Solutions to Exercises Week 4, Stochastic Calculus

1. Integration by parts: $\int_0^t f(s)dg(s) = f(t)g(t) - f(0)g(0) - \int_0^t g(s)df(s)$. When $f(\cdot) = g(\cdot)$, and $g(0) = 0$, we find $\int_0^t g(s)dg(s) = g(t)^2 - \int_0^t g(s)dg(s)$, and hence $\int_0^t g(s)dg(s) = \frac{1}{2}g(t)^2$.
Itô's Lemma applied to $f(t, B_t) = B_t^2$: The underlying process is B_t, with volatility $\sigma_t = 1$. The function f has derivatives $\partial f/\partial t = 0$, $\partial f/\partial B_t = 2B_t$, and $\partial^2 f/\partial B_t^2 = 2$, so that

$$df(t, B_t) = d(B_t^2) = 2B_t dB_t + \tfrac{1}{2}2dt = 2B_t dB_t + dt,$$

so that $B_t dB_t = \frac{1}{2}[d(B_t^2) - dt]$. Integrating from 0 to t gives

$$\int_0^t B_s dB_s = \tfrac{1}{2}\left(\int_0^t d(B_s)^2 - \int_0^t ds\right) = \tfrac{1}{2}\left(B_t^2 - t\right).$$

2. The underlying process X_t has volatility $\sigma_t = \sigma$. The function $f(t, X_t)$ has derivatives $\partial f/\partial t = 0$, $\partial f/\partial X_t = \exp(X_t) = S_t$, and $\partial^2 f/\partial X_t^2 = \exp(X_t) = S_t$, because the derivative of the exponential function is the exponential function itself. Therefore, Itô's Lemma gives

$$\begin{aligned} df(t, X_t) = dS_t &= S_t dX_t + \tfrac{1}{2}S_t\sigma^2 dt \\ &= \alpha S_t dt + \sigma S_t dB_t + \tfrac{1}{2}\sigma^2 S_t dt \\ &= \left(\alpha + \tfrac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dB_t. \end{aligned}$$

3. The underlying process S_t has volatility $\sigma_t = \sigma S_t$. The function $f(t, S_t) = \ln S_t$ has derivatives $\partial f/\partial t = 0$, $\partial f/\partial S_t = 1/S_t$, and $\partial^2 f/\partial S_t^2 = -1/S_t^2$. Therefore, Itô's Lemma gives

$$\begin{aligned} df(t, S_t) = dX_t &= \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(\sigma S_t)^2 dt \\ &= \mu dt + \sigma dB_t - \tfrac{1}{2}\sigma^2 dt \\ &= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dB_t. \end{aligned}$$

4. Note that $X_0 = e^{0t}Y_0 = Y_0$. Furthermore $\int_0^t e^{-\gamma(t-s)}dB_s = e^{-\gamma t}\int_0^t e^{\gamma s}dB_s$. This leads to

$$\begin{aligned} X_t = e^{\gamma t}Y_t &= e^{\gamma t}\mu + X_0 - \mu + \sigma\int_0^t e^{\gamma s}dB_s \\ &= X_0 + \mu[e^{\gamma t} - 1] + \sigma\int_0^t e^{\gamma s}dB_s. \end{aligned}$$

The function $\mu[e^{\gamma t} - 1]$ has derivative $\mu\gamma e^{\gamma t}$, or in other words $\mu[e^{\gamma t} - 1] = \mu\gamma\int_0^t e^{\gamma s}ds$. This means that, with drift $\mu_t = \mu\gamma e^{\gamma t}$ and volatility $\sigma_t = \sigma e^{\gamma t}$, X_t is the Itô process

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s \\ \iff \quad dX_t &= \mu_t dt + \sigma_t dB_t. \end{aligned}$$

The function $f(t, X_t) = e^{-\gamma t}X_t$ has derivatives $\partial f/\partial t = -\gamma e^{\gamma t}X_t = -\gamma Y_t$, $\partial f/\partial X_t = e^{-\gamma t}$, and $\partial^2 f/\partial X_t^2 = 0$. Therefore, Itô's Lemma gives

$$\begin{aligned} dY_t &= -\gamma Y_t dt + e^{-\gamma t}dX_t + \tfrac{1}{2}0\sigma_t^2 dt \\ &= -\gamma Y_t dt + \mu\gamma dt + \sigma dB_t \\ &= -\gamma(Y_t - \mu)dt + \sigma dB_t. \end{aligned}$$

5. Let $t_i = ih$. Using the original formula for Y_t, we have

$$y_i - \mu = Y_{t_i} - \mu = e^{-\gamma t_i}[Y_0 - \mu] + \sigma e^{-\gamma t_i} \int_0^{t_i} e^{\gamma s} dB_s.$$

Similarly,

$$\begin{aligned} \rho(y_{i-1} - \mu) = e^{-\gamma h}(Y_{t_{i-1}} - \mu) &= e^{-\gamma(t_{i-1}+h)}[Y_0 - \mu] + \sigma e^{-\gamma(t_{i-1}+h)} \int_0^{t_{i-1}} e^{\gamma s} dB_s \\ &= e^{-\gamma t_i}[Y_0 - \mu] + \sigma e^{-\gamma t_i} \int_0^{t_{i-1}} e^{\gamma s} dB_s. \end{aligned}$$

Therefore,

$$y_i - \mu - \rho(y_{i-1} - \mu) = \sigma e^{-\gamma t_i} \int_{t_{i-1}}^{t_i} e^{\gamma s} dB_s = \varepsilon_i.$$

Using the fact that for a general non-stochastic σ_t, $\int_{t_{i-1}}^{t_i} \sigma_s dB_s \sim N\left(0, \int_{t_{i-1}}^{t_i} \sigma_s^2 ds\right)$, we find that

$$\varepsilon_i = \sigma e^{-\gamma t_i} \int_{t_{i-1}}^{t_i} e^{\gamma s} dB_s \sim N\left(0, \left[\sigma e^{-\gamma t_i}\right]^2 \int_{t_{i-1}}^{t_i} \left[e^{\gamma s}\right]^2 ds\right).$$

The variance σ_ε^2 of ε_i can be further expressed as

$$\begin{aligned} \sigma_\varepsilon^2 = \sigma^2 e^{-2\gamma t_i} \int_{t_{i-1}}^{t_i} e^{2\gamma s} ds &= \sigma^2 e^{-2\gamma t_i} \left[\tfrac{1}{2\gamma} e^{2\gamma s}\right]_{t_{i-1}}^{t_i} \\ &= \sigma^2 e^{-2\gamma t_i} [\tfrac{1}{2\gamma} e^{2\gamma t_i} - \tfrac{1}{2\gamma} e^{2\gamma t_{i-1}}] \\ &= \sigma^2 [1 - e^{-2\gamma[t_i - t_{i-1}]}]/(2\gamma) \\ &= \sigma^2 (1 - e^{-2\gamma h})/(2\gamma) \\ &= \sigma^2 (1 - \rho^2)/(2\gamma). \end{aligned}$$

Hence $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$. What remains to be shown is that ε_i and ε_j are independent. This follows from the fact the ε_i's are increments, multiplied by $\sigma e^{-\gamma t_i}$, of the Itô process $\int_0^t e^{\gamma s} dB_s$ which has independent increments (because $e^{\gamma s}$ is non-stochastic).

Solutions to Exercises Week 5, Stochastic Calculus

1. X_t satisfies the SDE $dX_t = \mu dt + \sigma\sqrt{X_t} dB_t$, so that X_t has volatility $\sigma_t = \sigma\sqrt{X_t}$.

   (a) The function $Y_t = f(t, X_t) = X_t^{1/2}$ has derivatives $\partial f/\partial t = 0$, $\partial f/\partial X_t = \frac{1}{2} X_t^{-1/2}$, and $\partial^2 f/\partial X_t^2 = -\frac{1}{4} X_t^{-3/2}$. Therefore, Itô's Lemma gives

$$\begin{aligned} dY_t &= \frac{1}{2\sqrt{X_t}} dX_t + \frac{1}{2}\left(-\frac{1}{4X_t^{3/2}}\right)\left(\sigma\sqrt{X_t}\right)^2 dt \\ &= \frac{\mu}{2\sqrt{X_t}} dt + \frac{\sigma}{2} dB_t - \frac{\sigma^2}{8\sqrt{X_t}} dt \\ &= \frac{\mu}{2Y_t}\left(\mu - \frac{\sigma^2}{4}\right) dt + \frac{\sigma}{2} dB_t. \end{aligned}$$

(b) Since the square root of X_t is involved, both X_t and Y_t should be non-negative. If μ is negative, then the process will drift below zero and therefore the volatility $\sigma_t = \sigma\sqrt{X_t}$ will no longer be defined. If μ is zero, then a strongly negative increment in B_t may push it towards (or even below) zero, which leads to the same problem. Only if μ is sufficiently large, the process will be pushed away from zero.

(c) If $\mu = \frac{1}{4}\sigma^2$ then $dY_t = \frac{1}{2}\sigma dB_t$, so that $Y_t = Y_0 + \frac{1}{2}\sigma B_t$ and hence $X_t = \left(Y_0 + \frac{1}{2}\sigma B_t\right)^2$. This solution does not make any sense, because $Y_t = Y_0 + \frac{1}{2}\sigma B_t$ may become negative, and $Y_t = \sqrt{X_t}$ of course can never become negative.

2. Let's call the three approximations $S_t^{(1)}$, $S_t^{(2)}$ and $S_t^{(3)}$ (binomial, Euler level, Euler log) of the geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dB_t$. Furthermore assume that alle three approximations are defined with time steps $t_i = ih$, $h = \delta t = T/n$. Consider two time points $t_k < t_l$, and the increments of the logarithms of these three approximations. For $S_t^{(1)}$, we have

$$R^{(1)} = \log \frac{S_{t_l}^{(1)}}{S_{t_k}^{(1)}} = \sum_{i=k+1}^{l} Y_i,$$

where Y_i is either $-\sigma\sqrt{h}$ or $+\sigma\sqrt{h}$, the latter with probability $p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$. The distribution of $R^{(1)}$ is binomial, but approximately (by the central limit theorem) $N(\mu[t_l - t_k], \sigma^2[t_l - t_k])$ where the approximation gets better as n increases. For $S_t^{(2)}$, we have

$$S_{t_l}^{(2)} = S_{t_k}^{(2)} + \sum_{i=k+1}^{l} S_{t_{i-1}}^{(2)} \left[\mu h + \sigma(B_{t_i} - B_{t_{i-1}})\right].$$

This means that

$$\begin{aligned}
R^{(2)} = \log \frac{S_{t_l}^{(2)}}{S_{t_k}^{(2)}} &= \log\left(1 + \sum_{i=k+1}^{l} \frac{S_{t_{i-1}}^{(2)}}{S_{t_k}^{(2)}} \left[\mu h + \sigma(B_{t_i} - B_{t_{i-1}})\right]\right) \\
&\approx \sum_{i=k+1}^{l} \frac{S_{t_{i-1}}^{(2)}}{S_{t_k}^{(2)}} \left[\mu h + \sigma(B_{t_i} - B_{t_{i-1}})\right] \\
&\approx \sum_{i=k+1}^{l} \left[\mu h + \sigma(B_{t_i} - B_{t_{i-1}})\right] \\
&\sim N(\mu[t_l - t_k], \sigma^2[t_l - t_k]),
\end{aligned}$$

so again we need an approximation to get the right normal distribution for the log-return. Finally, for $S_t^{(3)}$ we have

$$R^{(3)} = \log \frac{S_{t_l}^{(3)}}{S_{t_k}^{(3)}} = \sum_{i=k+1}^{l} \left[\mu h + \sigma(B_{t_i} - B_{t_{i-1}})\right] \sim N(\mu[t_l - t_k], \sigma^2[t_l - t_k]),$$

exactly, without any approximation. Therefore, as long as we look at the process $S_t^{(3)}$ at the time points t_i, it has exactly the right distribution, whether n is small or large.