Midterm exam Stochastic Calculus

1. Consider the following two-period binomial tree for a stock price:

$$\begin{array}{ccc} (t=0) & (t=1) & (t=2) \\ & & 150 \\ & 120 & \\ 100 & & 100 \\ & 80 & \\ & & 65 \end{array}.$$

   Furthermore, assume that the (discretely compounded) interest rate on a bond per period is 10%. At all nodes, the probability p of going up is 0.8.

   (a) Show that these data imply that agents are risk-averse.

   (b) Determine the risk-neutral probabilities q, at time 0, and in the two possible states at time 1; note that q is different at each of these three nodes.

   (c) Consider a derivative X which pays off (at time $T = 2$) 50 if the first movement is up (i.e., if $S_1 = 120$), and an additional 50 if the second movement is up as well (if $S_2 = 150$); otherwise the payoff is zero. Set this up as a tree for X (Hint: the tree should not be recombining, i.e., it should have 4 possible outcomes at time $T = 2$). Also determine the value of the derivative at time 1 (in both possible situations separately) and 0.

2. Consider the process

$$dX_t = (a + bt)X_t dt + \sigma X_t dB_t,$$

   where a, b and $\sigma > 0$ are constants, and B_t is a standard Brownian motion.

   (a) Use Itô's lemma to obtain the differential equation for $Y_t = \ln X_t$.

   (b) For which values of a, b and σ is X_t a martingale?

   (c) What is the strong solution for X_t?

   (d) Let $g(t)$ be some differentiable function of time only, and let $Z_t = f(t, X_t) = \ln(g(t)X_t)$. Obtain the differential equation for Z_t (using Itô's lemma).

   (e) For which choice of $g(t)$ is Z_t a martingale (for general a, b and σ^2)?

3. Let $B_t, t \in [0, 1]$ be a Brownian motion process, and define

$$\sigma_t = \begin{cases} \sqrt{\frac{3}{4}} & \text{if} \quad 0 \le t \le \frac{2}{3} \\ \\ \sqrt{\frac{3}{2}} & \text{if} \quad \frac{2}{3} < t \le 1 \end{cases}.$$

   (a) Determine the function $\tau(t) = \int_0^t \sigma_s^2 ds$, $t \in [0, 1]$.

(b) Define $X_t = B_{\tau(t)}, t \in [0, 1]$. In other words, at time t the value of X is the same as the value of B at time $\tau(t)$. Note that X_t is like a Brownian motion on a "deformed time scale". Show that X_t is a martingale, but not a Brownian motion.

(c) Show that the quadratic variation of X_t is given by $\tau(t)$.

(d) Show that X_t has the same finite-dimensional distributions as $Y_t = \int_0^t \sigma_s dB_s$.