CAPM with Behavior Adjustment under Non-perfect Information

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Abstract

In this paper, I extend the traditional CAPM theory through the way of introducing a new concept of risk-reward measurement based on view tendency adjustment under non-perfect information. The main result is that the generalized excess return can still be described in a single beta representation, except that the systematic risk is the weighted average of exposed risk and potential risk. Meanwhile Non-perfect information can induce instantaneous profit by repackaging portfolios. Empirical study indicates that the new concept can explain the equity premium puzzle in a way that people are pessimistic when there is no perfect information in the postwar US, and it also explains the momentum by the fact that view tendency is a mean reversion process.

Keywords: CAPM, non-perfect information, beta coefficient

1. Introduction

The goal of this paper is to reexamine the traditional CAPM model with an assumption of non-perfect information. People only know the possible results of an uncertainty, they do not know the exact probabilities of each state, but just have a vague assessment. When people are pessimistic, they will relatively amplify the possibilities of left tail events, and vice versa. We establish a general framework to measure the risk and reward which

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embodies the traditional measurements - variance and mean as special cases in which people are neutral. If we explain that more pessimism or optimism comes from more unknowing, neutrality means people know the true probabilities or at least they have a clear subjective probability assessment.

We test if these new measurements are coherent, that is if they satisfy the additivity, homogeneity and Risk-free condition properties. The result is that when there is no risk-resources dimensional receding, three properties are satisfied. In other words, when portfolios are repackaged, the reward remains the same, if and only if people are neutral. When pessimism exists, people who know the probabilities of each individual security can repackage portfolios to do no-risk arbitrage; When optimism exist, they can split the package to earn non-zero excess return. There is no conflict between this result and the traditional no arbitrage theory, since under this general framework of risk-reward measurement, information is another component which influences the asset pricing besides time and risk. We define the premium as the information premium under non-perfect information environment.

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We analyze CAPM theory based on new risk-reward concepts, using the adjusted utility expectation as the optimization object instead. The pricing equation is

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$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f}{\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f} = \frac{\Theta^2\tilde{\sigma}_{iM} + \Phi\sigma_{iM}}{\Theta^2\tilde{\sigma}_M^2 + \Phi\sigma_M^2}$$

Where r^f denotes the risk-free rate, μ_i is the expected return on the ith portfolio, μ_M is the expected return on market portfolio, $\frac{\sigma_i}{\sqrt{dt}}$ and $\frac{\sigma_M}{\sqrt{dt}}$ are the standard deviation of annualized continuously compounded rate of return of the ith security and market portfolio respectively , σ_{iM} is the covariance of the ith security and market portfolio, $\tilde{\sigma}_M^2$ and $\tilde{\sigma}_{iM}$ are the potential variance and covariance. The item of Θ and Φ are two adjustors which are constant when the extent of unknowing is given.

We can still explain it general CAPM model, each adjusted excess return should be proportional to beta, which is a weighted average of exposed risk and potential risk. When perfect information exists, people will be view tendency neutral, the model degenerates into the classical CAPM model.

Expanding the stochastic discount factor theory a little bit, We get a beta pricing model which can explain the equity premium puzzle in a way that people are pessimistic when there is no perfect information in postwar US. The empirical study indicates that (……………)

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Literature review

The capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965) and ICAPM of Merton, R (1973b) have long been the backbone of academic finance, but recent empirical tests have challenged the CAPM by identifying several powerful anomalies. One of the strongest and most puzzling challenges is the equity premium puzzle by Mehra and Prescott (1985). Mehra (2003) summarizes the efforts in point, including alternative assumptions about preferences, disaster states, survivorship bias, incomplete markets, and market imperfections.

Another dramatic challenge is the momentum effect documented by Jegadeesh and Titman (1993), which shows that past winners continue to outperform the past losers, while the beta estimate for the winner portfolio is even lower. Brav and Heaton (2002), Lewellen and Shanken (2002) stress that parameter uncertainty can play an important role in explaining asset pricing anomalies. Andrew Ang and Joseph Chen (2005) show that under a conditional CAPM with time-varying betas, predictable market risk premium, and stochastic systematic volatility, book-to-market effect is eliminated.

Meanwhile distributional misspecification is researched by Ching Chih Lu(2004). Alternative distributions that allow tail dependence is used to capture return patterns to help explain pricing anomalies. Still many others devote themselves to find new measurement of risk and reward such as Gerardo Jose Lemus Rodriguez (1999).

2. A general framework of risk-reward measurement

2.1 Definitions and properties

Consider a random variable X whose probability density function is $f(x)$, and $\theta \in (0,1)$ is view tendency coefficient, the greater it is, the more optimism, and vice versa.

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Define the reward of an uncertainty, which is obtained by minimizing the following objective function with respect to q,

$$E_\theta(X) \equiv \arg\min_q \left[(1-\theta)\int_{X<q} (x-q)^2 f_X(x)dx + \theta \int_{X>q} (x-q)^2 f_X(x)dx \right] \qquad \textbf{2.1-1}$$

Simultaneously we get the risk definition

$$D_\theta(X) \equiv (1-\theta)\int_{X<E_\theta(X)} (x-E_\theta(X))^2 f_X(x)dx + \theta \int_{X>E_\theta(X)} (x-E_\theta(X))^2 f_X(x)dx \qquad \textbf{2.1-2}$$

Solving the minimization problem by taking the first order derivative, we can rewrite the definition this way

$$E_\theta(X) = q \equiv \int \pi_X(\theta) x f_X(x)dx \qquad \textbf{2.2-3}$$

Where

$$\pi_X(\theta) \equiv \frac{(1-\theta)1_{X<q} + \theta 1_{X>q}}{\int [(1-\theta)1_{X<q} + \theta 1_{X>q}] f_X(x)dx} \qquad \textbf{2.2-4}$$

At first glance, definite2.2-3 with 2.2-4 is very artificial, but understanding how it forms, we can accept it as result of an evolutive process. Virtually quantile is generated from a very similar minimization problem, but using $|x-q|$ other than $(x-q)^2$ in the objective function.

From2.2-4, we know that $\pi_X(\theta) f_X(x)$ is a probability measure, and $E_\theta(X)$ is a mathematical expectation of X being adjusted by view tendency θ. Since it is an implicit function, and there is no closed form, we can get it through numerical method.

We want to test if the measurement is coherent, that is if they satisfy the three properties: additivity, homogeneity and risk free condition.

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i. Additivity: $E_\theta(X+Y)=E_\theta(X)+E_\theta(Y)$

At present we only consider a rather simpler case, variables are independent. $\pi_X(\theta)f_X(x)$ is a probability measure, and the joint probability density of (X,Y) is $\pi_X(\theta)\pi_Y(\theta)f_X(x)f_Y(y)$, additivity is satisfied.

But from another perspective, we get a conflicting result. Since X and Y are random variables, $Z=X+Y$ is also a random variable. Figure out its' distribution function and take into the original definition, we get

$$E_\theta(Z)=\int \pi_Z(\theta)Zf_Z(z)dz$$

which is not equal to the former. So, we modify the definition as follows.

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Define $\mathrm{X}=(X_1,X_2,\cdots X_n)$ as a random variable vector, $Z=\varsigma(\mathrm{X})$ is any function of X

$$E_\theta^1(Z)=q=\int \pi_Z(\theta)Zf_Z(z)dz,\quad \forall Z$$

Where

$$\pi_Z(\theta)=\frac{(1-\theta)1_{Z<q}+\theta 1_{Z>q}}{\int[(1-\theta)1_{Z<q}+\theta 1_{Z>q}]f_Z(z)dz}$$

And

$$E_\theta^n(Z)=\int zf_{\theta,X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n)dx_1dx_2\cdots dx_n$$

where $f_{\theta,X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n)$ is the joint probability density function. We can prove that

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$$E_\theta^n(\sum_{i=1}^n X_i)=\sum_{i=1}^n E_\theta^1(X_i)$$

and

$$E_\theta^1(\sum_{i=1}^n X_i)\neq\sum_{i=1}^n E_\theta^1(X_i)$$

If there is no risk-resources dimensional receding, additivity is satisfied. We define the information premium as

$$\left| E_\theta^n(\sum_{i=1}^n X_i) - E_\theta^1(\sum_{i=1}^n X_i) \right|$$

The superscript is very useful to remind us if there is any risk resources dimensional receding. Since it is confusing, we mark the power outside the bracket from now on. For example, $\left[E_\theta^n(\sum_{i=1}^n X_i)\right]^2$ stands for the square of θ-adjusted n dimensional expectation of the sum of n random variables.

We understand it this way, when portfolios are repackaged, the reward remains the same, if and only if people are neutral. When pessimism exist, people who know the probabilities of each individual security can repackage the portfolios to do no-risk arbitrage; When optimism exist, they can split the package to earn non-zero excess return. (For more detail, please see the appendix II.) There is no conflict between this result and the traditional no arbitrage theory, since under this general framework of risk-reward measurement, information is another component that influences asset pricing besides time and risk.

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Within the general framework of risk-reward measurement, we reexamine the market completeness in contingent claim market, Arrow security market and ordinary security market. Market completeness expands itself from security level to portfolio level. To make sure each elementary adopted consumption process obtainable, there should be no portfolio repackaging constraints. Getting information premium through repackaging can improve the welfare of two parties, and it is a Pareto equilibrium allocation process.

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ii. Homogeneity:

$$E_\theta^n(\alpha Z) = \int (\alpha z) f_{\theta, X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n = \alpha E_\theta^n(Z)$$

$$E_\theta^1(\alpha Z) = \int \pi_Z(\theta) \alpha z f_Z(z) dz = \alpha E_\theta^1(Z)$$

Homogeneity is satisfied.

iii. Risk-free condition:

$$E_\theta^n(r+Z) = \int (r+z) f_{\theta, X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n = r + E_\theta^n(z)$$

$$E_\theta^1(r+Z) = \int \pi_Z(\theta)(r+z) f_Z(z) dz = r + E_\theta^1(Z)$$

Risk-free condition is satisfied.

We also prove that,

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i. Additivity: $D_\theta^2(X+Y)=D_\theta^1(X)+D_\theta^1(Y)+COV_\theta^2(X,Y)$

ii. Homogeneity: $D_\theta^n(\alpha Z)=\alpha^2 D_\theta^n(Z)$

iii. Risk-free condition: $D_\theta^n(r+Z)=D_\theta^n(Z)$

We find many properties that mean and variance holds are not held by θ-adjusted ones, this is merely because the former are special cases of the latter. Fox example,

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$$Y=\rho X \Rightarrow E_\theta^1(Y)=\rho E_\theta^1(X), \quad iff \quad \rho \ge 0$$

I will discuss more details in the following part where it is needed. We need to be very careful when performing mathematical calculations, and can not directly use θ-adjusted concepts into the formula that we get through the mean and variance process.

2.2 Motivation and Explanation

CAPM is the backbone of academic finance, but recent empirical tests have challenged the CAPM by identifying several powerful anomalies. For example, the momentum effect and the famous equity premium puzzle which attract a lot of research. We introduce a new component, view tendency under non-perfect information, besides risk aversion to try to give some explanation. We set out from a simple psychological test.

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Table 2.2-1 Perfect Information Based Decision Making

X state	s1	s2	s3	s4	Sum	Y state	s1	s2	s3	s4	Sum
payoff	1	3	50	100		payoff	-1000	3	50	100	
probability	0.05	0.15	0.7	0.1	1	probability	0.00001	0.09999	0.8	0.1	1
E(X)	0.05	0.45	35	10	45.5	E(Y)	-0.01	0.29997	40	10	50.28997
D(X)	99.0125	270.9375	14.175	297.025	681.15	D(X)	11.03109	223.6118	0.067266	247.1087	481.8188

There are four states in the world. The payoff and the probabilities of two strategies X and Y under each state are shown in table2.2-1. When people know both payoff and probability, risk aversion will prefer Y strategy, since the expectation of Y is greater than X, meanwhile variance is less. Risk lover is more concerned about how much can be earned when good states happen. As for the probability of good states, Y is greater than X. Once again, Y is preferred. That is, no matter they are risk averse or risk preference, strategy Y is preferred. Let go through another situation,

Table 2.2-2 Non- perfect Information Based Decision Making

X state	s1	s2	s3	s4	Sum	Y state	s1	s2	s3	s4	Sum
payoff	1	3	50	100		payoff	-1000	3	50	100	

When people are blind to the probabilities of each state, what they see can drive them pessimistic or optimistic. When they are pessimistic, they will choose the strategy following the minmax principle. The minimum of X is 1, and Y is -1000, people select the maximum of them. That is, strategy X is preferred.

However if the minimum payoffs of each strategy are no difference, for example, zero for the price of a security, or negative infinity for the growth rate, minmax principle is not available. So many people turn to quantile to measure risk and reward, but the cost is that the expectation does not exist, and the martingale theory is not available any more. That is why we introduce a θ-adjusted measurement which embodies the traditional measurements, variance and mean, as special cases when people are neutral. When people are pessimistic, they will relatively amplify the possibilities of left tail events, and vice versa. If we explain that more pessimism or optimism comes from more unknowing, neutrality means people know the true probabilities or at least they have a clear subjective probability assessment. Figure2.2-1 gives us intuition.

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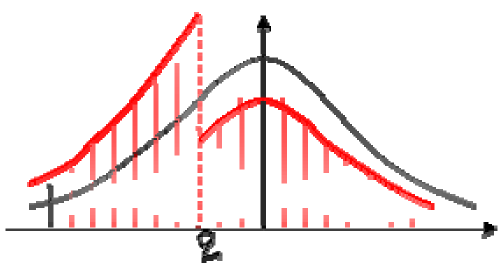
Figure2.2-1　Probability Adjustment under Non-perfect Information (Pessimistic Investor)



The relation between $E_\theta^1(x)$ and $E(x)$ is determined as follows,

$$E_\theta^1(x)\begin{cases} > E(x) & \theta > 50\% \\ = E(x) & \theta = 50\% \\ < E(x) & \theta < 50\% \end{cases} \qquad \textbf{2.2-1}$$

View tendency adjustment is an in-continuous change of measure actually, not only mean but also variance, so we can statistically separate it from risk aversion, which is another change of measure, but on mean only.

The concept of risk aversion or risk preference exists no matter information is perfect or not, it describes people's character which does not vary a lot. Pessimism or optimism is meaningful only under non-perfect information. It scales the attitude in face of unknowing, and it should vary when situation changes.

Risk aversion changes the measure from the real probabilities to the subjective probabilities. This is some kind of shift, which maintains all the probability information. Pessimism is also equivalent to paying more attention to unpleasant states, but through the way of amplifying the weight of unpleasant states, and shrinking the probability of the other states. The probability information of the latter is lost gradually as unknowing becomes more and more serious.

The total effect is the combination of those two components,

$$P(x) = E_{\theta}^{1}(mx) = \int_{\Omega} \pi(\theta) mx df(x)$$

where x is the payoff of an asset, m is the derivative or change of measure by risk aversion or preference, $\pi(\theta)$ is change of measure by view tendency.

3. CAPM Model Based on View Tendency Adjustment

3.1 θ-adjusted CAPM under constant state variables

In this section, I derive a θ-adjusted CAPM formula, assume that state variables, such as interest rate, climate, etc., which influence the floating rate or volatility of prices' diffusion processes are constant. Please see the appendix I, where a general case which state variables are also diffusion processes is derived. The main thought is still from Merton, R (1973b), but in general framework of risk-reward measurements.

Define:

$W(t) =$ Total wealth at time t

$P_i(t) =$ Price of the ith asset at time t (i=1, ⋯, n)

$C(t) =$ Consumption per unit time at time t

$w_i(t) =$ Proportion of total wealth in the ith asset at time t(i=1, ⋯, n)

Note $[\sum_{i=1}^{n} w_i(t)] \equiv 1$

Assumption1: time interval between each decision is infinitesimal.
Assumption2: prices are diffusion processes.
Assumption3: only consumption and portfolio process are controllable.
Assumption4: No exogenous endowment
Assumption5: Homogenous investors

I model the consumption and portfolio choosing process as follows,

$$J[W(t),t] \equiv \max_{\{C_{(\tau)},w_{(\tau)}\}} E^{m+n}_{(\theta,\ t)}\{\int_t^T U_1[C(\tau),\tau]d\tau + U_2[W(T),T]\} \qquad \textbf{3-1}$$

St: boundary condition: $J[W(T),T] = U_2[W(T),T]$

budget equations: $W(t) = \sum_{i=1}^{n} w_i(t_0)\frac{P_i(t)}{P_i(t_0)}[W(t_0) - C(t_0)h], \quad t \equiv t_0 + h, \quad h \to 0$ **3-2**

assumption2,4: $\frac{dP_i(t)}{P_i(t)} = \mu_i(t)dt + \sigma_i(t)\sqrt{dt}\omega_i, \quad i = 1,2,\cdots,n$

$V_{(n\times n)} = [\sigma_{il}], \quad \sigma_{il} = \sigma_i\sigma_l\rho_{il}, \quad i,l = 1,2,\cdots,n$

Using stochastic dynamic programming technique, I get the optimized relationship of θ -adjusted excess return and risk amount between an individual security and market portfolio.

$$\frac{\mu_i(t) + \frac{\sigma_i(t)}{\sqrt{dt}}E^1_{(\theta,\ t)}(\omega) - r^f}{\mu_M(t) + \frac{\sigma_M(t)}{\sqrt{dt}}E^1_{(\theta,\ t)}(\omega) - r^f} = \frac{\sigma^\theta_{iM}(t)}{\sigma^\theta_{MM}(t)} = \frac{[E^1_{(\theta,\ t)}(\omega)]^2\sigma_i(t)\sigma_M(t)\delta_{iM}(t) + [E^1_{(\theta,\ t)}(\omega^2)]\sigma_i(t)\sigma_M(t)\rho_{iM}(t)}{[E^1_{(\theta,\ t)}(\omega)]^2\sigma^2_M(t)\delta_{MM}(t) + [E^1_{(\theta,\ t)}(\omega^2)]\sigma^2_M(t)}, \quad i = 1,2,\cdots,n$$

3-3

Where

$$\sigma_i(t)\sigma_M(t)\delta_{iM}(t) = \sum_{j=1}^{n} w_j\sigma_i(t)\sigma_j(t)\sqrt{1-\rho^2_{ij}(t)} \times 1_{\rho_{ij}}; \quad \sigma_M{}^2(t)\delta_{iM}(t) = \sum_{j=1}^{n} w_j\sigma_M(t)\sigma_j(t)\sqrt{1-\rho^2_{jM}(t)} \times 1_{\rho_{jM}} \qquad \textbf{3-4}$$

and ω is any standard normal distributed random variable. Drop t, and define

$\Theta \equiv E^1_{(\theta,t)}(\omega)$; $\Phi \equiv E^1_{(\theta,t)}(\omega^2)$; $\tilde{\sigma}_{iM} \equiv \sigma_i\sigma_M\delta_{iM}$; $\tilde{\sigma}^2_M \equiv \sigma^2_M\delta_{MM}$; **3-5**

Equation 3-3 is simplified as follows,

$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f}{\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f} = \frac{\Theta^2\tilde{\sigma}_{iM} + \Phi\sigma_{iM}}{\Theta^2\tilde{\sigma}^2_M + \Phi\sigma^2_M} \equiv \beta^\theta, \quad i = 1,2,\cdots,n \qquad \textbf{3-6}$$

3.2 Model specification

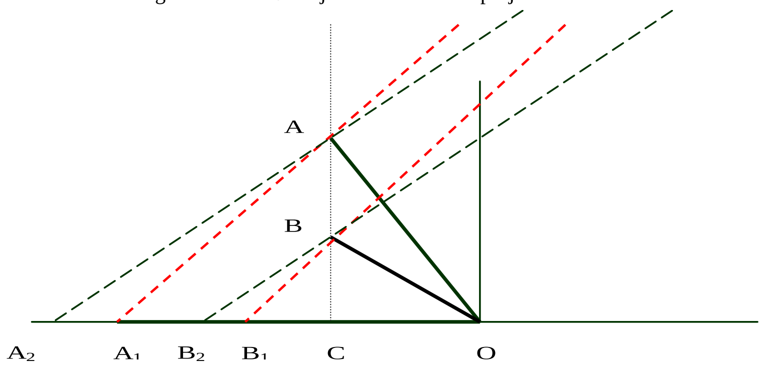
From equation3-6, expected excess return is adjusted from $\mu - r^f$ to $\mu + \frac{\sigma}{\sqrt{dt}}\Theta - r^f$, where Θ is affected by people's view tendency under non-perfect information. From Figure 2.2-1 and formula 2.2-1, if investors are pessimistic, $\Theta < 0$, they will devalue their expectation of excess return, and vise versa.

As for the risk measurement, great changes take place in this formula. If perfect information exists, we regard market variance as benchmark. The proportion of covariance σ_{iM} to the benchmark is the amount of risk. That is just a special case when $\Phi = 1$ and $\Theta = 0$, for people are neutral when perfect information exists. Please note that NEUTRAL here is different from the concept of risk neutral. It refers that people are neither pessimistic nor optimistic under perfect information.

If non-perfect information exists, $\Phi \neq 1$ and $\Theta^2 > 0$, the term $\tilde{\sigma}_{iM}$ and $\tilde{\sigma}_{MM}$ take effects in the formula. From definition 3-5 and equation 3-4, we can explain σ_{iM} as the exposed risk and $\tilde{\sigma}_{iM}$ the potential risk. The systematic risk is the weighted average of both of them. Divided by the new benchmark, systematic market risk, the general beta coefficient is obtained.

For the sake of not diluting the whole picture, also because the concepts come up during the course of CAPM proving, I put the detail in appendix II, although a deep understanding of potential risk is necessary. Here I just give an intuition explanation. Under perfect information, the risk can be divided into two parts. One is exposed to the market, we name it systematic risk. The other is exposed to some other random process regarded as the idiosyncratic risk. Although the second part has nothing to do with the market movement, the power (see figure3.2-1 AC, BC) still exists. When people's view tendency is changed, becoming pessimistic or optimistic, that kind of power will transform itself into additive risk as Θ^2 goes larger. So we consider it as the potential risk. The so-called systematic risk is the weighted average of those two parts. Figure3.2-1 is an intuitive sketch map, as matter of fact it is a simple affine change described as follows. However if θ is not a constant, if it is a mean reversion, from equation 3-6, β^{θ} is determined by more of exposed risk or potential risk now and again. It looks as if it is a mean reversion process, but virtually not. And the periodicity is just half as many.

Figure 3.2-1 θ-adjusted risk-reward projection



4. New approach to explain equity premium puzzle

In post war U.S. data, the slope of average return-beta lines is much higher than reasonable risk aversion and consumption volatility estimates suggest. Over the last 50 years, the real stock returns have averaged 9% with a standard deviation of about 16%. While the real return on treasury bills has been about 1%. Aggregate nondurable and services consumption growth had a mean and standard deviation of about 1%. These facts with

$$\left|\frac{E(R^{mv}-R^{f})}{\sigma(R^{mv})}\right| \approx a\sigma(\Delta \ln c) \qquad \textbf{4.1}$$

can only be reconciled if investors have a risk-aversion coefficient of 50. Considering the aggregate consumption has about 0.2 correlation with the market return, risk aversion needs to be 250 to explain the formula 4.1.

I try to explain it from the perspective of non-perfect information. By modifying the continuous model, I get a formula as follows,

$$\mu_P + \frac{D_t}{P_t} + \frac{\sigma_P}{\sqrt{dt}}\Theta - r^f = \alpha\sigma_C\sigma_P\Phi \qquad \textbf{4.2}$$

where Θ, Φ are two adjustors which are affected by view tendency θ. The great gap can be explained by people’s fear of unknowing in the postwar. If we don’t take the case that consumption growth and market returns are perfectly correlated, we have

$$\mu_P + \frac{D_t}{P_t} + \frac{\sigma_P}{\sqrt{dt}}\Theta - r^f = \alpha\sigma_C\sigma_P\left(\Theta^2\sqrt{1-\rho_{CP}^2} + \Phi\rho_{CP})\right) \qquad \textbf{4.3}$$

Extending the original derivation in some sort, equation4.2, 4.3 are obtained. Let a generic security have price P_t at any moment in time, and let it pay dividends at the rate $D_t dt$. In an interval dt , the security pays dividends $D_t dt$ .The instantaneous total return is,

$$\frac{dP_t}{P_t} + \frac{D_t}{P_t} dt \qquad \textbf{4.5}$$

We model the price of risky assets as diffusions, for example,

$$\frac{dP_t}{P_t} = \mu dt + \sigma \sqrt{dt} \omega \ , \qquad \textbf{4.6}$$

where μ and σ can be functions of state variables, we can think of a risk-free security as one that has a constant price equal to 1, and pays the risk-free rate as a dividend.

$$P_t = 1, D_t = r^f \qquad \textbf{4.7}$$

or as a security that pays no dividend but whose climbs deterministically at a rate

$$\frac{dP_t}{P_t} = r^f dt \qquad \textbf{4.8}$$

The utility function is

$$U(\{C_t\}) = E\left(\int_{t=0}^{\infty} e^{-\delta \times t} U(C_t) dt\right) \qquad \textbf{4.9}$$

where $U(\{C_t\})$ is the total utility of the whole period, $U(C_t)$ is the utility at t time. The future price and consumptions are random variables. People only choose buying the amount of a certain security at time t, to maximize his or her wealth of whole life. So we model the process this way.

$$\begin{aligned}
\max_{\{\xi_\tau\}} E^2_{(\theta,\ t)}\{\int_t^{\infty} e^{-\delta\tau} U(C_\tau) d\tau\} &= E^2_{(\theta,\ t)}\{\int_t^{t+h} e^{-\delta\tau} U(C_\tau) d\tau\} + E^2_{(\theta,\ t)}\{\int_{t+h}^{\infty} e^{-\delta\tau} U(C_\tau) d\tau\} \\
&= E^2_{(\theta,\ t)}\{\int_t^{t+h} e^{-\delta\tau} U(C_\tau) d\tau\} + E^2_{(\theta,\ t)}\{\int_{s=0}^{\infty} e^{-\delta\times(t+h+s)} U(C_{t+h+s}) ds\} \\
&= E^2_{(\theta,\ t)}\{e^{-\delta\times\hat{t}} U(C_{\hat{t}}) dt\} + E^2_{(\theta,\ t)}\{\int_{s=0}^{\infty} e^{-\delta\times(t+h+s)} U(C_{t+h+s}) ds\}
\end{aligned} \qquad \textbf{4.10}$$

St: $C_t = e_t - \xi_t P_t ; \quad C_{t+s+h} = e_{t+s+h} + \xi_t D_{t+s+h} h ; \quad h \to 0$

Although this is a continuous model, and it seems there are infinite periods, from the restrictions, we know it is a two period model virtually, since people only make decision at the starting point, and all the others can be regarded as future times as a whole. Take restrictions into objective, setting derivative with respect to ξ_t equal to zero,

$$F = E^2_{(\theta,\ t)}\{\int_t^{\infty} e^{-\delta\tau} U(C_\tau) d\tau\} = e^{-\delta \times t} U(C_t) dt + E^2_{(\theta,\ t)}\{\int_{s=0}^{\infty} e^{-\delta \times (t+s)} U(C_{t+s}) ds\} \quad \textbf{4.11}$$

$$\frac{\partial F}{\partial \xi_t} = e^{-\delta \times t} U'(C_t) dt \times (-P_t) + E^2_{(\theta,\ t)}\{\int_{s=0}^{\infty} [e^{-\delta \times (t+s)} U'(C_{t+s}) \times (D_{t+s} dt)] ds\} = 0 \quad \textbf{4.12}$$

We obtain the first order condition for optimal portfolio choice.

$$P_t U'(C_t) = E^2_{(\theta,\ t)}\{\int_{s=0}^{\infty} [e^{-\delta \times s} U'(C_{t+s}) D_{t+s}] ds\} \quad \textbf{4.13}$$

Define:

$$\Lambda_t \equiv e^{-\delta \times t} U'(C_t) \quad \textbf{4.14}$$

Take 4.10 into 4.10,

$$P_t \Lambda_t = E^2_{(\theta,\ t)}(\int_{s=0}^{\infty} \Lambda_{t+s} D_{t+s} ds) \quad \textbf{4.15}$$

In general, substitute $t+h$ for t, we push time forward an infinitesimal interval,

$$P_{t+h} \Lambda_{t+h} = E^2_{(\theta,\ t+h)}(\int_{s=0}^{\infty} \Lambda_{t+h+s} D_{t+h+s} ds) \quad \textbf{4.16}$$

Take the θ-adjusted expectation operator, since law of iterated expectation still available in this broader sense, we obtain

$$E^2_{(\theta,\ t)}(P_{t+h} \Lambda_{t+h}) = E^2_{(\theta,\ t)}(\int_{s=h}^{\infty} \Lambda_{t+s} D_{t+s} ds) \quad \textbf{4.17}$$

Integral as an additive function of the interval of integration, we have

$$P_t \Lambda_t = E^2_{(\theta,\ t)}(\int_{s=0}^{h} \Lambda_{t+s} D_{t+s} ds + \int_{s=h}^{\infty} \Lambda_{t+s} D_{t+s} ds) = E^2_{(\theta,\ t)}(\int_{s=0}^{h} \Lambda_{t+s} D_{t+s} ds) + E^2_{(\theta,\ t)}(P_{t+h} \Lambda_{t+h}) \quad \textbf{4.18}$$

Applying the mean value theorem for integrals,

$$P_t \Lambda_t = \Lambda_{t+\bar{h}} D_{t+\bar{h}} h + E^2_{(\theta,\ t)}(P_{t+h} \Lambda_{t+h}) \quad \textbf{4.19}$$

Introduce differences,

$$P_t \Lambda_t = \Lambda_{t+\bar{h}} D_{t+\bar{h}} h + E^2_{(\theta,\ t)}[P_t \Lambda_t + (P_{t+h} \Lambda_{t+h} - P_t \Lambda_t)] \quad \textbf{4.20}$$

And canceling $P_t \Lambda_t$ on both sides,

$$0 = \Lambda_{t+\bar{h}} D_{t+\bar{h}} h + E^2_{(\theta,\ t)}(P_{t+h} \Lambda_{t+h} - P_t \Lambda_t) \quad \textbf{4.21}$$

Taking the limit as $h \to 0$

$$0 = \Lambda_t D_t dt + E^2_{(\theta,\ t)}[d(P_t \Lambda_t)] \quad \textbf{4.22}$$

Break up the $d(P_t \Lambda_t)$ using Ito's lemma,

$$0 = \Lambda_t D_t dt + E^2_{(\theta,\ t)}(P_t d\Lambda_t + \Lambda_t dP_t + dP_t d\Lambda_t) \quad \textbf{4.23}$$

Dividing by $P_t \Lambda_t$

$$0 = \frac{D_t}{P_t} dt + E^2_{(\theta,\ t)}[\frac{d\Lambda_t}{\Lambda_t} + \frac{dP_t}{P_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}] \quad \textbf{4.24}$$

Taking $\frac{dP_t}{P_t} = r^f dt$ into equation 4.21, we obtain,

$$0 = E^1_{(\theta,\ t)}[\frac{d\Lambda_t}{\Lambda_t} + r^f dt + \frac{d\Lambda_t}{\Lambda_t} r^f dt] \quad \textbf{4.25}$$

$$r^f dt = -E^1_{(\theta,\ t)}[\frac{d\Lambda_t}{\Lambda_t}] \qquad \textbf{4.26}$$

Taking equation 4.23 into 4.21, 4.21 can be rearrange as

$$E^1_{(\theta,\ t)}[\frac{dP_t}{P_t}] + \frac{D_t}{P_t}dt = r^f dt - E^2_{(\theta,\ t)}[\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}] \qquad \textbf{4.27}$$

With definition $\Lambda_t \equiv e^{-\delta\times t}U'(C_t)$, we take Taylor expansion,

$$d\Lambda_t = -\delta e^{-\delta\times t}U'(C_t)dt + \delta e^{-\delta\times t}U''(C_t)dC_t + \frac{1}{2}e^{-\delta\times t}U'''(C_t)dC_t^{\ 2} \qquad \textbf{4.28}$$

$$\frac{d\Lambda_t}{\Lambda_t} = -\delta dt + \frac{C_tU''(C_t)}{U'(C_t)}\frac{dC_t}{C_t} + \frac{1}{2}\frac{C_t^{\ 2}U'''(C_t)}{U'(C_t)}\frac{dC_t^{\ 2}}{C_t^{\ 2}} \qquad \textbf{4.29}$$

Define：

$$\alpha_t \equiv -\frac{C_tU''(C_t)}{U'(C_t)} \text{ 为常数} \alpha\text{，} \qquad \upsilon_t \equiv \frac{C_t^{\ 2}U'''(C_t)}{U'(C_t)} \qquad \textbf{4.30}$$

Taking 4.27 into 4.26, then taking the result into 4.24. When α_t is constant, we drop the subscript, and obtain

$$E^1_{(\theta,\ t)}[\frac{dP_t}{P_t}] + \frac{D_t}{P_t}dt = r^f dt - E^2_{(\theta,\ t)}[(-\delta dt - \alpha\frac{dC_t}{C_t} + \frac{1}{2}\upsilon_t\frac{dC_t^{\ 2}}{C_t^{\ 2}})\times\frac{dP_t}{P_t}]$$

$$E^1_{(\theta,\ t)}[\frac{dP_t}{P_t}] + \frac{D_t}{P_t}dt - r^f dt = \alpha E^2_{(\theta,\ t)}[\frac{dC_t}{C_t}\frac{dP_t}{P_t}] \qquad \textbf{4.31}$$

Taking the diffusions processes into right side of above equation,

$$E^2_{(\theta,\ t)}[\frac{dC_t}{C_t}\frac{dP_t}{P_t}] = E^2_{(\theta,\ t,\rho_{CP})}[(\mu_C dt + \sigma_C\sqrt{dt}\varpi_C)\times(\mu_P dt + \sigma_P\sqrt{dt}\varpi_P)] \qquad \textbf{4.32}$$

where ϖ_C, ϖ_P are standard normal distributed, their correlation is ρ_{CP}.
Omitting the high order derivatives, we obtain

$$\begin{aligned}E^2_{(\theta,\ t)}[\frac{dC_t}{C_t}\frac{dP_t}{P_t}] &= E^2_{(\theta,\ t,\rho_{CP})}[(\sigma_C\sqrt{dt}\varpi_C)\times(\sigma_P\sqrt{dt}\varpi_P)] = \sigma_C\sigma_P dt E^2_{(\theta,\ t,\rho_{CP})}[\varpi_C\varpi_P] \\ &= \sigma_C\sigma_P dt\left([E^1_{(\theta,\ t)}(\varpi)]^2\sqrt{1-\rho^2_{CP}} + E^1_{(\theta,\ t)}(\varpi^2)\rho_{CP})\right)\end{aligned} \qquad \textbf{4.33}$$

Taking 4.31 into 4.29,

$$E^1_{(\theta,\ t)}[\frac{dP_t}{P_t}] + \frac{D_t}{P_t}dt - r^f dt = \alpha\sigma_C\sigma_P dt\left(\Theta^2\sqrt{1-\rho^2_{CP}}\times 1_{\rho_{CP}} + \Phi\rho_{CP})\right) \qquad \textbf{4.34}$$

Canceling dt on both sides,

$$\mu_P + \frac{D_t}{P_t} + \frac{\sigma_P}{\sqrt{dt}}\Theta - r^f = \alpha\sigma_C\sigma_P\left(\Theta^2\sqrt{1-\rho^2_{CP}} + \Phi\rho_{CP})\right) \qquad \textbf{4.35}$$

If the consumption and price are completely correlated, that is, $\rho_{CP} = 1$

$$\mu_P + \frac{D_t}{P_t} + \frac{\sigma_P}{\sqrt{dt}}\Theta - r^f = \alpha\sigma_C\sigma_P\Phi \qquad \textbf{4.36}$$

5. Empirical study (To be done.)

(1). Make tables of Θ and Φ, or get a formula to approximate the relations between those two and θ. The latter is preferred if possible, since without it, there would be an action of checking number during econometrics programming. θ's sensitivity analysis is also important. (If it is excessively sensitive, it would be of no use?)

(2).Solve equation

$$\mu_P + \frac{D_t}{P_t} + \frac{\sigma_P}{\sqrt{dt}}\Theta - r^f = \alpha\sigma_C\sigma_P\left(\Theta^2\sqrt{1-\rho_{CP}^2} + \Phi\rho_{CP})\right) \qquad \textbf{5-1}$$

and get people's view tendency. Taking it into θ-adjusted CAPM

$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f}{\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f} = \frac{\Theta^2\tilde{\sigma}_{iM} + \Phi\sigma_{iM}}{\Theta^2\tilde{\sigma}_M^2 + \Phi\sigma_M^2} \equiv \beta^\theta, \quad i = 1,2,\cdots,n \qquad \textbf{5-2}$$

to obtain β^θ. Is β^θ a constant? Does β^θ move in a way analogous to mean reversion? Further more, is θ a mean reversion? Can we estimate θ as a mean reversion? How to test its' consistency? May be some theoretical work can be done in econometrics field.

(3).We can also set out from θ-adjusted CAPM to estimate people's view tendency, and take it into equation 5-1 to test if equity premium puzzle is explained in a sense. But from the definition of $\tilde{\sigma}_{iM}, \tilde{\sigma}_M^2$, we know that each security information is needed, that make the approach infeasible. Stock indices by industry can be used as an approximation.

6. Conclusion

After view tendency adjustment, excess return can still be described in a single beta representation, except that the systematic risk is the weighted average of exposed risk and potential risk. Empirical study indicates that in the postwar US, because of people's fear of unknowing, there exits a great discrepancy between the predicted return and what consumption suggests. The momentum can be explained by the fact that view tendency is a mean reversion process.

Appendix I:

Define:

$W(t) =$ Total wealth at time t

$P_i(t) =$ Price of the i[th] asset at time t (i=1, ⋯, n)

$S_j(t) =$ Value of the j[th] state variable at time t , (j=1, ⋯, m)

$C(t) =$ Consumption per unit time at time t

$w_i(t) =$ Proportion of total wealth in the i[th] asset at time t (i=1, ⋯, n)

Note $[\sum_{i=1}^{n} w_i(t)] \equiv 1$

Assumption1: the time interval between each decision is infinitesimal.
Assumption2: price and state variables are diffusion processes.
Assumption3: only consumption and portfolio process are controllable.
Assumption4: No exogenous endowment
Assumption5: Homogenous investors

We model the consumption and portfolio choosing process as follows,

$$J[W(t), S(t), t] \equiv \max_{\{C_{(\tau)}, w_{(\tau)}\}} E_{(\theta,\ t)}^{m+n} \{\int_t^T U_1[C(\tau), \tau] d\tau + U_2[W(T), T]\} \qquad \textbf{AI-1}$$

St: boundary condition: $J[W(T), S(T), T] = U_2[W(T), T]$

budget equations: $W(t) = \sum_{i=1}^{n} w_i(t_0) \frac{P_i(t)}{P_i(t_0)} [W(t_0) - C(t_0)h], \quad t \equiv t_0 + h, \quad h \to 0$ **AI-2**

assumption2,4:

$$\frac{dP_i(t)}{P_i(t)} = \mu_i(S,t)dt + \sigma_i(S,t)\sqrt{dt}\omega_i, \qquad i = 1,2,\cdots,n$$

$$V_{(n\times n)} = [\sigma_{il}], \qquad \sigma_{il} = \sigma_i \sigma_l \rho_{il}, \qquad i,l = 1,2,\cdots,n$$

$$dS_j(t) = f_j(S,t)dt + g_j(S,t)\sqrt{dt}q_j, \qquad j = 1,2,\cdots,m$$

$$\Omega_{(m\times m)} = [g_j g_k \eta_{jk}], \qquad j,k = 1,2,\cdots,m$$

$$\Gamma_{(n\times m)} = [\varepsilon_{ij}], \qquad \varepsilon_{ij} = \sigma_i g_j \pi_{ij}, \qquad i = 1,2,\cdots,n; \qquad j = 1,2,\cdots,m$$

From definition 3-1,

$$J[W(t_0), S(t_0), t_0] = \max_{\{C_{(\tau)}, w_{(\tau)}\}} E_{(\theta,\ t_0)}^{m+n} \{\int_{t_0}^{t} U_1[C(\tau), \tau] d\tau + J[W(t), S(t), t]\} \qquad \textbf{AI-3}$$

Define $t \equiv t_0 + h$, by Taylor's theorem and the mean value theorem for integrals, 3-3 cane be rewritten as,

$$J[W(t_0),S(t_0),t_0] = \max_{\{C_{(\tau)},w_{(\tau)}\}} E^{m+n}_{(\theta,\ t_0)}\{U_1[C(\bar{t}),\bar{t}]h + J[W(t_0),S(t_0),t_0] + \frac{\partial J[W(t_0),S(t_0),t_0]}{\partial t}h$$
$$+ \frac{\partial J[W(t_0),S(t_0),t_0]}{\partial W}[W(t)-W(t_0)] + \sum_{j=1}^{m}\frac{\partial J[W(t_0),S(t_0),t_0]}{\partial S_j}[S_j(t)-S_j(t_0)]$$
$$+\frac{1}{2}\frac{\partial^2 J[W(t_0),S(t_0),t_0]}{\partial W^2}[W(t)-W(t_0)]^2 + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 J[W(t_0),S(t_0),t_0]}{\partial S_k S_j}[S_k(t)-S_k(t_0)][S_j(t)-S_j(t_0)]$$
$$+\sum_{j=1}^{m}\frac{\partial J[W(t_0),S(t_0),t_0]}{\partial W \partial S_j}[W(t)-W(t_0)][S_j(t)-S_j(t_0)] + O(h^2)\}$$

AI-4

where $\bar{t} \in [t_0, t]$, take limit as $h \to 0$, take the θ-adjusted expectation operators onto each term, and subtracting $J[W(t_0), S(t_0), t_0]$ of both sides,

$$0 = \max_{\{C_{(t)},w_{(t)}\}} \{U_1[C(t),t]dt + \frac{\partial J[W(t),S(t),t]}{\partial t}dt$$
$$+ \frac{\partial J[W(t),S(t),t]}{\partial W}E^n_{(\theta,\ t)}[dW(t)] + \sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial S_j(t)}E^1_{(\theta,\ t)}[dS_j(t)]$$
$$+\frac{1}{2}\frac{\partial^2 J[W(t),S(t),t]}{\partial W^2}E^n_{(\theta,\ t)}[dW(t)]^2 + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 J[W(t),S(t),t]}{\partial S_k \partial S_j}E^2_{(\theta,\ t,\eta_{jk})}[dS_k(t)dS_j(t)]$$
$$+\sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial W \partial S_j}E^{n+1}_{(\theta,\ t,\Gamma_j)}[dW(t)dS_j(t)] + O(dt^2)\} \qquad \Gamma_j \textit{ is the } j^{th} \textit{ row of } \Gamma \textit{ matrix}$$

AI-5

Now I specify how to get each θ-adjusted expectation term. By subtracting $W(t_0)$ on both sides, the budget equation is rewritten as,

$$W(t) - W(t_0) = [\sum_{i=1}^{n} w_i(t_0)\frac{P_i(t) - P_i(t_0)}{P_i(t_0)}][W(t_0) - C(t_0)h] - C(t_0)h \qquad \textbf{AI-6}$$

The θ-adjusted expectation of the limit process as $h \to 0$ is

$$E^n_{(\theta,\ t)}[dW(t)] = \{\sum_{i=1}^{n} w_i(t)W(t)E^1_{(\theta,\ t)}[\frac{dP_i(t)}{P_i(t)}] - C(t)dt\} + O(dt^2)$$
$$= \left\{\sum_{i=1}^{n}\left(w_i(t)W(t)[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}}E^1_{(\theta,\ t)}(\omega_i)]\right) - C(t)\right\}dt$$

AI-7

Applying the same limit process to other terms,

$$[W(t)-W(t_0)]^2 = \{[\sum_{i=1}^{n} w_i(t_0)\frac{P_i(t)-P_i(t_0)}{P_i(t_0)}][W(t_0)-C(t_0)h]-C(t_0)h\}^2$$

$$= \{\sum_{l=1}^{n}\sum_{i=1}^{n} w_i(t_0)w_l(t_0)\frac{P_i(t)-P_i(t_0)}{P_i(t_0)}\frac{P_l(t)-P_l(t_0)}{P_l(t_0)}(W(t_0)^2-2W(t_0)C(t_0)h+C(t_0)^2h^2)$$

$$-2\sum_{i=1}^{n} w_i(t_0)\frac{P_i(t)-P_i(t_0)}{P_i(t_0)}[W(t_0)-C(t_0)h]C(t_0)h+C(t_0)^2h^2\}$$

$$= \sum_{l=1}^{n}\sum_{i=1}^{n} w_i(t_0)w_l(t_0)(\mu_i(S,t)\mu_j(S,t)h^2+\sigma_i(S,t)\sigma_j(S,t)h\omega_i\omega_j+\mu_i(S,t)\sigma_j(S,t)h\sqrt{h}\omega_j$$

$$+\mu_j(S,t)\sigma_i(S,t)h\sqrt{h}\omega_i)(W(t_0)^2-2W(t_0)C(t_0)h+C(t_0)^2h^2)$$

$$-2\sum_{i=1}^{n} w_i(t_0)[\mu_i(S,t)h+\sigma_i(S,t)\sqrt{h}\omega_i][W(t_0)C(t_0)h-C(t_0)^2h^2]$$

$$= \sum_{l=1}^{n}\sum_{i=1}^{n}\{w_i(t_0)w_l(t_0)W(t_0)^2[\sigma_i(S,t)\sigma_j(S,t)h\omega_i\omega_j]\}+O(h^2)$$

$$E^n_{(\theta,\ t)}[dW(t)]^2 = \sum_{l=1}^{n}\sum_{i=1}^{n}\{w_i(t_0)w_l(t_0)W(t_0)^2\sigma_i(S,t)\sigma_l(S,t)E^2_{(\theta,\ t,\rho_{il})}[\omega_i\omega_l]\}dt \qquad \textbf{AI-8}$$

$$E^1_{(\theta,\ t)}[dS_j(t)] = f_j(S,t)dt \qquad \textbf{AI-9}$$

$$E^2_{(\theta,\ t)}[dS_k(t)dS_j(t)] = E^2_{(\theta,\ t)}\{[f_k(S,t)dt+g_k(S,t)\sqrt{dt}q_k(t)][f_j(S,t)dt+g_j(S,t)\sqrt{dt}q_j(t)]\}$$
$$= g_k(S,t)g_j(S,t)E^2_{(\theta,\ t,\eta_{jk})}[q_k(t)q_j(t)]dt+O(dt^2) \qquad \textbf{AI-10}$$

$$E^{n+1}_{(\theta,\ t)}[dW(t)dS_j(t)] = E^{n+1}_{(\theta,\ t)}\left(\{\sum_{i=1}^{n} w_i(t)W(t)[\mu_i(S,t)dt+\sigma_i(S,t)\sqrt{dt}\omega_i]-C(t_0)dt\}[f_j(S,t)dt+g_j(S,t)\sqrt{dt}q_j]\right)$$
$$= [\sum_{i=1}^{n} w_i(t)W(t)\sigma_i(S,t)g_j(S,t)E^2_{(\theta,\ t,\pi_{ij})}(\omega_i q_j)]dt \qquad \textbf{AI-11}$$

Take formula 3-7 to 3-11 into equation3-5, we get the following HJB function:

$$0 = \max_{\{C_{(t)},w_{(t)}\}}\{U_1[C(t),t]+\frac{\partial J[W(t),S(t),t]}{\partial t}$$
$$+\frac{\partial J[W(t),S(t),t]}{\partial W}\left\{\sum_{i=1}^{n}\left(w_i(t)W(t)[\mu_i(S,t)+\frac{\sigma_i(S,t)}{\sqrt{dt}}E^1_{(\theta,\ t)}(\omega_i)]\right)-C(t)\right\}+\sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial S_j(t)}f_j(S,t)$$
$$+\frac{1}{2}\frac{\partial^2 J[W(t),S(t),t]}{\partial W^2}\sum_{l=1}^{n}\sum_{i=1}^{n}\{w_i(t)w_l(t)W(t)^2\sigma_i(S,t)\sigma_l(S,t)E^2_{(\theta,\ t,\rho_{il})}[\omega_i\omega_l]\}$$
$$+\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 J[W(t),S(t),t]}{\partial S_k\partial S_j}g_k(S,t)g_j(S,t)E^2_{(\theta,\ t,\eta_{jk})}[q_k(t)q_j(t)]$$
$$+\sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial W\partial S_j}[\sum_{i=1}^{n} w_i(t)W(t)\sigma_i(S,t)g_j(S,t)E^2_{(\theta,\ t,\pi_{ij})}(\omega_i q_j)]\} \qquad \textbf{AI-12}$$

Suppose n^th asset is risk free asset, HJB is

$$0 = \max_{\{C_{(t)}, w_{(t)}\}} \{U_1[C(t), t] + \frac{\partial J[W(t), S(t), t]}{\partial t}$$
$$+ \frac{\partial J[W(t), S(t), t]}{\partial W}\left\{\left[\left(\sum_{i=1}^{n-1}\left(w_i(t)[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega_i)] - r^f\right) + r^f\right)W(t)\right] - C(t)\right\}$$
$$+ \sum_{j=1}^{m} \frac{\partial J[W(t), S(t), t]}{\partial S_j(t)} f_j(S,t)$$
$$+ \frac{1}{2}\frac{\partial^2 J[W(t), S(t), t]}{\partial W^2}\sum_{l=1}^{n-1}\sum_{i=1}^{n-1}\{w_i(t)w_l(t)W(t)^2\sigma_i(S,t)\sigma_l(S,t)E^2_{(\theta,\ t)}[\omega_i\omega_l]\}$$
$$+ \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 J[W(t), S(t), t]}{\partial S_k \partial S_j} g_k(S,t) g_j(S,t) E^2_{(\theta,\ t)}[q_k(t) q_j(t)]$$
$$+ \sum_{j=1}^{m}\frac{\partial J[W(t), S(t), t]}{\partial W \partial S_j}[\sum_{i=1}^{n-1} w_i(t) W(t)\sigma_i(S,t) g_j(S,t) E^2_{(\theta,\ t)}(\omega_i q_j)]\}$$

AI-13

Let the derivatives of HJB on consumption C(t) and proportion invested in the risky assets, $w_1(t)$ to $w_{n-1}(t)$, equal to zero, the first order conditions are,

$$U_{1,C}[C^*(t),t] - \frac{\partial J[W(t),S(t),t]}{\partial W} = 0$$ **AI-14**

$$\frac{\partial J[W(t),S(t),t]}{\partial W}[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega_i) - r^f] + \frac{\partial^2 J[W(t),S(t),t]}{\partial W^2}\sum_{l=1}^{n-1}\{w_l^*(t)W(t)\sigma_i(S,t)\sigma_l(S,t)E^2_{(\theta,\ t)}[\omega_i\omega_l]\}$$
$$+\sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial W \partial S_j}[\sigma_i(S,t) g_j(S,t) E^2_{(\theta,\ t)}(\omega_i q_j)] = 0, \qquad i = 1,2,\cdots,n-1$$

AI-15

Define a more compact expression in the following way,

$$V^{\theta}{}_{(n-1)\times(n-1)} = [\sigma^{\theta}{}_{il}], \quad \sigma^{\theta}{}_{il} = \sigma_i\sigma_l E^2_{(\theta,\ t)}(\omega_i\omega_l), \quad i,l = 1,2,\cdots,n-1$$
$$\Gamma^{\theta}{}_{(n-1)\times m} = [\varepsilon^{\theta}{}_{ij}], \quad \varepsilon^{\theta}{}_{ij} = \sigma_i g_j E^2_{(\theta,\ t)}(\omega_i q_j), \quad i = 1,2,\cdots,n-1; \quad j = 1,2,\cdots,m$$

We need to be very careful to differentiate θ-adjusted mean of the product of two standard normal distributed random variables and θ-adjusted covariance of those two, since θ-adjusted mean is not zero any more. Maybe the symbol σ is a little bit confusing, but they are definitely not θ-adjusted variance-covariance matrix here! Then, we can get the optimized portfolio process.

$$w^* = -\frac{J_W[W(t),S(t),t]}{W(t)J_{WW}[W(t),S(t),t]}(V^{\theta}{}_{(n-1)\times(n-1)})^{-1}[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega_i) - r^f]$$
$$-(V^{\theta}{}_{(n-1)\times(n-1)})^{-1}\Gamma^{\theta}{}_{(n-1)\times m}\frac{J_{SW}[W(t),S(t),t]}{W(t)J_{WW}[W(t),S(t),t]}$$

AI-16

Leave out the subscript i, write $\mu(S,t)$ and $\sigma(S,t)$ in form of vectors, and sum K homogeneity investors' portfolio proportions, we get the market portfolio proportion.

$$w_M = \frac{\sum_{k=1}^{K} w^k W^k}{\sum_{k=1}^{K} W^k} = \frac{A}{M}(V^{\theta}{}_{(n-1)\times(n-1)})^{-1}[\mu(S,t) + \frac{\sigma(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega) - \gamma^f] + (V^{\theta}{}_{(n-1)\times(n-1)})^{-1}\Gamma^{\theta}{}_{(n-1)\times m}\frac{B}{M}$$

AI-17

where $A = \sum_{k=1}^{K}(-\frac{J^k_W[W(t),S(t),t]}{J^k_{WW}[W(t),S(t),t]})$; $B = \sum_{k=1}^{K}(-\frac{J^k_{SW}[W(t),S(t),t]}{J^k_{WW}[W(t),S(t),t]})$; $M = \sum_{k=1}^{K} W^k$

The θ-adjusted excess return vector satisfies the following equation,

$$[\mu(S,t) + \frac{\sigma(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega) - \gamma^f] = w_M{}^T V^{\theta}_{(n-1)\times(n-1)}\frac{M}{A} - \Gamma^{\theta}{}_{(n-1)\times m}\frac{B}{M}$$ **AI-18**

$$w_M{}^T V^{\theta}_{(n-1)\times(n-1)} = \left(\sigma^{\theta}_{iM}\right)^T \qquad i = 1,2,\cdots \quad n-1$$

Write in scalar,

$$[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega_i) - r^f] = \frac{M}{A}\sigma^{\theta}_{iM}, \qquad i = 1,2, \quad n-1$$ **AI-19**

We specify the term of σ^{θ}_{iM}, and get

$$\begin{aligned}\sigma^{\theta}_{iM} &= \sum_{j=1}^{n} w_j \sigma^{\theta}_{ij} = \sum_{j=1}^{n} w_j \sigma_i \sigma_j E^2_{(\theta,\ t,\rho_{ij}>0)}(\omega_i \omega_j) \\ &= \sum_{j=1}^{n} w_j \sigma_i \sigma_j \{[E^1_{(\theta,\ t)}(\omega)]^2 \sqrt{1-\rho_{ij}^2} \times 1_{\rho_{ij>0}} + [E^1_{(\theta,\ t)}(\omega^2)]\rho_{ij}\} \\ &= [E^1_{(\theta,\ t)}(\omega)]^2 \sum_{j=1}^{n} w_j \sigma_i \sigma_j \sqrt{1-\rho_{ij}^2} \times 1_{\rho_{ij>0}} + [E^1_{(\theta,\ t)}(\omega^2)]\sum_{j=1}^{n} w_j \sigma_i \sigma_j \rho_{ij}\end{aligned}$$ **AI-20**

Define, $\delta_{ij} \equiv 1_{\rho_{ij}>0} \times \sqrt{1-\rho_{ij}{}^2}$, and take into3-20, we get

$$\begin{aligned}\sigma^{\theta}_{iM} &= [E^1_{(\theta,\ t)}(\omega)]^2 \sum_{j=1}^{n} w_j \sigma_i \sigma_j \delta_{ij} + [E^1_{(\theta,\ t)}(\omega^2)]\sum_{j=1}^{n} w_j \sigma_i \sigma_j \rho_{ij} \\ &= [E^1_{(\theta,\ t)}(\omega)]^2 \sigma_i \sigma_M \delta_{iM} + [E^1_{(\theta,\ t)}(\omega^2)]\sigma_i \sigma_M \rho_{iM}\end{aligned}$$ **AI-21**

If you are interested in details during the derivation of equation 3-20, please see appendix II. We consider a market portfolio as a whole, and then it is a one-dimension random variable. We use symbol $\overline{M}$ to distinguish it from n dimensional market portfolio,

$$[\mu_{\overline{M}}(S,t) + \frac{\sigma_{\overline{M}}(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega_{\overline{M}}) - r^f] = \frac{M}{A}\sigma^{\theta}_{\overline{M}M}, \qquad i = 1,2, \quad n-1$$ **AI-22**

Where,

$$\sigma^{\theta}_{\overline{M}M} = \sum_{j=1}^{n} w_j \sigma^{\theta}_{j\overline{M}} = \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j E^2_{(\theta,\ t,\rho)}(\omega_{\overline{M}} \omega_j)$$

$$= \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j \{[E^1_{(\theta,\ t)}(\omega)]^2 \sqrt{1-\rho^2_{j\overline{M}}} \times 1_{\rho_{j\overline{M}}>0} + [E^1_{(\theta,\ t)}(\omega^2)]\rho_{j\overline{M}}\} \qquad \textbf{AI-23}$$

$$= [E^1_{(\theta,\ t)}(\omega)]^2 \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j \sqrt{1-\rho^2_{j\overline{M}}} \times 1_{\rho_{j\overline{M}}>0} + [E^1_{(\theta,\ t)}(\omega^2)] \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j \rho_{j\overline{M}}$$

Define $\delta_{j\overline{M}} \equiv 1_{j\overline{M}>0} \times \sqrt{1-\rho_{j\overline{M}}{}^2}$, it can be rewritten as follow,

$$\sigma^{\theta}_{\overline{M}M} = [E^1_{(\theta,\ t)}(\omega)]^2 \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j \delta_{j\overline{M}} + [E^1_{(\theta,\ t)}(\omega^2)] \sum_{j=1}^{n} w_j \sigma_{\overline{M}} \sigma_j \rho_{j\overline{M}} \qquad \textbf{AI-24}$$

$$= [E^1_{(\theta,\ t)}(\omega)]^2 \sigma_M \sigma_{\overline{M}} \delta_{\overline{M}M} + [E^1_{(\theta,\ t)}(\omega^2)] \sigma_{\overline{M}} \sigma_M \rho_{\overline{M}M}$$

The reason why we use symbol $\overline{M}$ to distinguish one dimensional from n dimensional market portfolio is that the θ-adjusted mean does not satisfy the additivity when there is risk resources dimensional receding. After coefficients come out of θ-adjusted mean, there is no difference between $\overline{M}$ and M , so we can simplify the expression,

$$\sigma^{\theta}_{\overline{M}M} = [E^1_{(\theta,\ t)}(\omega)]^2 \sigma_M \sigma_M \delta_{MM} + [E^1_{(\theta,\ t)}(\omega^2)] \sigma_M \sigma_M \rho_{MM}$$

$$= [E^1_{(\theta,\ t)}(\omega)]^2 \sigma^2_M \delta_{MM} + [E^1_{(\theta,\ t)}(\omega^2)] \sigma^2_M \qquad \textbf{AI-25}$$

Take equations 3-23 and 3-25 into 3-22, and change $\mu_{\overline{M}}(S,t)$, $\sigma_{\overline{M}}(S,t)$ to $\mu_M(S,t)$, $\sigma_M(S,t)$, 3-22 is re-written as,

$$\frac{\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega) - r^f}{\mu_{\overline{M}}(S,t) + \frac{\sigma_{\overline{M}}(S,t)}{\sqrt{dt}} E^1_{(\theta,\ t)}(\omega) - r^f} = \frac{\sigma^{\theta}_{iM}}{\sigma^{\theta}_{\overline{M}M}} = \frac{[E^1_{(\theta,\ t)}(\omega)]^2 \sigma_i \sigma_M \delta_{iM} + [E^1_{(\theta,\ t)}(\omega^2)] \sigma_i \sigma_M \rho_{iM}}{[E^1_{(\theta,\ t)}(\omega)]^2 \sigma^2_M \delta_{MM} + [E^1_{(\theta,\ t)}(\omega^2)] \sigma^2_M}, \quad i = 1,2,\cdots,n$$

AI-26

Where ω is any standard normal distributed random variable. If S is constant, we define,

$$\mu_i(S,t) \equiv \mu_i, \quad \sigma_i(S,t) \equiv \sigma_i, \quad \mu_{\overline{M}}(S,t) \equiv \mu_{\overline{M}} \quad \sigma_{\overline{M}}(S,t) \equiv \sigma_{\overline{M}}$$

$$\Theta \equiv E^1_{(\theta,t)}(\omega) \quad \Phi \equiv E^1_{(\theta,t)}(\omega^2) \quad \tilde{\sigma}_{iM} \equiv \sigma_i \sigma_M \delta_{iM} \quad \tilde{\sigma}^2_M \equiv \sigma^2_M \delta_{MM} \qquad \textbf{AI-27}$$

Equation 3-26 is simplified as follows,

$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}} \Theta - r^f}{\mu_{\overline{M}} + \frac{\sigma_{\overline{M}}}{\sqrt{dt}} \Theta - r^f} = \frac{\Theta^2 \tilde{\sigma}_{iM} + \Phi \sigma_{iM}}{\Theta^2 \tilde{\sigma}^2_M + \Phi \sigma^2_M}, \quad i = 1,2,\cdots,n \qquad \textbf{AI-28}$$

Appendix II:

Here I want to explain within general expectation framework how to construct two standard normal distributions X, Y whose correlation is ρ. When perfect information exists, usually Y is structured as follows,

$X, Z \overset{iid}{\sim} N(0,1)$; $\quad Y = \rho X + \sqrt{1-\rho^2} Z$

We can prove that no matter $\rho > 0$ or $\rho < 0$, the character of Y, that is, $Y \sim N(0,1)$ is maintained, and $\rho_{XY} = \rho$, the point we want to carry, is satisfied.

But in θ-adjusted framework, $\rho > 0$ or $\rho < 0$ matters. For the maintenance of mathematical character needs the sign of the second term in accordance with the sign of ρ. In other words, it should be $Y = \rho X + 1_{\rho>0} \sqrt{1-\rho^2} Z$. The reason is as follows,

Case1: $\rho > 0$

Suppose $X, Z \overset{iid}{\sim} N(0,1)$, Y is structured as $Y = \rho X + \sqrt{1-\rho^2} Z$, then $Y \sim N(0,1)$, we get

$$E_\theta^1(Y) = \Theta \quad \textbf{AII-1}$$

$$E_\theta^2(\rho X + \sqrt{1-\rho^2} Z) = E_\theta^1(\rho X) + E_\theta^1(\sqrt{1-\rho^2} Z) = \rho E_\theta^1(X) + \sqrt{1-\rho^2} E_\theta^1(Z) = \rho\Theta + \sqrt{1-\rho^2}\Theta \quad \textbf{AII-2}$$

If Y is structured as $Y = \rho X - \sqrt{1-\rho^2} Z$, then still $Y \sim N(0,1)$, we get

$$E_\theta^1(Y) = \Theta \quad \textbf{AII-3}$$

$$E_\theta^2(\rho X - \sqrt{1-\rho^2} Z) = E_\theta^1(\rho X) - E_\theta^1(\sqrt{1-\rho^2} Z) = \rho E_\theta^1(X) - \sqrt{1-\rho^2} E_\theta^1(Z) = \rho\Theta - \sqrt{1-\rho^2}\Theta \quad \textbf{AII-4}$$

In the general expectation framework, we need to maintain more character other than $Y \sim N(0,1)$. If X and Y are positively correlated, people's view tendency are concordant. That is if he is pessimistic, he tends to amplify the left tails of X and Y, and vise versa. So one dimension distribution of $(\rho X \pm \sqrt{1-\rho^2} Z)$ and the joint distribution of X and Y are coherent. $E_\theta^1(Y)$ is the expectation of Y, $E_\theta^2(\rho X \pm \sqrt{1-\rho^2} Z)$ is the expectation of $(\rho X \pm \sqrt{1-\rho^2} Z)$, which is integrated by the joint probability of X and Z. From section2.1, if there is risk-resources dimensional receding, additivity is not satisfied. And it is sure enough that $(\rho \pm \sqrt{1-\rho^2}) \neq 1$, unless $\rho = 1$, and joint distribution recedes to one dimension. However, they should be of the same sign. Since both of them are being used to measure the reward of an uncertainty. Obviously,

$$Y = (\rho X + \sqrt{1-\rho^2} Z) \qquad \textbf{AII-5}$$

should be the only choice.

Case2: $\rho = -1$

$Y = -X$, equation $E_\theta^1(Y) = E_\theta^1(X) = -E_\theta^1(-X)$ exists. Now I am explaining it. Since $X \sim N(0,1)$, normal distribution is symmetry, $Y \sim N(0,1)$, so we obtain, $E_\theta^1(X) = \Theta$ as well as $E_\theta^1(Y) = \Theta$. From the property Homogeneity in section2, we obtain $E_\theta^1(-X) = -E_\theta^1(X)$. It is amazing and confusing. But the following equation show that the main problem lies in $\rho = -1$, people's view tendency are not concordant any more.

$$E_\theta^1(Y) = \int_{-\infty}^{\infty} y\pi_y(\theta) f(y) dy = \int_{-\infty}^{\infty} -x\pi_{-x}(\theta) f(-x) d(-x) = \int_{-\infty}^{\infty} -w\pi_w(1-\theta) f(w) dw = -(-\Theta) = \Theta = E_\theta^1(X) \qquad \textbf{AII-6}$$

Looking at X while thinking of Y is not the same as looking at Y itself. That is why a joke says that the pessimistic father worry about his elder son when the sun is shining and his younger son when it is raining, because the former is selling umbrella, and the latter is a fisher.

Case3: $\rho < 0, \quad \rho \neq -1$

$$E_\theta^1(Y) = -[-\rho E_\theta^1(X) + \sqrt{1-(-\rho)^2} E_\theta^1(Z)] = \rho\Theta - \sqrt{1-\rho^2}\Theta \qquad \textbf{AII-7}$$

$$Y = (\rho X - \sqrt{1-\rho^2} Z) \qquad \textbf{AII-8}$$

Combine equation AII-5 and AII-8,

$$Y = \rho X + 1_{\rho>0}\sqrt{1-\rho^2} Z$$

We can see that the risk of Y is generated from two parts. One is correlated with X , the other part is uncorrelated with X . In other words, when variance is standardized to 1, risk ρ is exposed to X , $\sqrt{1-\rho^2}$ is exposed to Z . Since the second part of power does exist, and taken into formula A.I-20, term $\sigma_X \sigma_Y \sqrt{1-\rho_{XY}^2} \times 1_{\rho_{XY}>0}$ comes up, that is why I name this term as the potential risk of Y to X .