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Heterogeneous Expectations, Restrictions on Short Sales, and Equilibrium Asset Prices

ROBERT JARROW*

ABSTRACT

Under heterogeneous expectations, the mean-variance model of capital market equilibrium is employed to determine the effect restricting short sales has on equilibrium asset prices. Two equivalent markets differing only with respect to short sale restrictions are compared. It is shown that, in general, risky asset prices can either rise or fall due to short sale constraints. However, under a homogeneity of beliefs for the covariance matrix of future prices, short sale constraints will only increase risky asset prices.

UNRESTRICTED SHORT SALES WITH full use of the proceeds is a crucial assumption underlying both arbitrage and equilibrium models of capital asset prices.¹ In the U.S. equity markets, this assumption is only a rough approximation. Short sales are allowed, but only when the current price is higher than the price of the previous trade (an uptick), or when the current price is unchanged from the previous trade but higher than that of the last trade at a different price (zero-plus tick). This prohibits short sales when the stock price has fallen continuously over an interval of time. In addition, the proceeds from a short sale are normally not available for use by the investor. The security broker will hold the proceeds from a short sale in escrow, paying no interest. Any subsequent price rise requires additional deposits to cover the loss. This procedure increases the costs of shortselling, inhibiting its use.

On the micro side, these restrictions can affect an investor's portfolio decision. Under homogeneous beliefs, using the capital asset pricing model (CAPM), it has been shown by Sharpe [9], Lintner [3], and Ross [7] that these restrictions are nonbinding, and the market portfolio is efficient. The traditional form of the CAPM holds. However, the assumption of homogeneous beliefs is strong, and under heterogeneous beliefs, these restrictions will affect an investor's portfolio strategy. In fact, it is known that if an investor is constrained by the short selling restrictions, then the market portfolio is in general inefficient, and the standard form of the CAPM does not go through.

On the macro side, short sale restrictions could also affect equilibrium asset prices in a systematic fashion. A recent paper by Miller [5] argues that short selling restrictions increase the price of risky assets above those which would occur in a world with no restrictions. He argues that an increase in aggregate

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¹ A review of asset pricing models can be found in Jensen [2] and arbitrage models in Merton [4], Ross [8].

demand occurs because those investors who would have short sold the stock with no restrictions must now hold zero quantities. In addition, he asserts that the larger the dispersion of opinions concerning the stock's future value, the larger will be the price increase. Only those investors who have the most optimistic expectations for the security's future value can register their opinions by purchasing the stock.

Although reasonable at first examination, a careful analysis of Miller's arguments reveals alternate considerations. The first argument supporting a rise in a particular security's price considers only those investors who are constrained by the short sale restrictions. It neglects changes in the aggregate demand of those investors for whom the constraint is nonbinding. A decline in the last group's aggregate demand could result because of binding short sale constraints on *other* substitute securities. Consequently, due to a "substitution" effect, the net change in aggregate demand is ambiguous a priori.

Miller's second argument concerning the relations between the dispersion of beliefs and equilibrium prices considers only the distribution (across individuals) for the mean vector of future asset prices. Including expectations for the covariance matrix of future asset prices could easily change the result. Clearly it is possible to have a wide dispersion of beliefs for the mean vector, but a correspondingly wide dispersion of beliefs for the covariance matrix so that on the "risk-adjusted" basis, all investors agree on the relative valuation of securities.

This paper formalizes these qualitative arguments and examines changes in equilibrium prices due to restrictions on short sales. The single period mean variance model of capital market equilibrium contained in Lintner [3] is employed and extended. The paper is divided into three sections. Section I introduces the assumptions underlying the model. Section II solves for equilibrium asset prices with and without short sales. If there is agreement about risk, Miller's intuition is supported; however different results are obtained if there is disagreement. Section III provides a conclusion.

I. Preliminaries

The specific assumptions which define the capital markets are contained in (A.1)-(A.7).

- (A.1) There are no transaction costs or taxes, and asset shares are infinitely divisible. Buyers and sellers act as price takers.
- (A.2) Asset returns are multivariate normal.
- (A.3) The riskless rate is exogeneous, and there is unrestricted borrowing and lending at the riskless rate.
- (A.4) Investors act as if they maximize the expected utility of terminal wealth. Risk aversion and nonsatiation is assumed.
- (A.5) Short sales of risky assets are not allowed.
- (A.6) Investors have heterogeneous expectations for the mean vector and the covariance matrix of next period's risky asset prices.
- (A.7) Investor's exhibit constant absolute risk aversion.

Assumptions (A.1)-(A.4) are standard in the literature. Restrictions on short

sales, the theme of this paper, is given in (A.5). Investors are assumed to have heterogeneous beliefs concerning the joint distribution for risky asset prices, (A.6). The formation of these beliefs is exogenous to the model. At the start of the model, investors enter with their beliefs already formed. Next period occurs, prices are realized, and the model ends. No reevaluation of expectations based on past forecasting errors is made. This is due to the single period nature of the model. Although the model could be expanded along the lines of Williams [10], an intertemporal model with endogeneous expectation formation, the added complexity is unnecessary for the purposes of this paper. Finally, investors are posited to exhibit constant absolute risk aversion, (A.7).² This assumption is included for tractability in the comparison of two distinct market equilibriums. It is also included in Lintner [3].

II. The Model

Let there be J risky assets, whose prices at time t are denoted $P_j(t)$ for $j = 1, \dots, J$. There are only two periods, $t = 0$ and $t = 1$. The riskless asset is the zeroth security. Its price is exogenously determined, so for ease of notation define $P_o(1) = 1$. Given equilibrium prices, the k th investor solves the following constrained portfolio selection problem

$$\max_{\{N_j^K; j=0,1,\dots,J\}} E^K U^K(W^K(1)) \tag{1}$$

subject to

$$W^K(0) = \sum_{j=1}^J N_j^K P_j(0) + N_o^K P_o(0)$$

$$W^K(1) = \sum_{j=1}^J N_j^K P_j(1) + N_o^K$$

$$N_j^K \geq 0 \quad j = 1, \dots, J$$

Each investor maximizes the expected utility of period one's wealth, $W^K(1)$. The K superscript denotes the K th investor. Each investor has a distinct utility function, $U^K(\cdot)$, and distinct beliefs reflected by the expectation operator $E^K(\cdot)$. The number of shares of asset j the investor demands is denoted N_j^K .

The investor cannot demand more shares than he can afford given his initial wealth, $W^K(0)$. His initial wealth is determined by his endowed shares, \bar{N}_j^K for $j = 0, \dots, J$ as in

$$W^K(0) = \sum_{j=1}^J \bar{N}_j^K P_j(0) + \bar{N}_o^K P_o(0) \tag{2}$$

Short selling restrictions are included by restricting the shares demanded to be non-negative.

² Pratt [6] demonstrates that this assumption is equivalent to assuming utility functions take the form

$$U(X) \sim c_1 - c_2 e^{-\alpha X} \quad \text{where } c_1 > 0, c_2 > 0 \text{ are constants.}$$

Under (A.7), Lintner [3; p. 352] shows that (1) is equivalent to solving

$$\max_{\{N_j^K, j=0, \dots, J\}} \sum_{j=1}^J N_j^K E^K P_j(1) + N_o^K - \frac{\alpha^K}{2} \sum_{i=1}^J \sum_{j=1}^J N_j^K N_i^K \sigma_{ji}^K \quad (3)$$

subject to

$$W^K(0) = \sum_{j=1}^J N_j^K P_j(0) + N_o^K P_o(0)$$

$$N_j^K \geq 0 \quad j = 1, \dots, J$$

where $[\sigma_{ji}^K]$ is the covariance matrix and α^K is the Pratt-Arrow absolute risk aversion coefficient for individual K . The Lagrangean of (3) is given by

$$L = \sum_{j=1}^J N_j^K E^K P_j(1) + N_o^K + \frac{\alpha^K}{2} \sum_{j=1}^J \sum_{i=1}^J N_j^K N_i^K \sigma_{ji}^K + \lambda_o^K (W^K(0) - \sum_{j=1}^J N_j^K P_j(0) - N_o^K P_o(0)) + \sum_{j=1}^J \lambda_j^K (N_j^K - S_j^K) \quad (4)$$

where $\lambda_j^K, j = 0, 1, \dots, J$ are non-negative Lagrangean multipliers and $S_j^K, j = 1, \dots, J$ are non-negative slack variables. As usual with slack variables, $\lambda_j^K S_j^K = 0$.

The first order conditions to (4) are

$$W^K(0) = \sum_{j=1}^J N_j^K P_j(0) + N_o^K P_o(0) \quad (5)$$

$$N_j^K \lambda_j^K = 0, \quad \lambda_j^K \geq 0, \quad N_j^K \geq 0 \quad \text{for } j = 1, \dots, J \quad (6)$$

$$E^K P_j(1) - P_j(0)/P_o(0) + \lambda_j^K = \alpha^K \sum_{i=1}^J N_i^K \sigma_{ij}^K \quad j = 1, \dots, J \quad (7)$$

Expression (5) is the wealth constraint, while (6) is the Kuhn-Tucker condition (see Cooper and Steinberg [1]). If the constraint is *nonbinding* ($N_j^K > 0$) then the shadow cost of the constraint is zero ($\lambda_j^K = 0$). The possibility exists that the optimal demand for asset j without the short selling constraint is zero ($N_j^K = 0$), and in this case the shadow cost is zero as well. If the constraint is *binding* ($N_j^K = 0$), then the shadow cost of the constraint is positive ($\lambda_j^K > 0$). This exhausts all the possible cases.

The last expression represents J equations in J unknowns (N_i^K) which can be solved

$$N_j^K = \frac{1}{\alpha^K} \sum_{i=1}^J V_{ji}^K (E^K P_i(1) - P_i(0)/P_o(0)) + \frac{1}{\alpha^K} \sum_{i=1}^J V_{ji}^K \lambda_i^K \quad j = 1, \dots, J \quad (8)$$

where

$$[V_{ji}^K] = [\sigma_{ij}^K]^{-1}$$

This is the demand for the j th risky asset by the k th individual. The first term on the right hand side of (8) represents the demand for asset j *without* short sale constraints. The second term represents the differential demand due to the short sale restrictions. Although both α^K, λ_i^K for all $i = 1, \dots, J$ are non-negative, the sign of this last term is ambiguous due to the coefficients from the inverse matrix (V_{ji}^K).

It is asserted that the sign of the differential demand could be positive, zero, or even negative. To illustrate this, let us examine the simple case where there are only 2 risky assets. In this case, (8) can be written for asset 1 as

$$N_1^K = \frac{1}{a^K} \left(\frac{\sigma_{22}^K}{D} (E^K P_1(1) - P_1(0)/P_o(0)) - \frac{\sigma_{12}^K}{D} (E^K P_2(1) - P_2(0)/P_o(0)) \right) + \frac{\sigma_2^K}{D} (\sigma_2^K \lambda_1^K - \sigma_1^K \rho_{12}^K \lambda_2^K) \quad (9)$$

where

$$D = \sigma_{22}^K \sigma_{11}^K - (\sigma_{12}^K)^2 > 0 \quad \text{and} \quad \sigma_i^K = \sqrt{\sigma_{ii}^K}$$

If asset 1 and asset 2 are substitutes ($\rho_{12}^K > 0$), and if the constraint on asset 1 is nonbinding ($N_1^K > 0$) so that the shadow cost is zero ($\lambda_1^K = 0$), then the differential demand on asset 1 is *negative* if the constraint on asset 2 is binding (so that the shadow cost is positive ($\lambda_2^K > 0$)). This is the possibility discussed in the introduction. Although not constrained on asset 1, the k th individual holds less of asset 1 with restrictions because he is constrained to hold more of a substitute security than he would desire. This is subsequently called the “substitution effect.”

Completing the k th individual’s demand analysis, the demand for the riskless asset is given by (5) using (8) for all $j = 1, \dots, J$. The next step is to aggregate (8) over all individuals to get the demand for this entire market. However, before proceeding, for simplicity we switch to matrix notation. Define the following matrices (by dropping subscripts).

$$\begin{aligned} E^K P(1) &\equiv (E^K P_1(1), \dots, E^K P_J(1))' \\ P(0)/P_o(0) &\equiv (P_1(0)/P_o(0), \dots, P_J(0)/P_o(0))' \\ \lambda^K &\equiv (\lambda_1^K, \dots, \lambda_J^K)' \\ N^K &\equiv (N_1^K, \dots, N_J^K)' \\ \Omega_{J \times J}^K &= [\sigma_{ij}^K] \end{aligned}$$

With this notation, the conservation equations for market equilibrium are (after summing over all K individuals)

$$\sum_K \bar{N}^K = \sum_K \frac{1}{a^K} (\Omega^K)^{-1} (E^K P(1) - P(0)/P_o(0)) + \sum_K \frac{1}{a^K} (\Omega^K)^{-1} \lambda^K \quad (10)$$

$$N_j^K \lambda_j^K = 0, \quad \lambda_j^K \geq 0, \quad N_j^K \geq 0 \quad j = 1, \dots, J \quad \text{all } K \quad (11)$$

There are an equal number of unknowns ($P_1(0)/P_o(0), \dots, P_J(0)/P_o(0), \lambda_j^K \forall j, \forall K$) and equations. The vector \bar{N}^K represents the endowed shares of risky assets.

Let there be two identical markets (i.e. $\bar{N}^K, a^K, \Omega^K, E^K P(1)$ for all K are the

same across markets), with the exception that one market has short sales restrictions while the other does not. Note that by (A.7), $\{a^K\}$ will not change with different equilibrium prices so this comparison is meaningful. Under these circumstances, the two terms on the right hand side of (10) can be interpreted.

The first term represents the aggregate demand in the market with no restrictions on short sales. The second term is the net change in aggregate demand due to restrictions on short sales. This term could be positive, zero, or even negative. If this term is positive for all j , then aggregate demand shifts outward, increasing relative risky asset prices above those which would clear the market with no restrictions.

Expression (10) can be solved for the vector of relative risky asset prices.

$$P(0)/P_o(0) = \left[\sum_K \frac{1}{a^K} (\Omega^K)^{-1} \right]^{-1} \left(\sum_K \frac{1}{a^K} (\Omega^K)^{-1} E^K P(1) - \sum_K \bar{N}^K \right) + \left[\sum_K \frac{1}{a^K} (\Omega^K)^{-1} \right]^{-1} \sum_K \frac{1}{a^K} (\Omega^K)^{-1} \lambda^K \quad (12)$$

The matrix $\sum_K \frac{1}{a^K} (\Omega^K)^{-1}$ is positive definite and consequently invertible.³ Analogous to the discussion following (10), the first term on the right hand side of (12) represents the relative price vector in a market with no short sale restrictions. The change in prices due to short sale restrictions is given by the second term. Due to the general form of expectations for the covariance matrix Ω^K , although a^K and λ_j^K are non-negative for all j, K ; the signs for the elements in this vector are ambiguous. In different worlds, changes in prices could be positive, zero, or negative.

Case 1: A Simple Example

To demonstrate that some relative risky asset prices could decline, consider the following simple example. Suppose a market consists of two individuals and two risky assets. Both individuals have identical risk aversion coefficients which are set equal to unity ($a^1 = a^2 = 1$). The first individual prefers asset 1, while the

$$^3 \sum_K \frac{1}{a^K} (\Omega^K)^{-1} \text{ is invertible}$$

Proof: (Ω^K) positive definite implies $(\Omega^K)^{-1} \equiv V^K$ is positive definite.

Look at $\sum_K \frac{1}{a^K} V^K$. Choose an arbitrary $(J \times 1)$ vector h , and form

$$h' \left(\sum_K \frac{1}{a^K} V^K \right) h = \sum_K \frac{1}{a^K} (h' V^K h) > 0$$

since $h' V^K h > 0$ for all K . Thus $\sum_K \frac{1}{a^K} V^K$ is positive definite and consequently invertible.

second individual prefers asset 2. They have the following beliefs and endowed shares.

$$\begin{aligned} \bar{N}^1 &= (1, 0) & \bar{N}^2 &= (0, 1) \\ E^1 P(1) &= (2, 1) & E^2 P(1) &= (1, 3) \\ \Omega^1 &= \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} & \Omega^2 &= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Both individuals have the same correlation coefficient between asset 1 and 2 ($\rho_{12} = 1/\sqrt{3}$). Individual 1 holds asset 1 at the start, and believes asset 1 to have a higher expectation and lower variance than individual 2. However, individual 2 likes asset 2 *more* than individual 1 likes asset 1 (see the mean vector).

Using expression (9) with $\lambda_1^K \equiv \lambda_2^K \equiv 0$, it can be shown that with no restrictions on short sales, equilibrium prices and demands are

$$P_1(0)/P_o(0) = 7/6, \quad P_2(0)/P_o(0) = 11/6$$

and

$$N^1 = (10/6, -5/6), \quad N^2 = (-4/6, 11/6).$$

Individual 1 shorts asset 2 to individual 2, and individual 2 shorts asset 1 to individual 1.

If we impose short sale restrictions, clearly the equilibrium demand vectors will be

$$N^1 = (1, 0), \quad N^2 = (0, 1)$$

since the individuals cannot short the undesirable asset. By the Kuhn-Tucker conditions, (11), the shadow prices are ($\lambda_1^1 = 0, \lambda_2^1 > 0$) for individual 1 and ($\lambda_1^2 > 0, \lambda_2^2 = 0$) for individual 2. Using expression (9), equilibrium prices can be shown by checking the Kuhn-Tucker conditions to be

$$P_1(0)/P_o(0) = 1, \quad P_2(0)/P_o(0) = 2.$$

The equilibrium price of asset 1 falls while the equilibrium price of asset 2 rises above those in a world of no restrictions.

This simple world gives a counter example to Miller's [5] arguments. Furthermore it emphasizes that without additional knowledge of $\{a^K, \Omega^K, \bar{N}^K, E^K P(1)\}$, one cannot say a priori whether relative risky assets rise or fall due to short sale restrictions.

Case 2: Uncorrelated Securities

A polar example is the case where all individuals agree that risky assets are pairwise uncorrelated. In this situation, the covariance matrix is diagonal but differs across individuals ($\Omega^K = \text{diag}(\sigma_{11}^K, \dots, \sigma_{JJ}^K)$). Under this assumption it can be shown that

$$P_j(0)/P_o(0) = \frac{1}{\sum_K \left(\frac{1}{a^K \sigma_{jj}^K} \right)} \left[\sum_K \left(\frac{1}{a^K \sigma_{jj}^K} \right) E^K P_j(1) - \sum_K \bar{N}_j^K \right] \\ + \frac{1}{\sum_K \left(\frac{1}{a^K \sigma_{jj}^K} \right)} \left[\sum_K \left(\frac{1}{a^K \sigma_{jj}^K} \right) \lambda_j^K \right] \quad \text{for } j = 1, \dots, J \quad (13)$$

The price of each risky asset is determined *independently* of the prices of other risky assets. Each asset is neither a substitute nor a complement to another risky asset in the investor's portfolio.

In this case, if at least one investor is constrained by no short sales, risky asset prices increase above the equilibrium prices which would occur in a world with no short sale restriction. Here the demand for each security is determined in isolation of the demand for other risky securities. Restricting short sales effectively increases aggregate demand by exchanging negative quantities demanded with zero quantities. The "substitution effect" discussed previously is eliminated.

Case 3: Williams' [10] Steady State Economy

The last restriction considered is a world where investors agree upon the covariance matrix ($\Omega^K \equiv \Omega$ for all K) and differ only upon the mean vector ($E^K P(1)$). This world can be characterized as a single steady state period in Williams' [10] intertemporal model. Williams shows under additional assumptions, which are consistent with (A.1)-(A.7), that with an initial heterogeneity of beliefs for Ω^K and $E^K P(1)$ in steady state (in a stationary economy) investors will agree upon the covariance matrix, but still disagree upon the mean vector.

In this case, risky asset prices are equal to

$$P_j(0)/P_o(0) = \frac{1}{\left(\sum_K \frac{1}{a^K} \right)} \left[\sum_K \left(\frac{1}{a^K} \right) E^K P_j(1) - \sum_{i=1}^J \sigma_{ij} \left(\sum_K \bar{N}_i^K \right) \right] \\ + \frac{1}{\left(\sum_K \frac{1}{a^K} \right)} \left[\sum_K \left(\frac{1}{a^K} \right) \lambda_j^K \right] \quad j = 1, \dots, J \quad (14)$$

Examining the sign of the second term on the right hand side of (14) we see that it is always positive, if at least one investor is constrained by short sale restrictions.

Analogous to Case 2, the "substitution effect" is eliminated in this world (no asset i appears in the second term except j). This result can be understood as follows. In this world, all investors agree upon Ω and therefore, they agree about each security's portfolio risk. In other words, they agree upon the expected return "required" to hold the asset in their portfolio. If they also agreed upon $E^K P(1)$, they would hold identical portfolios. Since they disagree upon $E^K P(1)$, they will adjust their holdings of each asset in isolation of the other assets, by determining whether it is under/over valued in comparison to the "required" return (the first term in the first bracketed expression on the right hand side of (14) only involves

j). Effectively, after adjusting for risk, the demand for each security is determined by an individual independently of the demand for the other securities.

III. Conclusion

This paper examined the influence that short sale restrictions have on relative risky asset prices. A single period mean variance model was employed. It was demonstrated that relative risky asset prices could rise or fall depending upon the underlying parameters in the economy, as long as investors disagree about the covariance matrix of next period's asset prices. If they agree upon the covariance matrix of next period's asset prices, relative risky asset prices will always rise.

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