market price, and $W_{SRCEV}$ is the square root *CEV* model price. *T*-statistics in parenthesis are computed using the standard deviation of the quarterly observations divided by the square root of the number of quarterly observations minus one. The *t*-statistics test whether the difference is significantly different from zero.

| Within Quarter Observations Pooled Across Warrants | Mean $\lvert W_{BS} - W_{MKT}\rvert - \lvert W_{SRCEV} - W_{MKT}\rvert$ | Mean $\left\lvert \frac{W_{BS} - W_{MKT}}{W_{MKT}} \right\rvert - \left\lvert \frac{W_{SRCEV} - W_{MKT}}{W_{MKT}} \right\rvert$ | Number of Observations[a] |
|---|---|---|---|
| Average Across Quarters When All Warrants are Included and All Days Within the Quarter are Used | .0599 (7.81) | .0217 (8.91) | 39 |
| Average Across Quarters When All Warrants are Included but Only Observations from the Third Month of the Quarter are Used | .0679 (6.02) | .0260 (7.11) | 39 |
| Average Across Quarters When All Warrants are Included but Only Observations from the First Month of the Quarter are Used | .0479 (7.48) | .0171 (7.52) | 39 |
| Average Across Quarters of Error Differences in the Third Month of the Quarter Minus Error Differences in the First Month of the Same Quarter | .0200 (1.83) | .0090 (2.47) | 39 |

**Table VIII**—*Continued*

| Within Quarter Observations Pooled Across Warrants | Mean $\lvert W_{BS} - W_{MKT} \rvert - \lvert W_{SRCEV} - W_{MKT} \rvert$ | Mean $\left\lvert \frac{W_{BS} - W_{MKT}}{W_{MKT}} \right\rvert - \left\lvert \frac{W_{SRCEV} - W_{MKT}}{W_{MKT}} \right\rvert$ | Number of Observations[a] |
|---|---|---|---|
| Average Across Quarters When Only Warrants with More than Four Years to Expiration are Included | .0830<br>(3.84) | .0259<br>(4.65) | 24 |
| Average Across Quarters When Only Warrants with Less than Two Years to Expiration are Included | .0349<br>(3.59) | .0244<br>(5.37) | 24 |
| Average Across Quarters of Error Differences of Warrants with More than Four Years to Expiration Minus Error Differences of Warrants with Less Than Two Years to Expiration | .0481<br>(2.10) | .0015<br>(.21) | 24 |

Of the 40 quarters in our sample, one is lost because price differences are computed using an ISD computed the previous quarter. In 15 quarters there are no observations of warrants with less than two years to expiration or no observations of warrants with more than four years to expiration.

centage differences confirms that the CEV model forecasts market prices of warrants more accurately than the Black-Scholes model.

In Table VIII, we examine some of the reasons why the SRCEV model prices warrants more accurately than the Black-Scholes model. To adjust for cross-sectional correlation, we first pool all the absolute dollar pricing errors from each model for all warrants during a quarter and all the absolute percentage pricing errors from each model for all the warrants during a quarter. We then compute average differences between the errors of the two models for each of the 39 quarters from April 1971 through December 1980. We thus have a series of 39 mean quarterly dollar pricing error differences and a series of 39 mean quarterly percentage pricing error differences. We examine the autocorrelations of these two series (and the other series in Table VIII) and do not find evidence of first or second order autocorrelation that is different from zero. So, we treat the 39 means quarterly pricing error differences in each series as independent observations.

The first row of Table VIII shows that the average of the mean quarterly dollar differences is .0599 and the average of the mean quarterly percentage differences is .0217. So, when all observations within quarters are pooled, Black-Scholes model prices are off by about 5.99¢ (or 2.17%) more than SRCEV prices.

This analysis is repeated in the second row of the table but now errors are computed using only the third month following estimation of the ISDs. These Black-Scholes model errors are 6.79¢ (or 2.6%) larger than the SRCEV errors. In the third row average mean quarterly percentage differences are computed using only the first month following ISD calculation. Here Black-Scholes errors are only 4.79¢ (or 1.71%) larger than the SRCEV errors. This demonstrates that the superiority of the SRCEV model over the Black-Scholes model becomes more pronounced as time passes from the ISD estimation. Thus the "automatic adjustment" of variances in the CEV model reduces the need for the type of ad-hoc adjustments frequently made by practitioners. This is shown explicitly in the fourth line of the table when we subtract mean pricing error differences in the first month of the quarter from mean pricing error differences in the third month of the quarter. On average, the SRCEV model beats the Black-Scholes model by 2¢ (or .9%) more during the third month of a quarter than during the first month of a quarter. The $t$-statistics of 1.83 and 2.47 indicate the deterioration of Black-Scholes prices relative to SRCEV prices is not quite significant at the 5% level in dollar price difference terms but is significant at the 1% level in percentage error terms.

Rows 5 and 6 of Table VIII show, respectively, differences between the average pricing errors of the Black-Scholes model and SRCEV model for warrants with more than four years to expiration and warrants with less than two years to expiration. The seventh row of the table shows the pricing error differences for the warrants with at least four years to expiration minus the pricing error differences of the warrants with less than two years to expiration. The value .0481 indicates that the SRCEV model outperformed the Black-Scholes model by 4.81¢ more on warrants with at least four years to expiration than it did on warrants with less than two years to expiration. The $t$-statistic of 2.10 is significant at the 5% level. Similar findings are reported for percentage price differences but the $t$-statistic is not significant. Besides indicating that the SRCEV model is particularly advantageous for pricing long-lived warrants, these results could be interpreted as indirect evidence that CEV models may be more important for pricing warrants than shorter-lived options.

## V. Summary and Conclusions

Perhaps the most common technique for pricing warrants is to use a version of the Black-Scholes model that includes an adjustment for dilution. There are a number of possible problems with using this model for warrant pricing. They include problems associated with i) early exercise prior to dividend payments, ii) the possibility of extension, iii) stochastic interest rates, iv) stochastic equity volatilities, and v) equity volatilities that are inversely related to equity values. While not ruling out other problems, our findings indicate that the constant equity variance assumption is the most serious deficiency in the Black-Scholes model.

We compare Black-Scholes model forecasts of warrant prices with forecasts from the Cox (1975) CEV model. Because it reduces to the Black-Scholes model as a special case, the CEV model cannot help but price warrants more accurately given good estimates of the elasticity. However, what is significant about our findings is that we use a particularly tractable version of the CEV model (the square root CEV model of Beckers (1980)) and find it outperforms the Black-Scholes model in predicting warrant prices. This leaves no reason for using the Black-Scholes model for pricing warrants.

Convertible securities are commonly priced by adding the value of the securities "warrant portion" to the value of an otherwise equivalent straight bond or preferred share. Our results imply that CEV models may be useful for pricing the warrant portion of convertibles. An important extension of the work here would be to examine the use of CEV models for pricing convertible securities.

## Appendix

The effects of dilution on warrant prices can be easily seen by comparing the value of a warrant with the value of an otherwise identical call. Consider two firms with identical assets. Firm A has $N$ shares of stock outstanding, no debt, and no warrants. The value of one share of stock can be expressed as $S = V/N$ where $V$ = the firm value. Now consider the value at expiration of a European call on this stock with exercise price $x$. The calls value is given by

$$\frac{V/N < x}{0} \qquad \frac{x < V/N}{V/N - x}.$$

For comparison, suppose that firm B is identical to firm A but has $N$ shares of stock, $M$ warrants that allow purchase of $\gamma$ shares each, and no debt. The warrants issued by firm B expire at the same time as the calls written on firms A's stock and also have an exercise price of $x$. Like the call option holders, the warrant owners will exercise their warrants if $V/N > x$ at expiration and let the warrants expire if $V/N \le x$. To see this, consider the case where $V/N = x$ at expiration. If warrant holders exercise, firm value increases by $\gamma Mx$ dollars and the firm will be split among $N + \gamma M$ shares of stock. Each of these shares will have a value of $(V + \gamma Mx)/(N + \gamma M)$, or given that $V/N = x$, $(Nx + \gamma Mx)/(N + \gamma M) = x$. Thus when $V/N = x$, the cost of exercising a warrant equals its exercise value and warrant holders will be indifferent between exercising their warrants and letting them expire.

When $V/N > x$, every warrant will be exercised for $\gamma$ shares worth $(V + \gamma Mx)/(N + \gamma M)$ apiece. The per share cost of exercise is $x$ and thus, at expiration, each warrant will be worth $\gamma((V + \gamma Mx)/(N + \gamma M) - x)$ dollars or, equivalently, $V/(N/\gamma + M) - (N/(N/\gamma + M))x$ dollars. To summarize, at expiration warrants issued by firm B will be worth

| $V/N \le x$ | $V/N > x$ |
|---|---|
| 0 | $[V/(N/\gamma + M)] - [(N/(N/\gamma + M))x]$ |

Note that at expiration, the warrants issued by firm B are worth $N/(N/\gamma + M)$ times as much as the calls issued for firm A regardless of the stock price. Prior to expiration then, the warrants should also have a value of $N/(N/\gamma + M)$ times the call value.

This is a general result. Any model for pricing European calls can be converted to a warrant pricing model by making use of the fact that the value of a warrant should be $N/(N/\gamma + M)$ times the value of a similar call. Notice however that the call price and warrant price are expressed in terms of the firm value. In the case of the firm without warrants, $V = NS$ or $V/N = S$. For the firm with warrants, $V = NS + MW$ or $V/N = S + (M/N)W$. So, if a model prices calls as a function of the stock price rather than the firm value, the model must be adjusted in two ways before being used to price warrants. As before, dilution implies that a warrant is worth $N/(N/\gamma + M)$ times as much as a similar call, and $S + (M/N)W$ must be substituted for $S$ when pricing warrants rather than calls.

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