Solutions to Exercises of Week 2, St.Calc.

1. Y is defined as

$$Y = \#heads = \begin{cases} 2 & \text{if} \quad \omega = HH = \omega_1 \\ 1 & \text{if} \quad \omega \in \{HT, TH\} = \{\omega_2, \omega_3\} \\ 0 & \text{if} \quad \omega = TT = \omega_4 \end{cases}$$

Hence

$$\sigma(Y) = \{\varnothing, \Omega, \{\omega_1\}, \{\omega_4\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_2, \omega_3, \omega_4\}\}.$$

2. $\mathbb{E}(Y) = \sum_{i=1}^{4} Y(\omega_i)\mathbb{P}(\omega_i) = 2p^2 + 1p(1-p) + 1(1-p)p + 0(1-p)^2 = 2p.$
The conditional expectation $\mathbb{E}(Y|\mathcal{F}_1)$ is a random variable; its realization depends on whether or not $\omega \in \{\omega_1, \omega_2\}$ (first toss is heads). Therefore:

$$\begin{aligned} \mathbb{E}(Y|\mathcal{F}_1) &= \begin{cases} \sum_{i=1}^{4} Y(\omega_i)\mathbb{P}(\omega_i|\{\omega_1,\omega_2\}) = 2\mathbb{P}(\omega_1|\{\omega_1,\omega_2\}) + 1\mathbb{P}(\omega_2|\{\omega_1,\omega_2\}) & \text{if} \quad \omega \in \{\omega_1,\omega_2\} \\ \\ \sum_{i=1}^{4} Y(\omega_i)\mathbb{P}(\omega_i|\{\omega_3,\omega_4\}) = 1\mathbb{P}(\omega_3|\{\omega_3,\omega_4\}) + 0\mathbb{P}(\omega_4|\{\omega_3,\omega_4\}) & \text{if} \quad \omega \in \{\omega_3,\omega_4\} \end{cases} \\ &= \begin{cases} 2\frac{p^2}{p^2+p(1-p)} + 1\frac{p(1-p)}{p^2+p(1-p)} = p+1 & \text{if} \quad \omega \in \{\omega_1,\omega_2\} \\ \\ 1\frac{(1-p)p}{(1-p)p+(1-p)^2} = p & \text{if} \quad \omega \in \{\omega_3,\omega_4\} \end{cases} \\ &= \begin{cases} p+1 & \text{if} \quad X_1(\omega) = a \\ \\ p & \text{if} \quad X_1(\omega) = -a \end{cases} \end{aligned}$$

This implies $\mathbb{E}[\mathbb{E}(Y|\mathcal{F}_1)] = (p+1)p + p(1-p) = 2p = \mathbb{E}(Y).$

3. Y is defined as

$$Y = \begin{cases} 1 & \text{if} \quad \omega \in \{\omega_1, \omega_4\} \\ \\ 0 & \text{if} \quad \omega \in \{\omega_2, \omega_3\} \end{cases}$$

So $\sigma(Y) = \{\varnothing, \Omega, \{\omega_1, \omega_4\}\{\omega_2, \omega_3\}\}$. To check independence of Y and X_1, we can check wether $\sigma(Y)$ is independent of $\sigma(X_1) = \mathcal{F}_1$, i.e., $\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) = 0$ for all $A \in \sigma(Y), B \in \sigma(X_1)$. Let's take an example:

$$\begin{aligned} \mathbb{P}(Y=1, X_1=a) - \mathbb{P}(Y=1)\mathbb{P}(X_1=a) &= \mathbb{P}(\{\omega_1,\omega_4\} \cap \{\omega_1,\omega_2\}) - \mathbb{P}(\{\omega_1,\omega_4\})\mathbb{P}(\{\omega_1,\omega_2\}) \\ &= p^2 - [p^2 + (1-p)^2]p, \end{aligned}$$

which is zero only if $p = 0.5$. The same follows for the other three possibilities: Y and X_1 are independent only if $p = 0.5$.
Very analogously one can check independence between Y and X_2; for example

$$\begin{aligned} \mathbb{P}(Y=1, X_2=a) - \mathbb{P}(Y=1)\mathbb{P}(X_2=a) &= \mathbb{P}(\{\omega_1,\omega_4\} \cap \{\omega_1,\omega_3\}) - \mathbb{P}(\{\omega_1,\omega_4\})\mathbb{P}(\{\omega_1,\omega_3\}) \\ &= p^2 - [p^2 + (1-p)^2]p, \end{aligned}$$

which again is zero only if $p = 0.5$.

Even if $p = 0.5$, so that Y is indepdent of X_1 and also Y is independent of X_2, it is not true that Y is independent of (X_1, X_2), simply because it can be expressed as a function of (X_1, X_2):

$$Y = \begin{cases} 1 & \text{if} \quad (X_1, X_2) \in \{(a, a), (-a, -a)\} \\ \\ 0 & \text{if} \quad (X_1, X_2) \in \{(a, -a), (-a, a)\} \end{cases}$$

4. $\mathbb{E}(S_2^2|S_1) = S_1^2\mathbb{E}(\exp(2X_2)|S_1) = S_1^2[pu^2 + (1-p)d^2]$. Next, $\mathbb{E}(S_2|S_1)^2 = S_1^2[pu + (1-p)d]^2$. Hence

$$\mathrm{Var}(S_2|S_1) = S_1^2(u-d)^2p(1-p)$$

5. $X_i = \mathbb{E}(X|\mathcal{F}_i)$ implies

$$\mathbb{E}(X_i|\mathcal{F}_{i-1}) = \mathbb{E}[\mathbb{E}(X|\mathcal{F}_i)|\mathcal{F}_{i-1}] = \mathbb{E}(X|\mathcal{F}_{i-1}) = X_{i-1}.$$

Hence X_i is a martingale.