Chapter 27
More on Models and Numerical Procedures

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BSM 期权定价公式的精确度评价

- BSM 期权定价公式在定价方面存在一定偏差
  - 期权市场价格偏离均衡;
  - 使用错误的参数;
  - BSM 期权定价公式建立在众多假定的基础上。

- 但它依然是迄今为止解释期权价格动态的最佳模型之一，应用广泛，影响深远

BSM 期权定价公式的缺陷与拓展

- 无交易成本
- 常数波动率
- 其他参数假设
- 资产价格连续变动
- ……

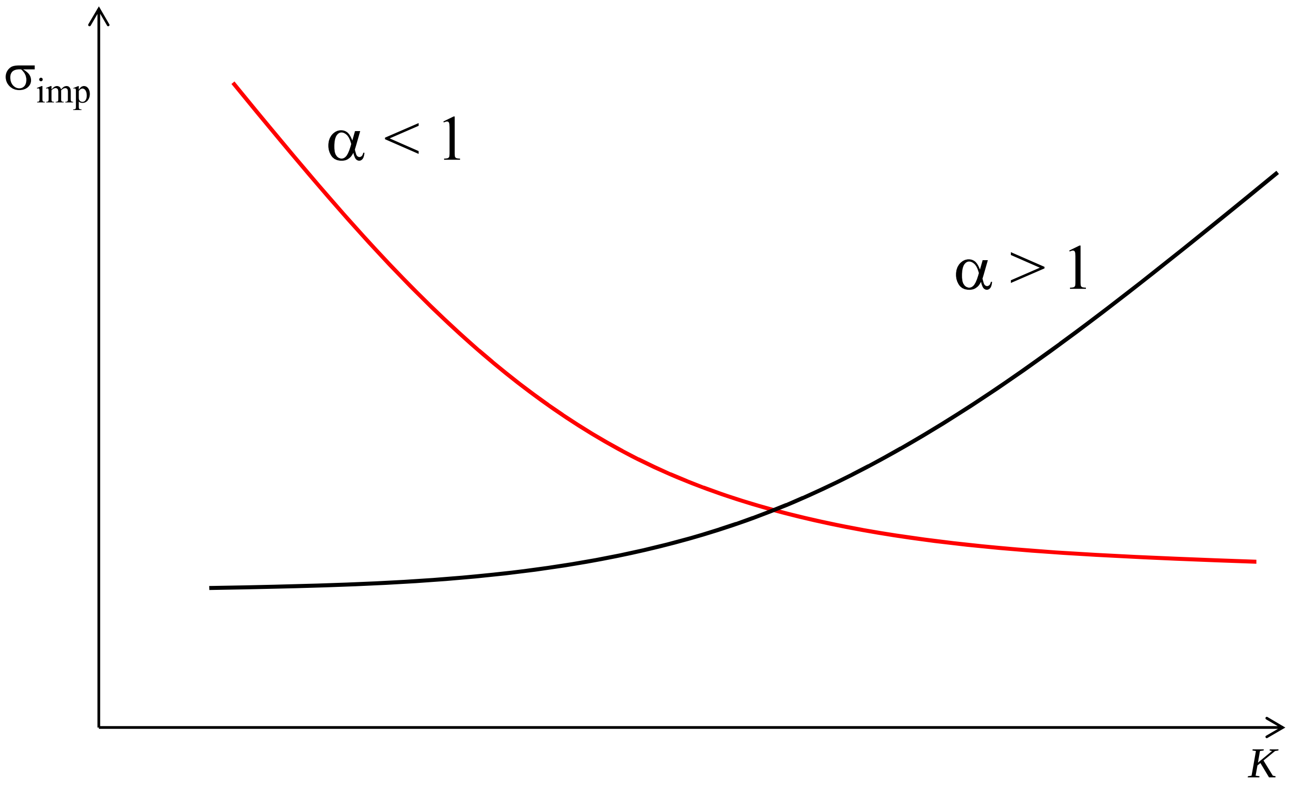
CEV Model

$$dS = (r - q)Sdt + \sigma S^{\alpha} dz$$

- When α = 1 the model is Black-Scholes
- When α > 1 volatility rises as stock price rises
- When α < 1 volatility falls as stock price rises

European options can be value analytically in terms of the cumulative non-central chi square distribution

CEV Models Implied Volatilities



随机波动率模型（Stochastic Volatility）

- 基本模型形式

$$\frac{dS}{S} = \mu_S dt + \sigma_S dz_S$$

$$V = \sigma_S^2$$

$$dV = \kappa\left(\theta - V\right)dt + \sigma_V V^{\alpha} dz_V$$

Hull-White模型（1987）

- 波动率服从最简单的几何布朗运动，是一个非平稳过程

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S(t) dz_S$$

$$V(t) = \sigma_S^2(t)$$

$$\frac{dV(t)}{V(t)} = \mu_V dt + \sigma_V dz_V$$

$$V(0) > 0, corr(dz_S, dz_V) = \rho dt(\text{常数})$$

  - 当$\rho = 0$时，有解析解
  - 当$\rho \neq 0$时，有数值解

Stein-Stein模型（1991）

- 波动率服从Gaussian OU过程，平稳但可能出现负值

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S(t) dz_S$$

$$V(t) = \sigma_S^2(t)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_V dz_V$$

Heston模型（1993）

- 波动率服从平方根过程(square root process，又称CIR过程)，避免了波动率出现负数的情况

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S(t) dz_S$$

$$V(t) = \sigma_S^2(t)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_V \sqrt{V(t)} dz_V$$

$$2k\theta > \sigma_V^2, V(0) > 0, \text{以保证方差为正}$$

$$corr(dz_S, dz_V) = \rho dt (\text{常数})$$

- 有解析解

各种随机波动率模型比较

- 当BS模型不断的更新波动率时，其定价与SV模型的定价很接近
- SV模型不能匹配波动率微笑
- 相对而言，Heston实证效果较好

跳跃模型（Merton,1976）

- dq为独立泊松过程，λ为跳跃发生概率，J为跳跃幅度，可为常数或随机过程

$$\frac{dS(t)}{S(t)} = (\mu - \lambda\mu_J)dt + \sigma dz + J(t)dq,\ dq = \begin{cases} 0, \text{概率} 1-\lambda dt \\ 1, \text{概率} \lambda dt \end{cases}$$

$$\mu_J \equiv E(J)$$

其中dq和dz独立

- 通常没有解析解

Simulating a Jump Process

- In each simulation trial
  - Sample from a binomial distribution to determine the number of jumps
  - Sample to determine the size of each jump

Jumps and the Smile

- Jumps have a big effect on the implied volatility of short term options
- They have a much smaller effect on the implied volatility of long term options

SVJ模型（Bates,1996）

- 随机波动率－跳跃扩散模型：有解析解

$$\frac{dS(t)}{S(t)} = (\mu_S - \lambda\mu_J)dt + \sigma_S(t)dz_S + J(t)dq$$

$$V(t) = \sigma_S^2(t)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma_V\sqrt{V(t)}dz_V$$

$$2k\theta > \sigma_V^2, V(0) > 0, \text{以保证方差为正}$$

$$corr(dz_S, dz_V) = \rho dt(\text{常数})$$

$$\ln J(t) \sim N\left(\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2\right)$$

综合改进模型（Bakshi, Cao and Chen，1997）

- 波动率、利率、跳跃均为随机

$$\frac{dS(t)}{S(t)}=\left(R(t)-\lambda u_J\right)dt+\sigma_S(t)dz_S+J(t)dq$$

$$dV(t)=\kappa_V\left(\theta_V-V(t)\right)dt+\sigma_V\sqrt{V(t)}dz_V$$

$$dR(t)=\kappa_R\left(\theta_R-R(t)\right)dt+\sigma_R\sqrt{R(t)}dz_R$$

$$\ln J(t)\sim N\left(\ln\left(1+\mu_J\right)-\frac{1}{2}\sigma_J^2,\sigma_J^2\right)$$

$$2k_V\theta_V>\sigma_V^2,V(0)>0,\text{以保证方差为正}$$

$$2k_R\theta_R>\sigma_R^2,R(0)>0,\text{以保证利率为正}$$

$corr\left(dz_S,dz_V\right)=\rho dt$（常数），$z_R$与其他过程不相关

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The IVF Model

The implied volatility function model is designed to create a process for the asset price that exactly matches observed option prices. The usual geomeric Brownian motion model

$$dS = (r - q)Sdt + \sigma Sdz$$

is replaced by

$$dS = [r(t) - q(t)]Sdt + \sigma(S, t)Sdz$$

The Volatility Function

The volatility function that leads to the model matching all European option prices is

$$[\sigma(K,t)]^2 = 2\frac{\partial c_{mkt}/\partial t + q(t)c_{mkt} + K[r(t)-q(t)]\partial c_{mkt}/\partial K}{K^2(\partial^2 c_{mkt}/\partial K^2)}$$

Strengths and Weaknesses of the IVF Model

- The model matches the probability distribution of asset prices assumed by the market at each future time
- The models does not necessarily get the joint probability distribution of asset prices at two or more times correct

Convertible Bonds

- Often valued with a tree where during a time interval Δt there is
  - a probability p_u of an up movement
  - A probability p_d of a down movement
  - A probability 1-exp(-λt) that there will be a default (λ is the hazard rate)
- In the event of a default the stock price falls to zero and there is a recovery on the bond

The Probabilities

$$p_u = \frac{a - de^{-\lambda \Delta t}}{u - d}$$

$$p_d = \frac{ue^{-\lambda \Delta t} - a}{u - d}$$

$$u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}}$$

$$d = \frac{1}{u}$$

Node Calculations

Define:

$Q_{1:}$ value of bond if neither converted nor called

Q_2: value of bond if called

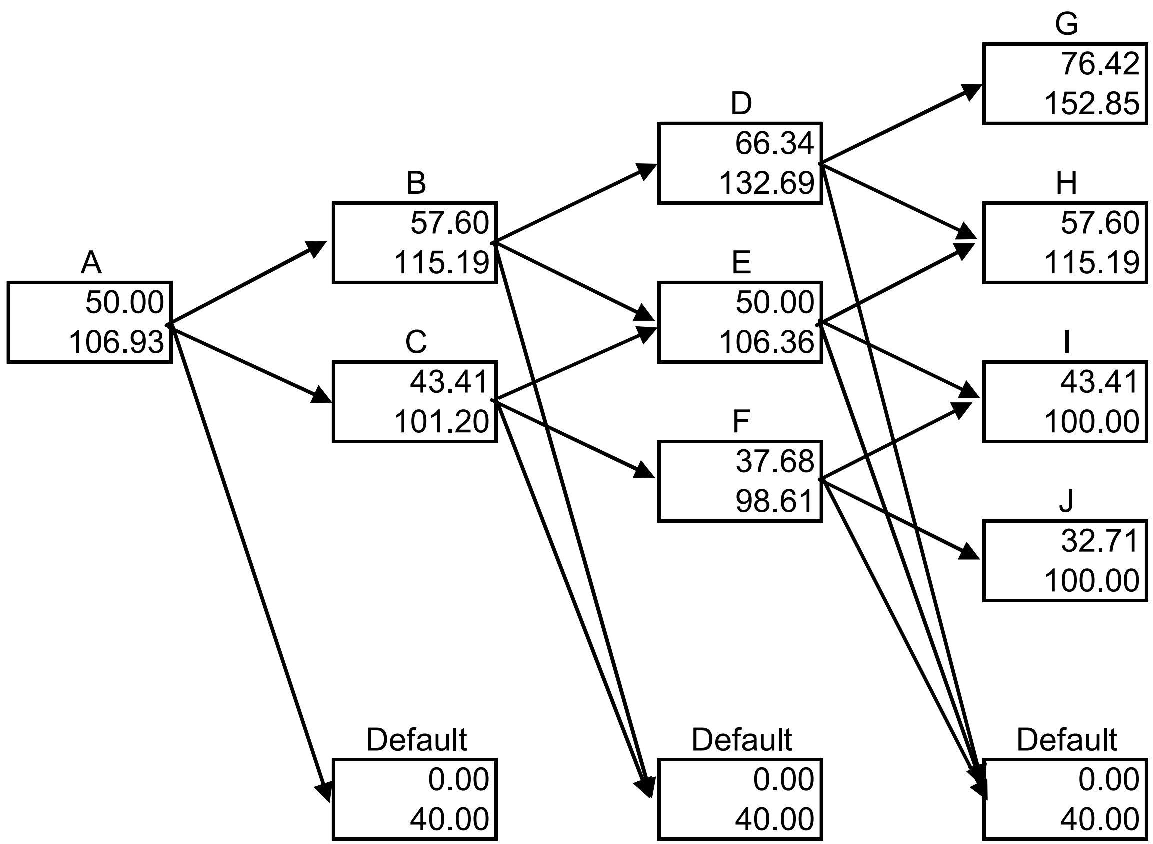
Q_3: value of bond if converted

Value at a node =max[min(Q_1,Q_2),Q_3]

Example 27.1

- 9-month zero-coupon bond with face value of $100
- Convertible into 2 shares
- Callable for $113 at any time
- Initial stock price = $50,
- volatility = 30%,
- no dividends
- Risk-free rates all 5%
- Default intensity, λ, is 1%
- Recovery rate=40%

The Tree



A
50.00
106.93
B
57.60
115.19
C
43.41
101.20
D
66.34
132.69
E
50.00
106.36
F
37.68
98.61
G
76.42
152.85
H
57.60
115.19
I
43.41
100.00
J
32.71
100.00
Default
0.00
40.00
Default
0.00
40.00
Default
0.00
40.00

Numerical Procedures

Topics:

- Path dependent options using tree
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo

Path Dependence: The Traditional View

- Trees work well for American options. They cannot be used for path-dependent options
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options

Extending the Use of Trees

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options

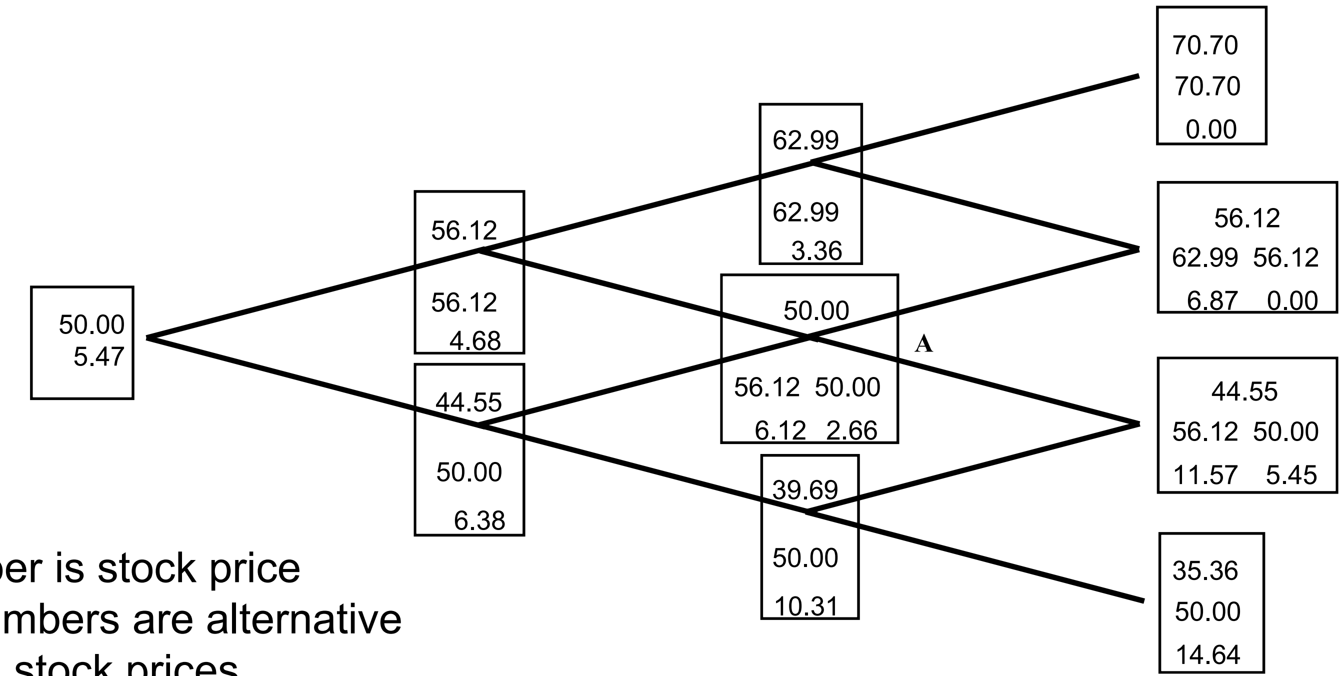
Lookback Example

- Consider an American lookback put on a stock where
  $S = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month & the life of the option is 3 months
- Payoff is $S_{max}-S_T$
- We can value the deal by considering all possible values of the maximum stock price at each node

(This example is presented to illustrate the methodology. It is not the most efficient way of handling American lookbacks (See Technical Note 13)

Example: An American Lookback Put Option

S_0 = 50, σ = 40%, r = 10%, Δt = 1 month,



Top number is stock price
Middle numbers are alternative maximum stock prices
Lower numbers are option prices

Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a "path function")
- The value of the path function at a node can be calculated from the stock price at the node and from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree

Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node
- The basic idea is that it is not necessary to consider every possible value of the path function
- It is sufficient to consider a relatively small number of representative values of the function at each node

Working Forward

- First work forward through the tree calculating the max and min values of the “path function” at each node
- Next choose representative values of the path function that span the range between the min and the max
  - Simplest approach: choose the min, the max, and N equally spaced values between the min and max

Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation

Part of Tree to Calculate Value of an Option on the Arithmetic Average

Node Y: S = 54.68

Average S	Option Price
47.99	7.575
51.12	8.101
54.26	8.635
57.39	9.178

Node X: S = 50.00

Average S	Option Price
46.65	5.642
49.04	5.923
51.44	6.206
53.83	6.492

X → Y: 0.5056

X → Z: 0.4944

Node Z: S = 45.72

Average S	Option Price
43.88	3.430
46.75	3.750
49.61	4.079
52.48	4.416

S=50, X=50, σ=40%, r =10%, T=1yr, Δt=0.05yr. We are at time $4\Delta t$

Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node X when the average of 5 observations is 51.44

Node Y: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node Z: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

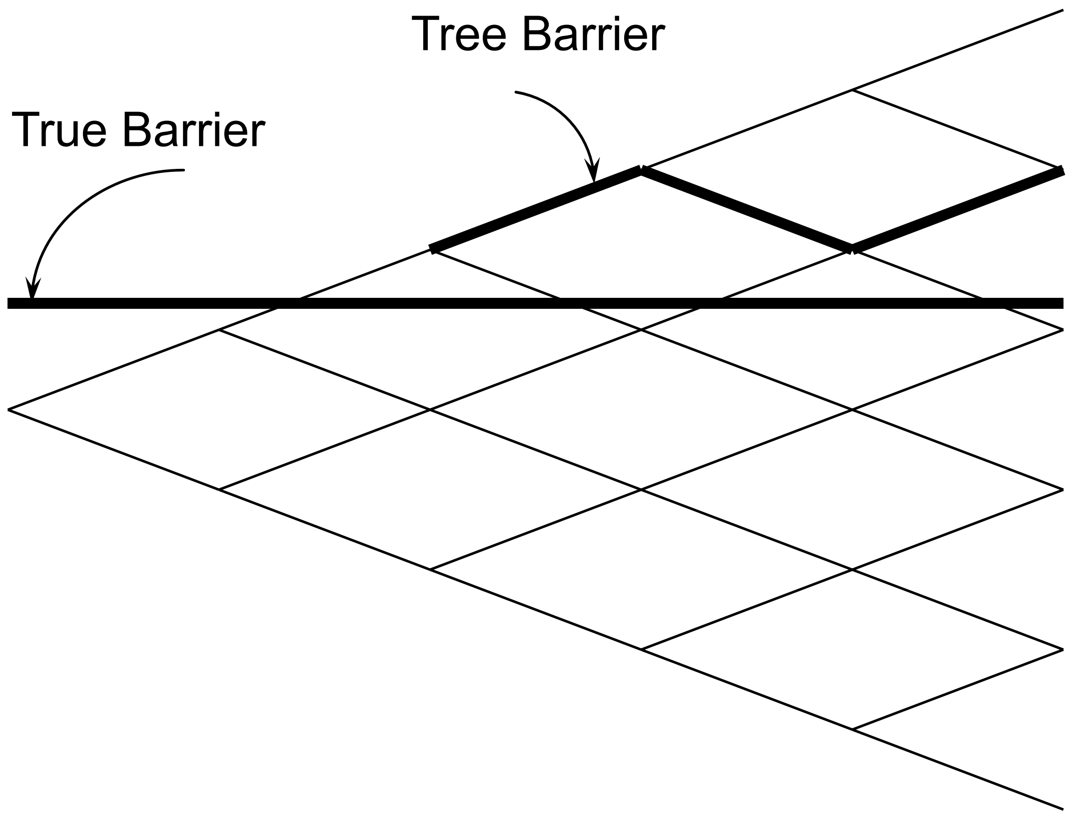
Node X: value is

$$(0.5056 \times 8.247 + 0.4944 \times 4.182)e^{-0.1 \times 0.05} = 6.206$$

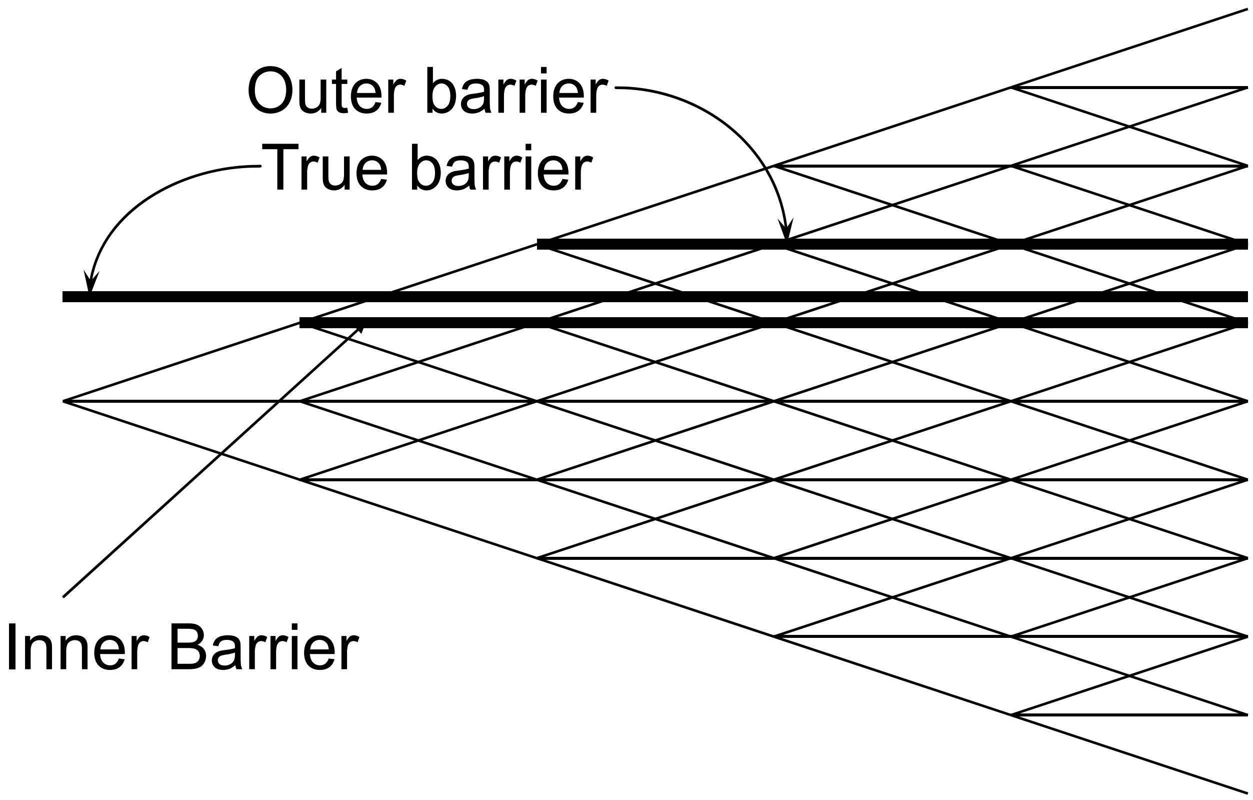
Using Trees with Barriers

- When trees are used to value options with barriers, convergence tends to be slow
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree

True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case



Inner and Outer Barriers for Trinomial Trees



Alternative Solutions to Valuing Barrier Options

- Interpolate between value when inner barrier is assumed and value when outer barrier is assumed
- Ensure that nodes always lie on the barriers
- Use adaptive mesh methodology

In all cases a trinomial tree is preferable to a binomial tree

Modeling Two Correlated Variables Using a 3-Dimensional Tree

- Approaches
  - Transform variables so that they are not correlated and build the tree in the transformed variables
  - Take the correlation into account by adjusting the position of the nodes
  - Take the correlation into account by adjusting the probabilities

Transforming Vaviables

- Suppose:

$$d \ln S_1 = \left(r - q_1 - \sigma_1^2 / 2\right) dt + \sigma_1 dz_1$$
$$d \ln S_2 = \left(r - q_2 - \sigma_2^2 / 2\right) dt + \sigma_2 dz_2$$
$$E\left(dz_1 dz_2\right) = \rho dt$$

- We define two new uncorrelated variables:

$$x_1 = \sigma_2 \ln S_1 + \sigma_1 \ln S_2$$
$$x_2 = \sigma_2 \ln S_1 - \sigma_1 \ln S_2$$

定理

- 定理：若dz_1和dz_2相关，则dz_1+dz_2与dz_1-dz_2不相关。
- 证明：由于E（dz_1+dz_2）和E（dz_1-dz_2）＝0，所以$cov(dz_1+dz_2, dz_1-dz_2)=E$（$dz_1+dz_2$）（$dz_1-dz_2$）= E（$dz_1^2-dz_2^2$）$=dt-dt=0$

$$dx_1 = \left[\sigma_2\left(r - q_1 - \sigma_1^2 / 2\right) + \sigma_1\left(r - q_2 - \sigma_2^2 / 2\right)\right]dt$$
$$+\sigma_1\sigma_2\sqrt{2(1+\rho)}dz_A$$
$$dx_2 = \left[\sigma_2\left(r - q_1 - \sigma_1^2 / 2\right) - \sigma_1\left(r - q_2 - \sigma_2^2 / 2\right)\right]dt$$
$$+\sigma_1\sigma_2\sqrt{2(1-\rho)}dz_B$$

证明：

$dz_1 + dz_2 = \sqrt{2(1+\rho)}dz_A$

$\because dz_1$和dz_2均为正态分布，其和也是正态分布。

它们之和的方差$= E(dz_1 + dz_2)^2 = E(dz_1^2 + dz_2^2 + 2dz_1dz_2)$

$= 2(1+\rho)\ dt$

$\therefore dz_1 + dz_2 = \sqrt{2(1+\rho)dt} \times \varepsilon = \sqrt{2(1+\rho)}dz_A$

同理可证$dz_1 - dz_2 = \sqrt{2(1-\rho)}dz_A$

Monte Carlo Simulation and American Options

- Two approaches:
  - The least squares approach
  - The exercise boundary parameterization approach
- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income

Sampled Paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The Least Squares Approach

- We work back from the end using a least squares approach to calculate the continuation value at each time
- Consider year 2. The option is in the money for five paths. These give observations on S of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00, $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$

The Least Squares Approach continued

- Fitting a model of the form $V=a+bS+cS^2$ we get a best fit relation

$$V=-1.070+2.983S-1.813S^2$$

  for the continuation value V
- This defines the early exercise decision at $t=2$. We carry out a similar analysis at $t=1$

The Least Squares Approach continued

In practice more complex functional forms can be used for the continuation value and many more paths are sampled

The Early Exercise Boundary Parametrization Approach

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^*, below which the option is exercised.
- At t =3 the optimal S^* for the eight paths is 1.10. At t =2 the optimal S^* is 0.84. At t =1 the optimal S^* is 0.88.
- In practice we would use many more paths to calculate the S^*