# Chapter 24
# Credit Risk

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# Credit Ratings

- In the S&P rating system, AAA is the best rating.  After that comes AA, A, BBB, BB, B, CCC, CC, and C
- The corresponding Moody's ratings are Aaa, Aa, A, Baa, Ba, B,Caa, Ca, and C
- Bonds with ratings of BBB (or Baa) and above are considered to be "investment grade"

# 机构评级

- 穆迪
- 标普
- 惠誉
- ……

| Moody's Ratings |
|---|
| Aaa |
| Aa1 |
| Aa |
| Aa2 |
| Aa3 |
| A1 |
| A2 |
| A |
| A3 |
| Baa1 |
| Baa2 |
| Baa |
| Baa3 |
| Ba1 |
| Ba2 |
| Ba3 |
| B1 |
| B2 |
| B3 |
| Caa1 |
| Caa2 |
| Caa3 |
| Ca |
| C |

# Historical Data

Historical data provided by rating agencies are also used to estimate the probability of default

# Cumulative Ave Default Rates (%) (1970-2012, Moody's)

| | 1 | 2 | 3 | 4 | 5 | 7 | 10 |
|---|---|---|---|---|---|---|---|
| Aaa | 0.000 | 0.013 | 0.013 | 0.037 | 0.106 | 0.247 | 0.503 |
| Aa | 0.022 | 0.069 | 0.139 | 0.256 | 0.383 | 0.621 | 0.922 |
| A | 0.063 | 0.203 | 0.414 | 0.625 | 0.870 | 1.441 | 2.480 |
| Baa | 0.177 | 0.495 | 0.894 | 1.369 | 1.877 | 2.927 | 4.740 |
| Ba | 1.112 | 3.083 | 5.424 | 7.934 | 10.189 | 14.117 | 19.708 |
| B | 4.051 | 9.608 | 15.216 | 20.134 | 24.613 | 32.747 | 41.947 |
| Caa-C | 16.448 | 27.867 | 36.908 | 44.128 | 50.366 | 58.302 | 69.483 |

# Interpretation

- The table shows the probability of default for companies starting with a particular credit rating
- A company with an initial credit rating of Baa has a probability of 0.177% of defaulting by the end of the first year, 0.495% by the end of the second year, and so on

# Do Default Probabilities Increase with Time?

- For a company that starts with a good credit rating default probabilities tend to increase with time
- For a company that starts with a poor credit rating default probabilities tend to decrease with time

# Hazard Rates vs Unconditional Default Probabilities

- The hazard rate (also called default intensity, conditional default probability) is the probability of default for a certain time period conditional on no earlier default
- The unconditional default probability is the probability of default for a certain time period  as seen at time zero
- What are the default intensities and unconditional default probabilities for a Caa rated company in the third year?

# Hazard Rate

- The hazard rate that is usually quoted is an instantaneous rate
- If $V(t)$ is the probability of a company surviving to time $t$

$$\frac{(1-V(t+\Delta t))-(1-V(t))}{V(t)}=\lambda\Delta t$$

$$V(t+\Delta t)-V(t)=-\lambda(t)V(t)$$

This leads to

$$V(t)=e^{-\int_0^t \lambda(t)dt}$$

The cumulative probability of default by time $t$ is

$$Q(t)=1-e^{-\bar{\lambda}(t)t}$$

# Recovery Rate

- The recovery rate for a bond is usually defined as the price of the bond a few days after default as a percent of its face value
- Recovery rates tend to decrease as default rates increase

# Recovery Rates; Moody's: 1982 to 2012

| Class | Mean(%) |
|---|---|
| Senior Secured | 51.6 |
| Senior Unsecured | 37.0 |
| Senior Subordinated | 30.9 |
| Subordinated | 31.5 |
| Junior Subordinated | 24.7 |

# Estimating Default Probabilities

- Alternatives:
  - Use Historical Data
  - Use Bond Prices
  - Use Merton’s Model
  - Use Option Prices
  - Use CDS spreads(下一章）

# Using Credit Spreads

- Suppose $s(T)$ is the credit spread for maturity $T$
- Average hazard rate between time zero and time $T$ is approximately
$$\frac{s(T)}{1-R}$$
where $R$ is the recovery rate
- This estimate is very accurate in most situations
- 这样计算出来的显然是风险中性世界

# Explanation

- Loss rate at time $t$ is $\lambda(t)(1-R)$
- If the credit spread is compensation for this loss rate it should approximately equal

$$\bar{\lambda}(t)(1-R)$$

# Matching Bond Prices

- For more accuracy we can work forward in time choosing hazard rates that match bond prices
- This is another application of the bootstrap method

# The Risk-Free Rate

- The risk-free rate when credit spreads and default probabilities are estimated is usually assumed to be the LIBOR/swap rate (or sometimes 10 bps below the LIBOR/swap rate)
- Asset swaps provide a direct estimates of the spread of bond yields over swap rates

# Real World vs Risk-Neutral Default Probabilities

- The default probabilities backed out of bond prices or credit default swap spreads are risk-neutral default probabilities
- The default probabilities backed out of historical data are real-world default probabilities

# A Comparison

- Calculate 7-year default intensities from the Moody's data, 1970-2012, (These are real world default probabilities)
- Use Merrill Lynch data to estimate average 7-year default intensities from bond prices, 1996 to 2007 (these are risk-neutral default intensities)
- Assume a risk-free rate equal to the 7-year swap rate minus 10 basis points

# Data from Moody's and Merrill Lynch

| | Cumulative 7-year default probability (Moody's: 1970-2012) | Average bond yield spread in bps* (Merrill Lynch: 1996 to June 2007) |
|---|---|---|
| Aaa | 0.247% | 35.74 |
| Aa | 0.621% | 43.67 |
| A | 1.441% | 68.68 |
| Baa | 2.927% | 127.53 |
| Ba | 14.117% | 280.28 |
| B | 32.747% | 481.04 |
| Caa | 58.302% | 1103.70 |

*The benchmark risk-free rate for calculating spreads is assumed to be the swap rate minus 10 basis points. Bonds are corporate bonds with a life of approximately 7 years.

# Real World vs Risk Neutral Hazard Rates

| Rating | Historical hazard rate[1] % per annum | Hazard rate from bond prices[2] (% per annum) | Ratio | Difference |
|---|---|---|---|---|
| Aaa | 0.04 | 0.60 | 17.0 | 0.56 |
| Aa | 0.09 | 0.73 | 8.2 | 0.64 |
| A | 0.21 | 1.15 | 5.5 | 0.94 |
| Baa | 0.42 | 2.13 | 5.0 | 1.71 |
| Ba | 2.27 | 4.67 | 2.1 | 2.50 |
| B | 5.67 | 8.02 | 1.4 | 2.35 |
| Caa | 12.50 | 18.39 | 1.5 | 5.89 |

[1] Calculated as $-[\ln(1-d)]/7$ where $d$ is the Moody's 7 yr default rate. For example, in the case of Aaa companies, $d$=0.00247 and $-\ln(0.99753)/7=0.0004$ or 4bps. For investment grade companies the historical hazard rate is approximately $d/7$.

[2] Calculated as $s/(1-R)$ where $s$ is the bond yield spread and $R$ is the recovery rate (assumed to be 40%).

# Average Risk Premiums Earned By Bond Traders

| Rating | Bond Yield Spread over Treasuries (bps) | Spread of risk-free rate over Treasuries (bps)[1] | Spread to compensate for historical default rate (bps)[2] | Extra Risk Premium (bps) |
|---|---|---|---|---|
| Aaa | 78 | 42 | 2 | 34 |
| Aa | 86 | 42 | 5 | 39 |
| A | 111 | 42 | 12 | 57 |
| Baa | 169 | 42 | 25 | 102 |
| Ba | 322 | 42 | 130 | 150 |
| B | 523 | 42 | 340 | 141 |
| Caa | 1146 | 42 | 750 | 323 |

[1] Equals average spread of our benchmark risk-free rate over Treasuries.

[2] Equals historical hazard rate times (1-$R$) where $R$ is the recovery rate. For example, in the case of Baa, 25bps is 0.6 times 42bps.

# Possible Reasons for the Extra Risk Premium (The third reason is the most important)

- Corporate bonds are relatively illiquid
- The subjective default probabilities of bond traders may be much higher than the estimates from Moody's historical data
- **Bonds do not default independently of each other. This leads to systematic risk that cannot be diversified away.**
- Bond returns are highly skewed with limited upside. The non-systematic risk is difficult to diversify away and may be priced by the market

# Which World Should We Use?

- We should use risk-neutral estimates for valuing credit derivatives and estimating the present value of the cost of default
- We should use real world estimates for calculating credit VaR and scenario analysis

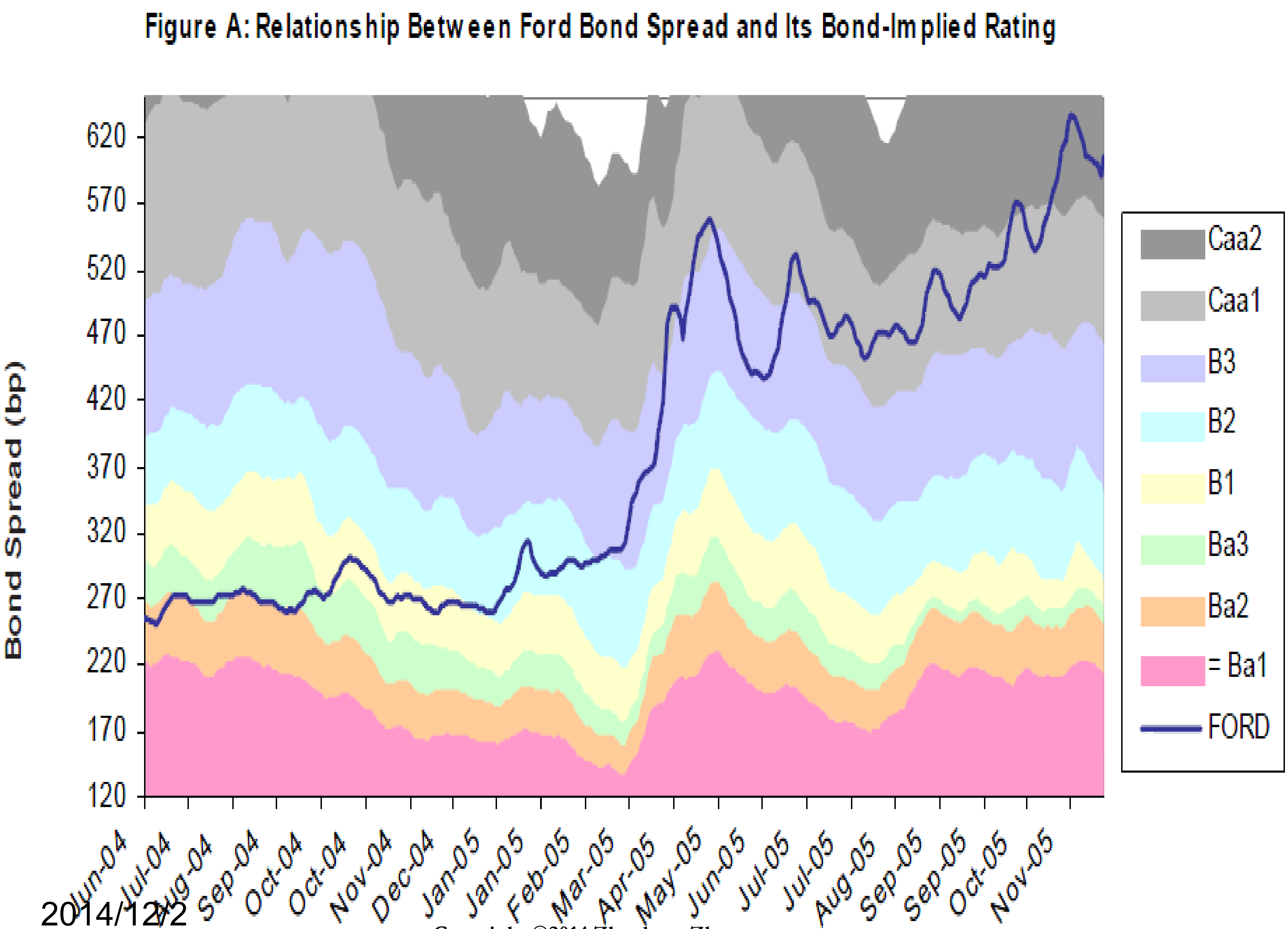
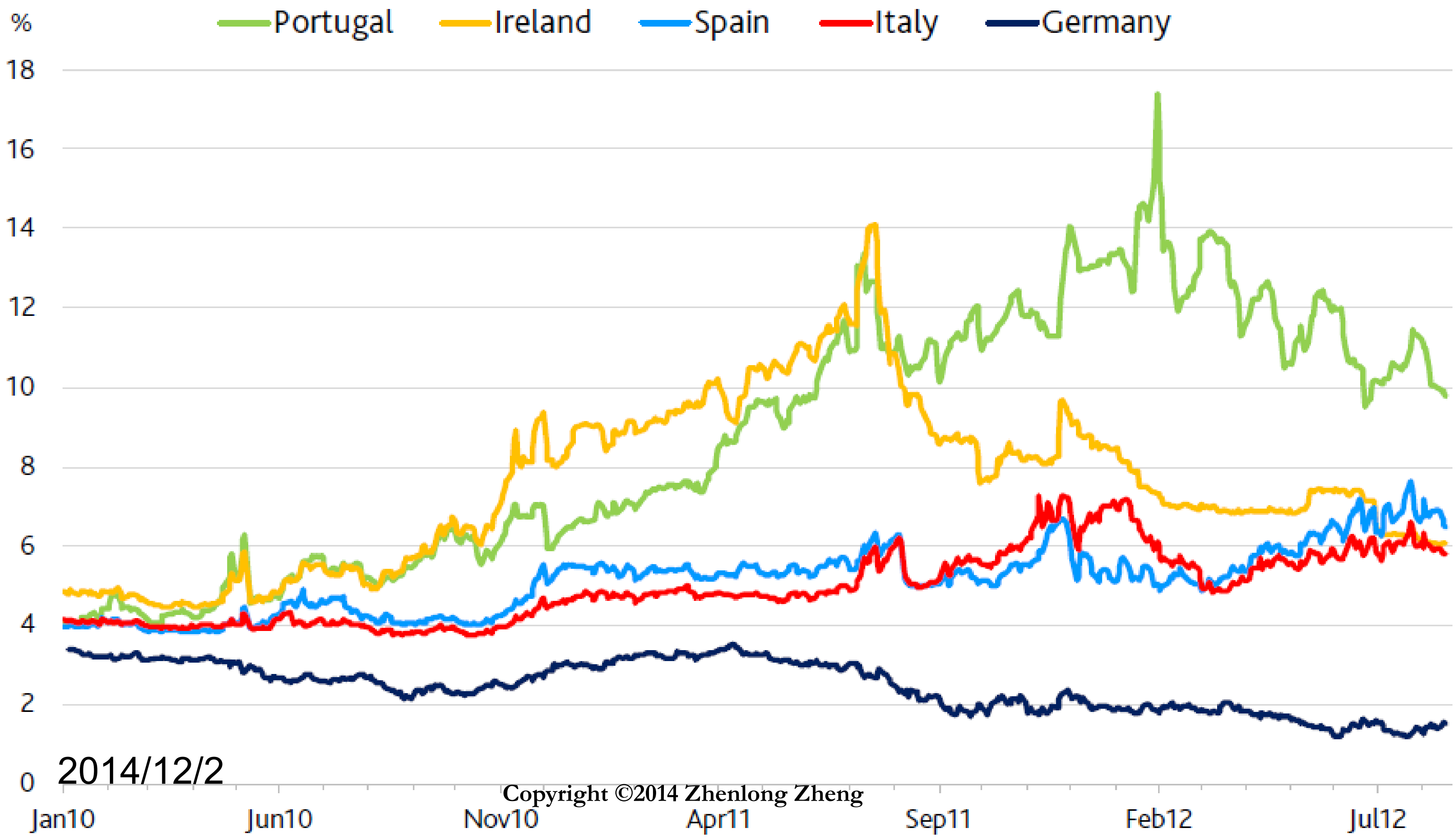


Figure A: Relationship Between Ford Bond Spread and Its Bond-Implied Rating
Bond Spread (bp)
620
570
520
470
420
370
320
270
220
170
120
Jun-04
Jul-04
Aug-04
Sep-04
Oct-04
Oct-04
Nov-04
Dec-04
Jan-05
Jan-05
Feb-05
Mar-05
Apr-05
May-05
Jun-05
Jul-05
Jul-05
Aug-05
Sep-05
Sep-05
Oct-05
Nov-05
Caa2
Caa1
B3
B2
B1
Ba3
Ba2
= Ba1
FORD

# 欧债危机国家10年国债收益率

Figure 3. 10-year generic government bond yields of selected countries



# Using Equity Prices: Merton's Model

- Merton's model regards the equity as an option on the assets of the firm
- In a simple situation the equity value is

$$\max(V_T - D, 0)$$

where $V_T$ is the value of the firm and $D$ is the debt repayment required

# Equity vs. Assets

The Black-Scholes-Merton option pricing model enables the value of the firm's equity today, $E_0$, to be related to the value of its assets today, $V_0$, and the volatility of its assets, $\sigma_V$

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}; \quad d_2 = d_1 - \sigma_V \sqrt{T}$$

# Volatilities

$$\sigma_E E_0 = \frac{\partial E}{\partial V}\sigma_V V_0 = N(d_1)\sigma_V V_0$$

This equation together with the option pricing relationship enables $V_0$ and $\sigma_V$ to be determined from $E_0$ and $\sigma_E$

# Example

- A company's equity is \$3 million and the volatility of the equity is 80%
- The risk-free rate is 5%, the debt is \$10 million and time to debt maturity is 1 year
- Solving the two equations yields $V_0$=12.40 and $\sigma_v$=21.23%
- The probability of default is $N(-d_2)$ or 12.7%

# The Implementation of Merton's Model (e.g. Moody's KMV)

- Moody利用股票可视为公司资产期权这一思想计算出风险中性世界的违约距离（如图所示），之后再利用其拥有的海量历史违约数据库，建立起风险中性违约距离与现实世界违约率之间的对应关系，从而得到预期违约频率（Expected Default Frequency, EDF），作为违约概率的预测指标。
- 下图2就是Moody公司用这种方法计算出来的贝尔斯登预期违约频率时间序列。从图上可以看出，在2008年3月14日贝尔斯登被摩根大通接管前后，其预期违约频率最高飙升到80%左右。可见，从股票价格中提炼出来的违约概率具有很强的信息功能。

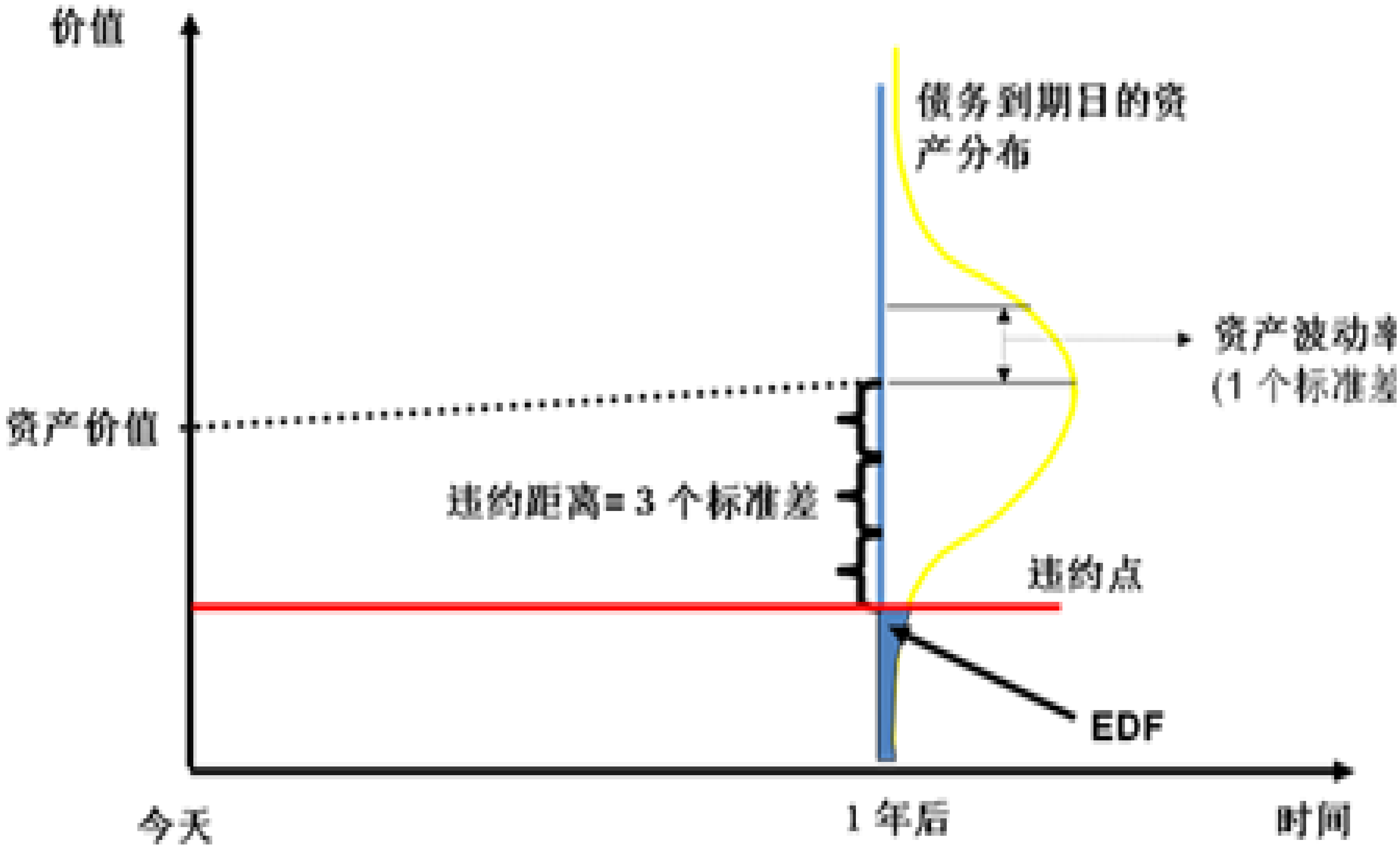



图1 穆迪的预期违约距离

资料来源: Moody's Creditedge。

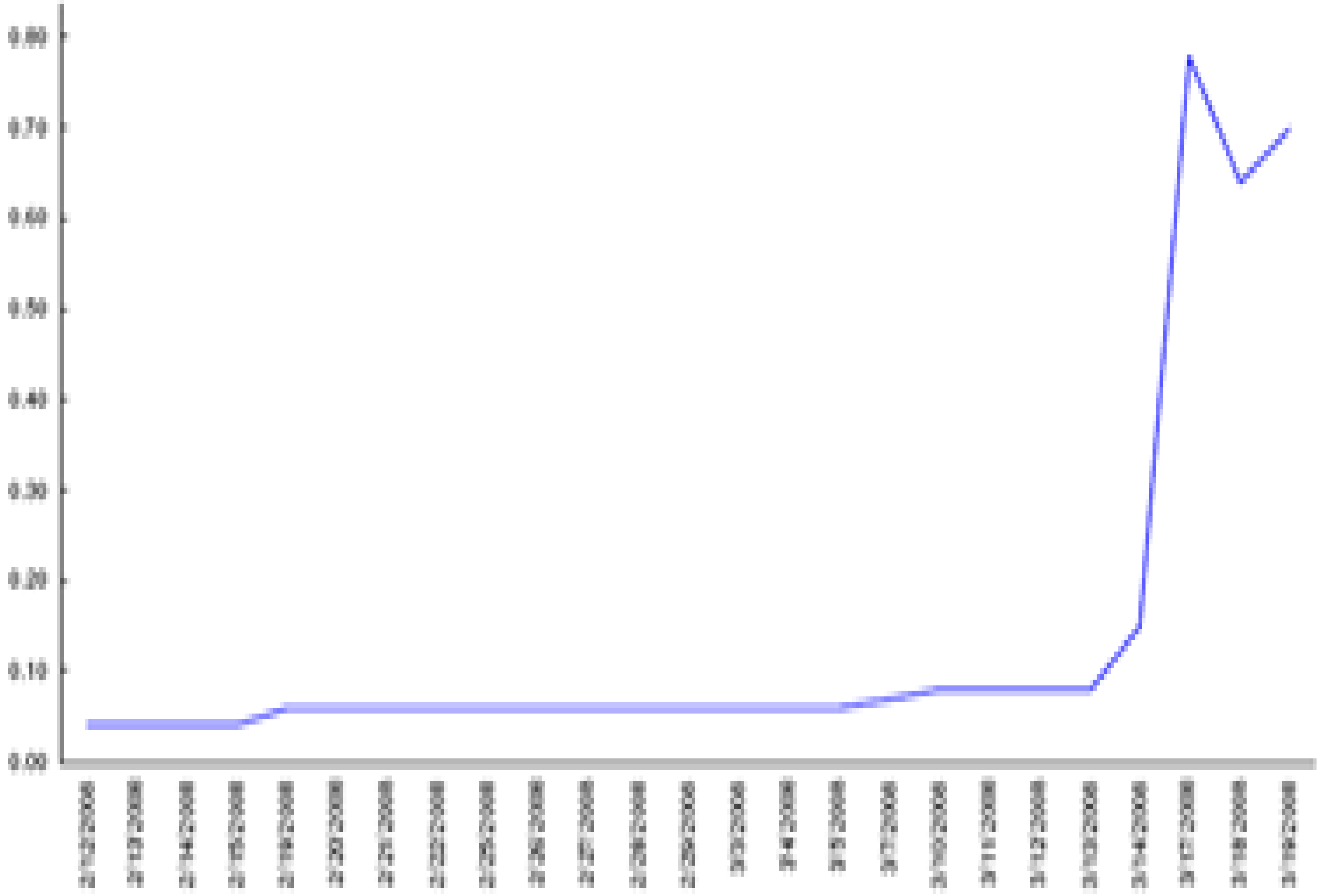



图2 贝尔斯登的预期违约频率

资料来源：Moody's Creditedge。

# 从期权价格中可以推导出风险中性违约概率

由于股票可以看作是公司资产的期权，这样股票期权就可以视为期权的期权，其价格可以表达为：

$$C_0^i = e^{-rT}\int_{V_T=0}^{\infty}\max(V_T - D - K_i;0)f(V_T)dV_T = e^{-rT}\int_{V_T=D+K_i}^{\infty}(V_T - D - K_i)f(V_T)dV_T \quad (10)$$

其中 $C_0^i$ 表示当前时刻协议价格为 $K_i$、期限为 T 的看涨期权价格，$V_T$ 表示 T 时刻的公司价值，D 表示 T 时刻公司债务，$f(V_T)$ 表示 T 时刻公司价值的风险中性概率密度。

运用最大熵的办法(Capuano, 2008)就可以从公司同期限的所有期权价格中估计出 $f(V_T)$ 和 D：

$$\min_{D}\left\{\min_{f(V_T)}\int_{V_T=0}^{\infty} f(V_T)\log\left[\frac{f(V_T)}{f^0(V_T)}\right]dV_T\right\} \quad (11)$$

其中 $f^0(V_T)$ 表示先验概率密度。

# 从期权价格中可以推导出风险中性违约概率

- 运用上述方法，我们就可根据2008年3月14日贝尔斯登将于2008年3月22日到期的期权价格，计算出贝尔斯登的风险中性违约概率和公司价值的概率分布（如图所示）。贝尔斯登于2008年3月14日被摩根大通接管。图显示，市场对贝尔斯登一周后的命运产生巨大分歧，公司价值大涨大跌的概率远远大于小幅变动的概率，这样的分布与正常情况的分布有天壤之别。可见期权价格可以让我们清楚地看出市场在非常时期对未来的特殊看法。

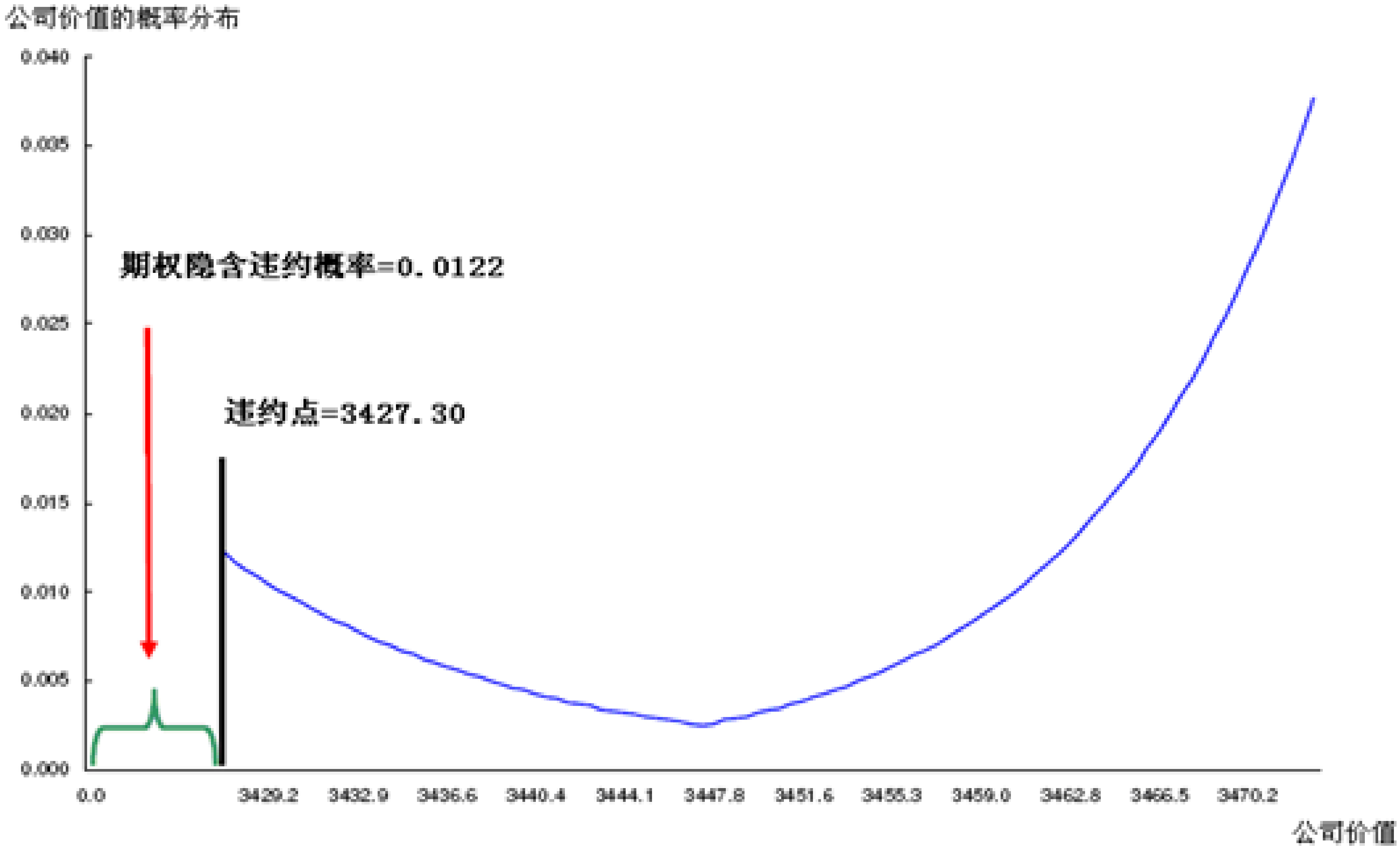
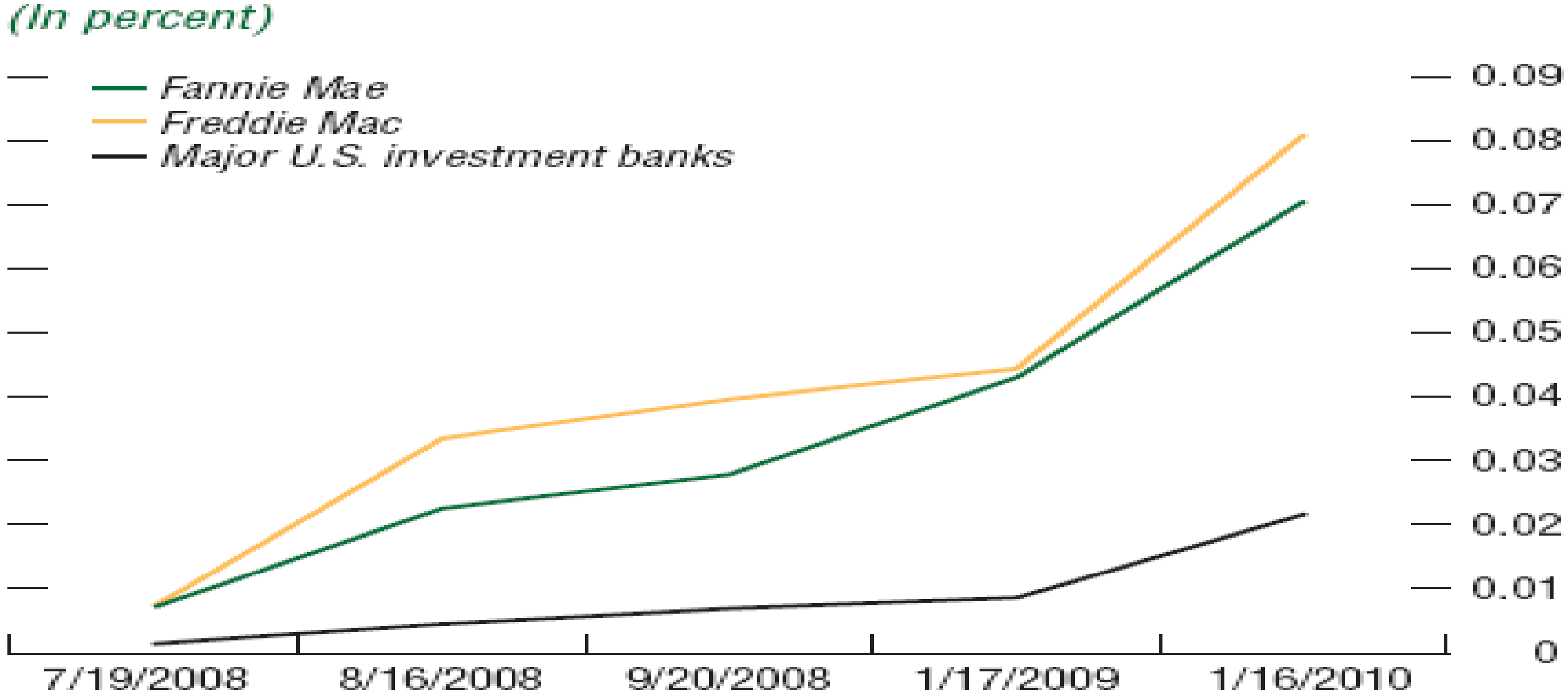



**图 13 贝尔斯登风险中性违约概率和公司价值概率分布（2008 年 3 月 14 日）**

**资料来源：Capuano（2008）。**

# 用其他到期日的期权价格
# 就可以算出违约概率的期限结构




Sources: Bloomberg L.P.; IMF staff estimates; see Capuano (2008) for further details.

# 风险中性违约概率

- 风险中性违约概率虽然不同于现实概率，但其变化可以反映现实世界违约概率的变化。在金融危机时期，它可能比信用违约互换（CDS）的价差能更敏感地反映出违约概率的变化（如图所示）。在贝尔斯登于2008年3月14日被接管前后，根据上述方法计算出来的风险中性概率每天的变化比CDS的价差更敏感。这是因为在金融危机期间，金融机构自身的信用度大幅降低，造成在场外（OTC）市场交易的CDS交易量急剧萎缩，价差大幅扩大，信号失真。

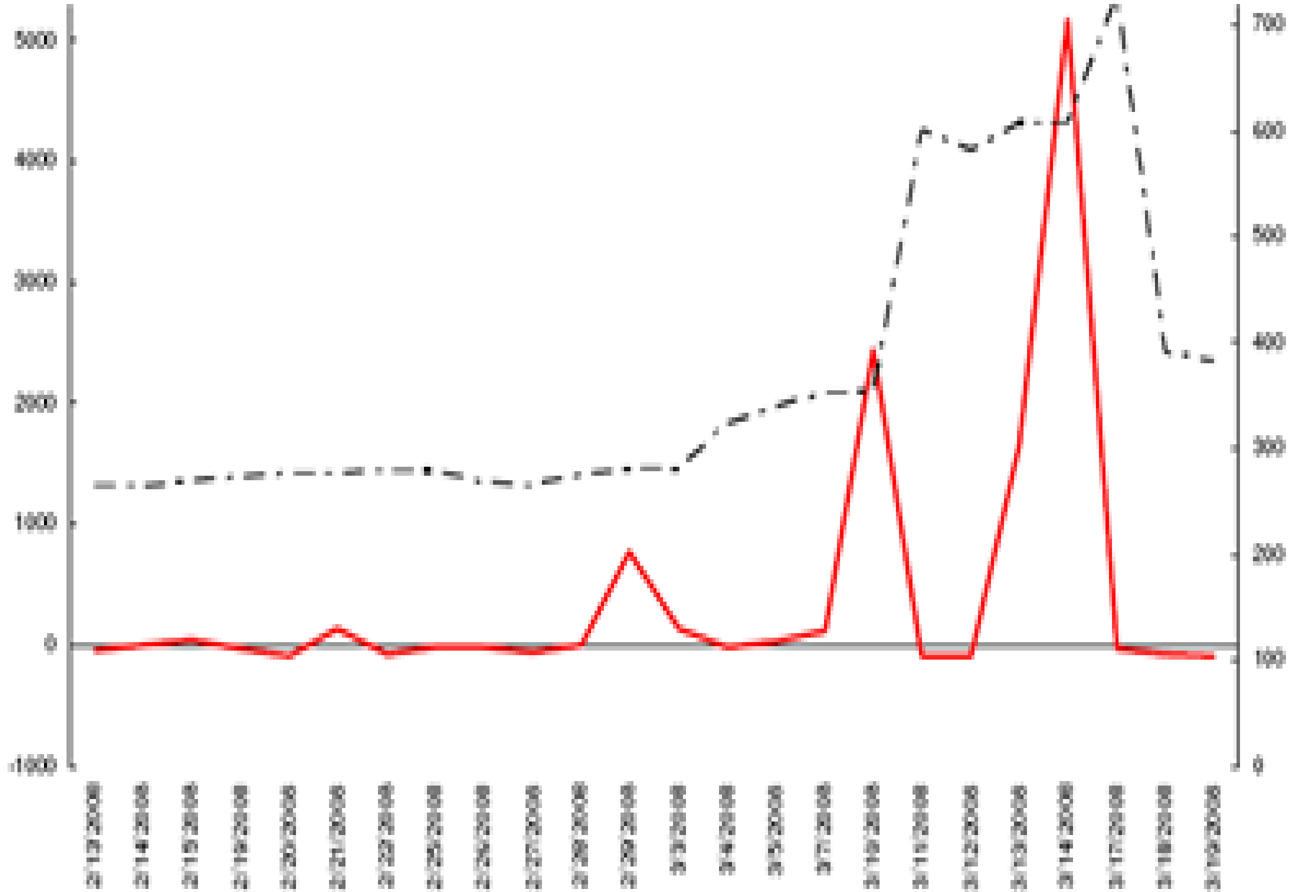



图 14　期权隐含风险中性违约概率与 CDS 价差

注：图中实线表示期权隐含的风险中性概率每天的变化幅度，左坐标轴的单位为%；虚线表示 CDS 价差，右坐标轴的单位为基点。

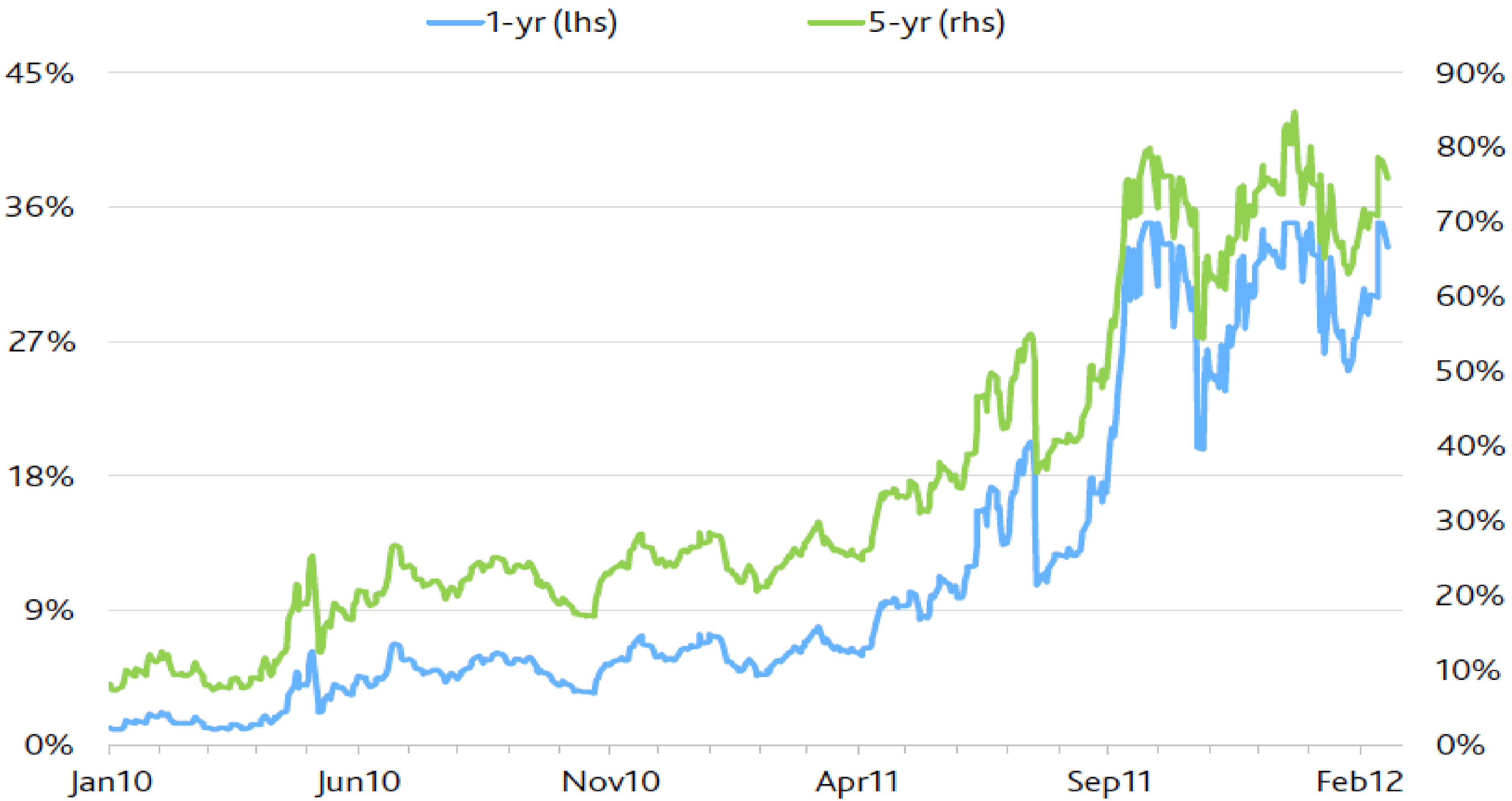
资料来源：Capuano（2008）。

# 信用违约互换（CDS）

- 从CDS的Spread中可以直接估计出风险中性违约概率。由于债券价格会受流动性和税收因素影响，因此从CDS价格提取的违约概率可能更准确。

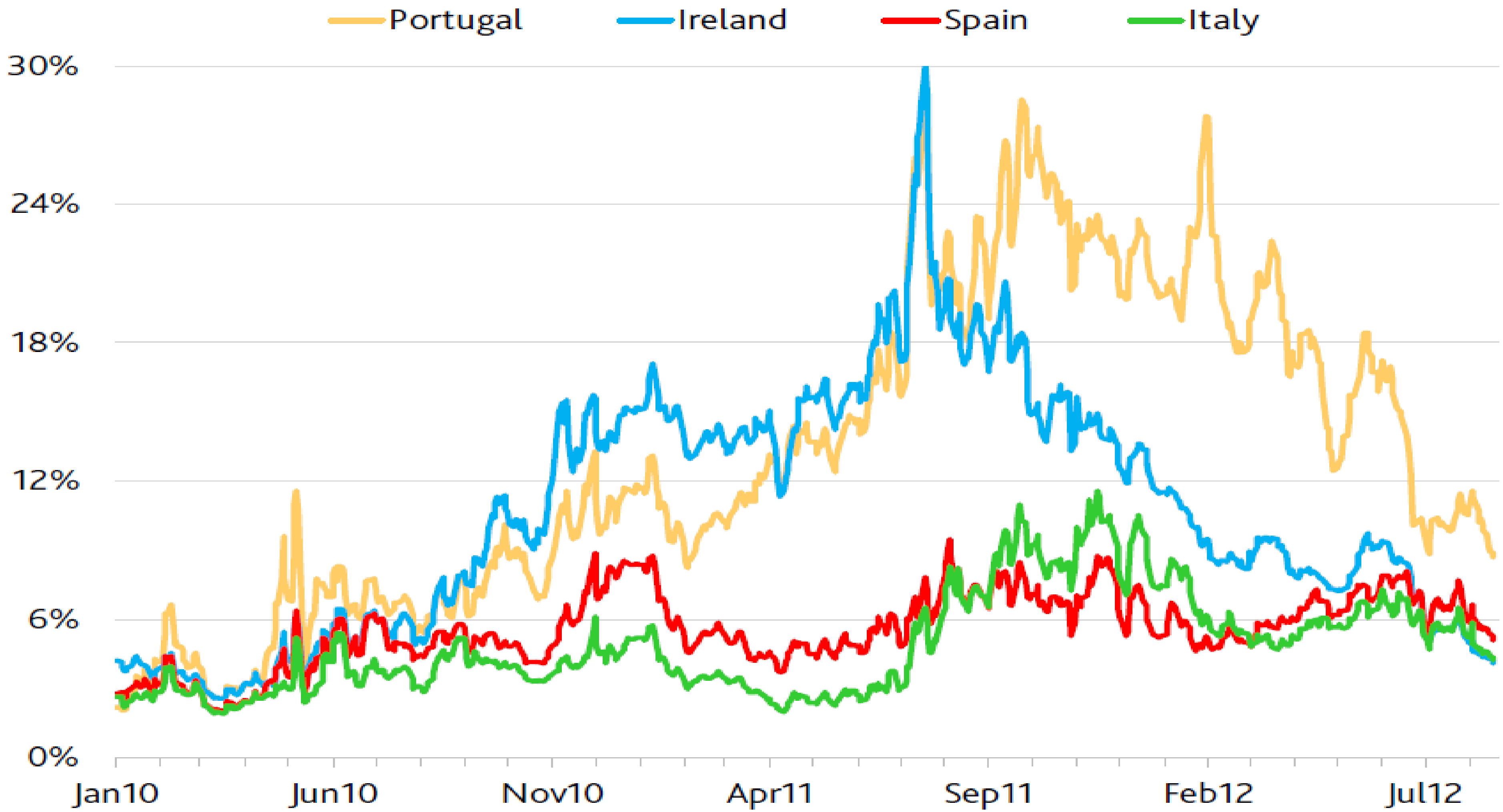
# 希腊1年期和5年期CDS隐含的累积违约概率

Figure 2. One- and five-year cumulative CDS-implied EDF™ metric of Greece

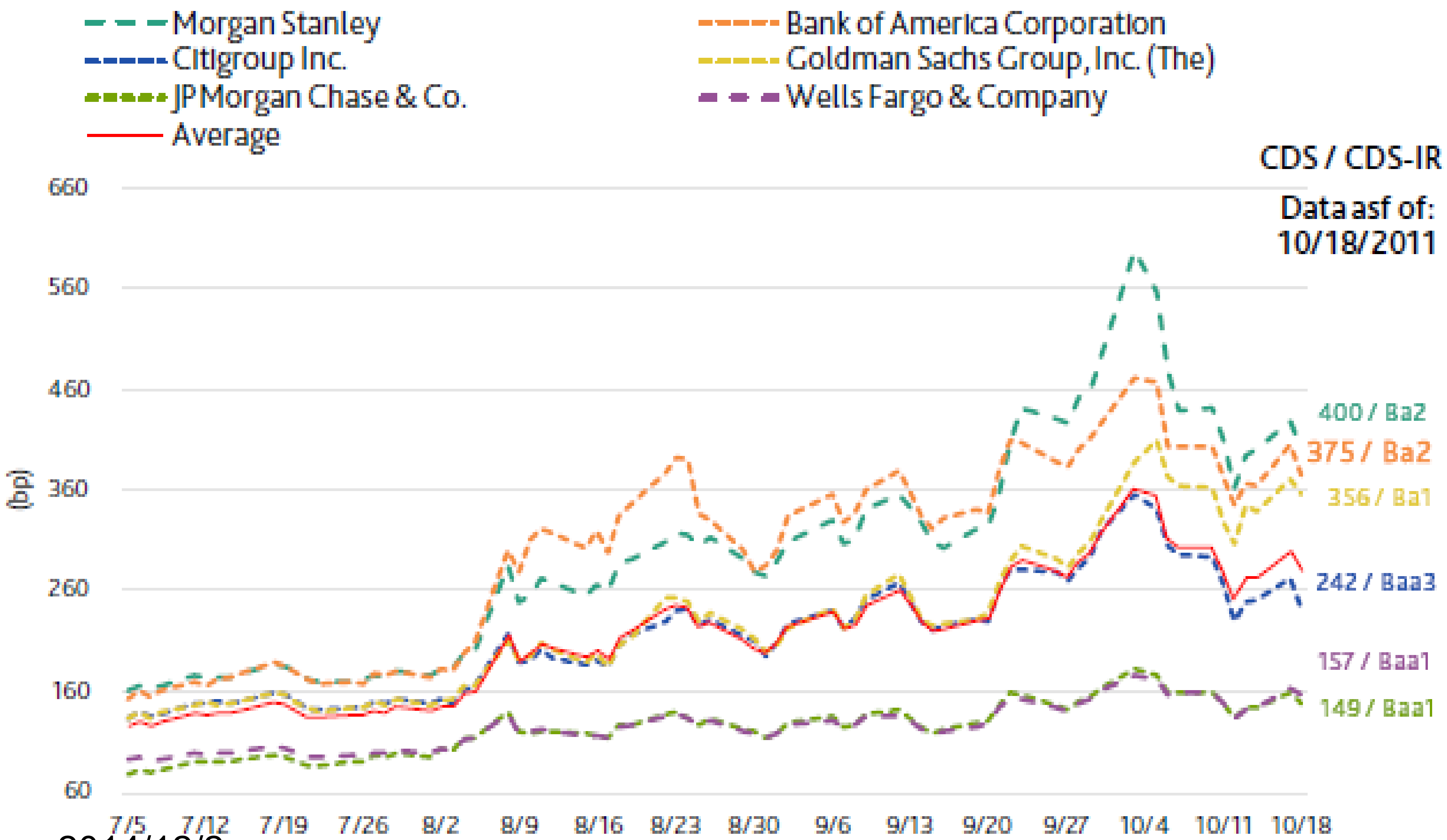


# 欧债危机国家5年期CDS隐含的预期违约概率

Figure 1. Five-year cumulative EDF™ measure of selected sovereign issuers



# 美国主要银行的市场隐含评级



# 机构评级PK市场隐含评级

- 机构评级的依据：基本面信息、统计信息、会计信息（统称统计信息）
- 市场隐含评级的依据：市场隐含信息

# 统计信息与隐含信息的对比

- 统计信息：滞后、水分、反映历史
- 隐含信息：及时、真实、反映未来

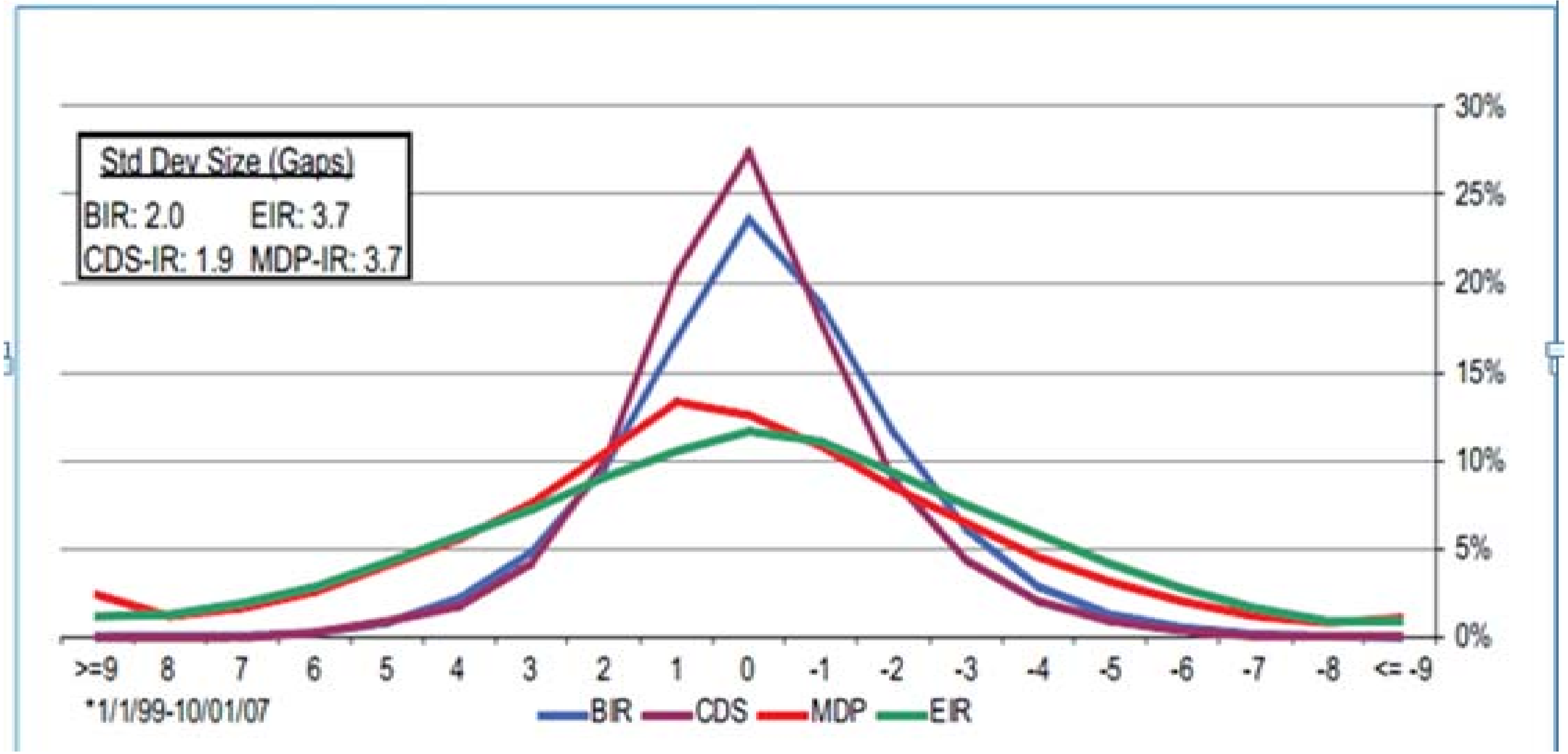
# 差异多大？（1）

| Bank | Moody's Sr. Rating | 5-Year CDS Spread | CDS Implied Rating | CDS Implied Gap | Bond Implied Rating | Bond Implied Gap | Equity Implied Rating | Equity Implied Gap | Expected Default Frequency |
|---|---|---|---|---|---|---|---|---|---|
| Citigroup Inc. | A3 | 242 | Baa3 | -3 | Baa3 | -3 | Ba1 | -4 | 0.47% |
| Wells Fargo & Company | A2 | 157 | Baa1 | -2 | Baa1 | -2 | Ba1 | -5 | 0.50% |
| Goldman Sachs Group, Inc. (The) | A1 | 356 | Ba1 | -6 | Ba1 | -6 | Ba2 | -7 | 0.74% |
| Bank of America Corporation | Baa1 | 375 | Ba2 | -4 | Ba2 | -4 | B2 | -7 | 3.16% |
| Morgan Stanley | A2 | 400 | Ba2 | -6 | Ba2 | -6 | Ba3 | -7 | 1.11% |
| JPMorgan Chase & Co. | Aa3 | 149 | Baa1 | -4 | Baa1 | -4 | Ba2 | -8 | 0.71% |

*As of 10/18/11*

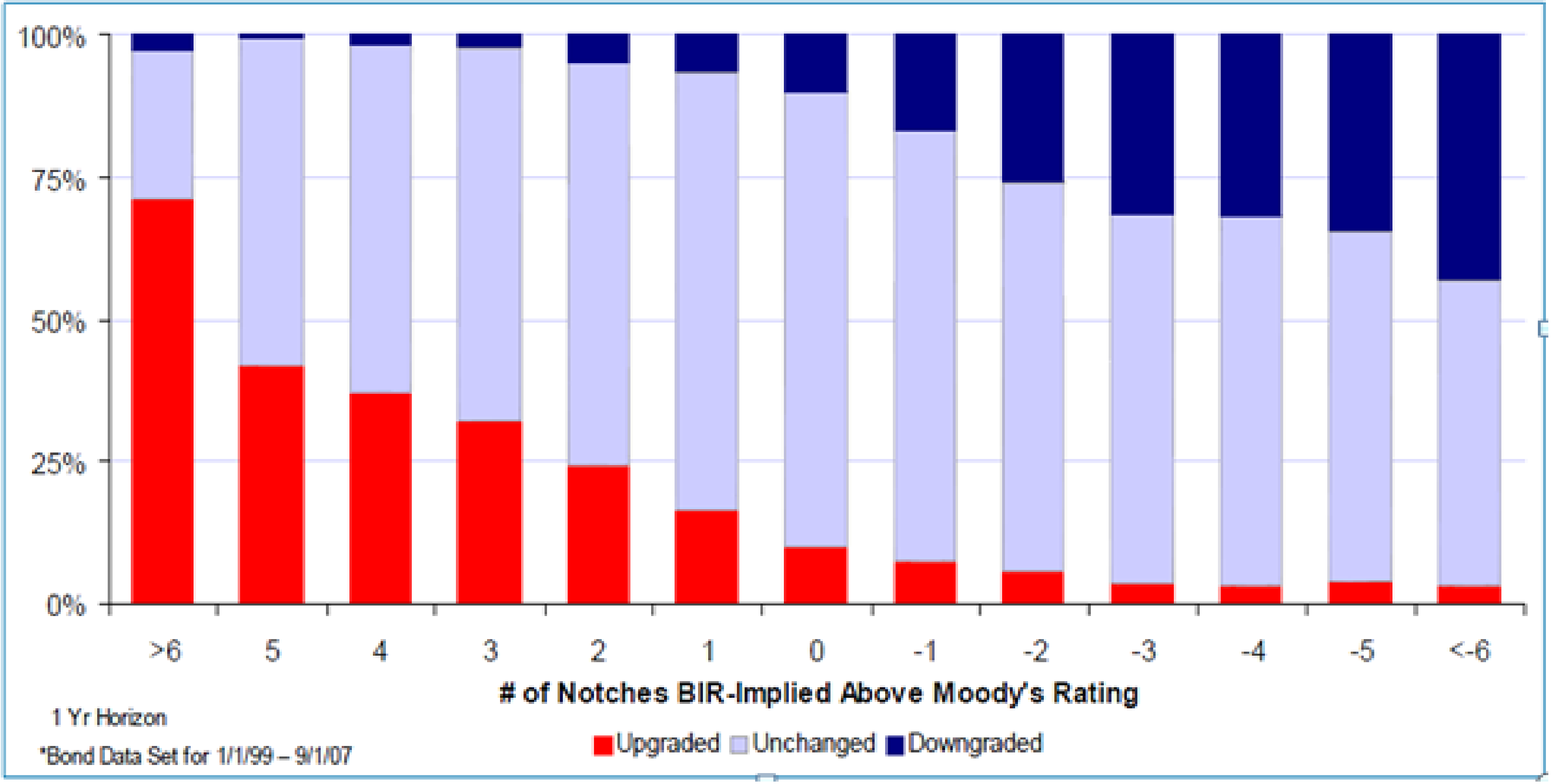
*Source: Markit; Moody's Analytics*

# 差异多大？（2）
# 评级差异的分布（1999－2007）



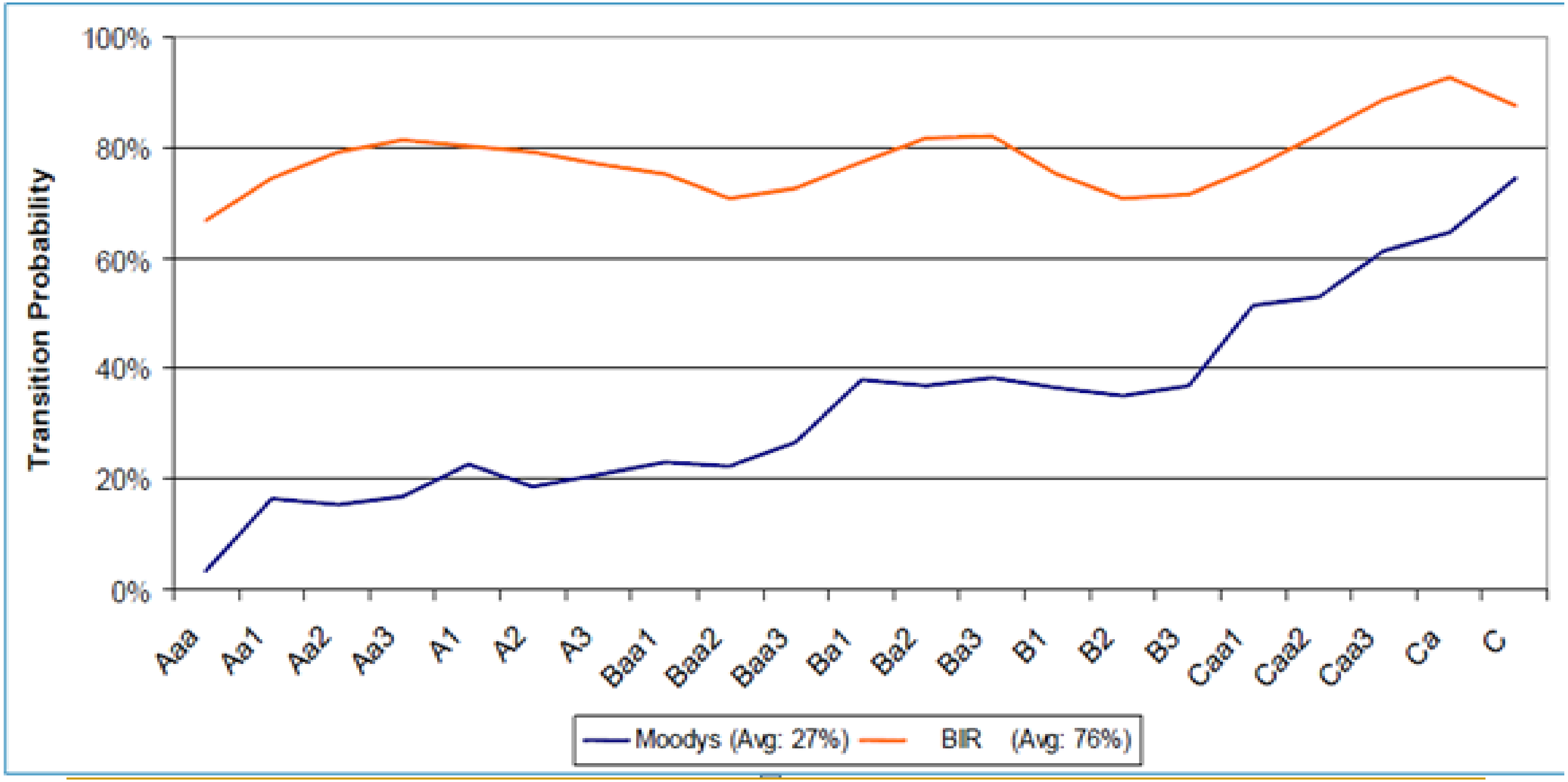
# 谁引领谁？（1）
# 穆迪评级与市场隐含评级：希腊

# 谁引领谁？（2）
# 穆迪评级变动频率与评级差距的关系



# 谁更灵敏？

## 穆迪评级与债券隐含评级1年转移概率比较



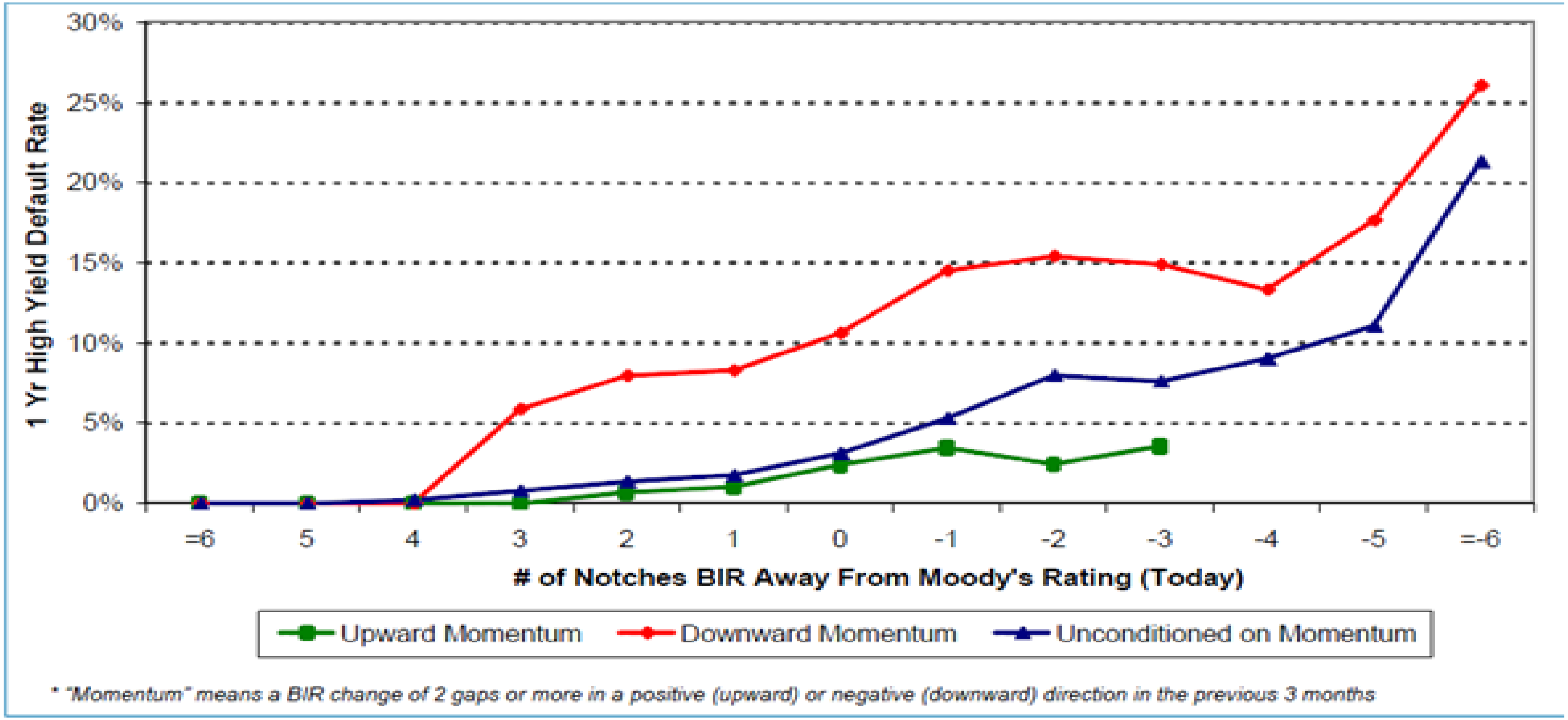
# 谁更准确？（1）
# 1年违约比率与评级差距的关系

| Moody's Rating | Default Rates 1999-2007 | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | All Issuers | Issuers Grouped by their Ratings Gaps | | | | | | |
| | | 3 | 2 | 1 | 0 | -1 | -2 | -3 |
| Baa1 | 0.3% | 0.1% | 0.2% | 0.2% | 0.1% | 0.5% | 0.4% | 0.7% |
| Baa2 | 0.3% | 0.0% | 0.0% | 0.3% | 0.1% | 0.0% | 0.2% | 0.6% |
| Baa3 | 0.4% | 0.0% | 0.1% | 0.0% | 0.1% | 0.6% | 0.6% | 1.3% |
| Ba1 | 0.8% | 0.0% | 0.0% | 0.1% | 0.2% | 0.1% | 0.7% | 0.9% |
| Ba2 | 1.6% | 0.0% | 0.0% | 0.2% | 0.4% | 0.5% | 0.7% | 3.5% |
| Ba3 | 1.0% | 0.0% | 0.0% | 0.1% | 0.7% | 1.0% | 0.8% | 2.2% |
| B1 | 2.4% | 0.4% | 0.1% | 0.2% | 0.6% | 2.5% | 6.6% | 8.2% |
| B2 | 4.7% | 0.2% | 0.6% | 1.1% | 2.9% | 5.7% | 13.1% | 19.7% |
| B3 | 7.5% | 0.7% | 1.8% | 1.3% | 4.4% | 9.8% | 19.9% | 25.7% |
| Caa1 - C | 19.7% | 6.0% | 11.3% | 13.0% | 16.9% | 24.2% | 35.6% | 29.7% |

*Bond dataset

# 谁更准确？（2）

## 高收益债券违约概率与评级差距及债券隐含评级动量的关系



# Credit Risk in Derivatives Transactions (page 531-534)

- Three cases
  - Contract always an asset
  - Contract always a liability
  - Contract can be an asset or a liability

# CVA

- Credit value adjustment (CVA) is the amount by which a dealer must reduce the total value of transactions with a counterparty because of counterparty default risk

# The CVA Calculation

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Time | $0$ | $t_1$ | $t_2$ | $t_3$ | $t_4$ | ..................... | $t_n=\mathrm{T}$ |
| Default probability for counterparty | | $q_1$ | $q_2$ | $q_3$ | $q_4$ | ..................... | $q_n$ |
| PV of expected loss given default | | $v_1$ | $v_2$ | $v_3$ | $v_4$ | ..................... | $v_n$ |

$$\text{CVA} = \sum_{i=1}^{n} q_i v_i$$

# Calculation of $q_i$'s

- Default probabilities are calculated from credit spreads

$$q_i = \exp\left(-\frac{s(t_{i-1})t_{i-1}}{1-R}\right) - \exp\left(-\frac{s(t_i)t_i}{1-R}\right)$$

# Calculation of $v_i$'s

- The $v_i$ are calculated by simulating the market variables underlying the portfolio in a risk-neutral world
- If no collateral is posted the loss on a particular simulation trial during the $i$th interval is the PV of $(1-R)\max(V_i, 0)$ where $V_i$ is the value of the portfolio at the mid point of the interval
- $v_i$ is the average of the losses across all simulation trials

# Collateral

- It is usually assumed that the collateral is posted as agreed, and returned as agreed, until $N$ days before a default. The $N$ days are referred to as the "cure period" or "margin period at risk." Usually $N$ is 10 or 20.
- Suppose that that a portfolio is fully collateralized with no initial margin and its value moves in favor of the dealer during the cure period. Then $v_i$ is positive because
  - If the portfolio has a positive value to the dealer at the default time, collateral posted by the counterparty is insufficient
  - If the portfolio has a negative value to the dealer at the default time, excess collateral posted by the dealer will not be returned

# Incremental CVA

- Results from Monte Carlo are stored so that the incremental impact of a new trade can be calculated without simulating all the other trades.

# CVA Risk

- The CVA for a counterparty can be regarded as a complex derivative
- Increasingly, dealers are managing it like any other derivative
- Two sources of risk:
  - Changes in counterparty spreads
  - Changes in market variables underlying the portfolio

# Wrong Way/Right Way Risk

- Simplest assumption is that probability of default $q_i$ is independent of net exposure $v_i$.
- Wrong-way risk occurs when $q_i$ is positively dependent on $v_i$
- Right-way risk occurs when $q_i$ is negatively dependent on $v_i$

# DVA

- Debit (or debt) value adjustment (DVA) is an estimate of the cost to the counterparty of a default by the dealer
- Same formulas apply except that $v$ is counterparty's loss given a dealer default and $q$ is dealer's probability of default
- Value of transactions with counterparty = No default value – CVA + DVA

# DVA continued

- What happens to the reported value of transactions as dealer’s credit spread increases?

# Credit Risk Mitigation

- Netting
- Collateralization
- Downgrade triggers

# Simple Situation

- Suppose portfolio with a counterparty consists of a single uncollateralized transaction that always a positive value to the dealer and provides a payoff at time $T$
- The CVA adjustment has the effect of multiplying the value of the transaction by $e^{-s(T)T}$ where $s(T)$ is the counterparty's credit spread for maturity $T$

# Example 25.5 (page 560)

- A 2-year uncollateralized option sold by a new counterparty to the dealer has a Black-Scholes-Merton value of $3
- Assume a 2 year zero coupon bond issued by the counterparty has a yield of 1.5% greater than the risk free rate
- If there is no collateral and there are no other transactions between the parties, value of option is $3e^{-0.015\times 2}=2.91$

# Uncollateralized Long Forward with Counterparty (page 560)

For a long forward contract that matures at time $T$, the expected exposure at time $t$ is

$$w(t) = \hat{E}(\max[(F_t - K)e^{-r(T-t)}, 0] = e^{-r(T-t)}\hat{E}(\max[(F_t - K), 0]$$
$$= e^{-r(T-t)}\left[F_0 N(d_1(t)) - KN(d_2(t))\right]$$

where

$$d_1(t) = \frac{\ln(F_0 / K) + \sigma^2 t / 2}{\sigma\sqrt{t}} \qquad d_2(t) = d_1(t) - \sigma\sqrt{t}$$

so that $v_i = w(t_i)e^{-rt_i}(1-R)$

where $F_0$ is the forward price today, $K$ is the delivery price, σ is the volatility of the forward price, $T$ is the time to maturity of the forward contract, and $r$ is the risk-free rate

# Example 24.6 (page 561)

- 2 year forward. Current forward price is \$1,600 per ounce. Two one-year intervals
- $K$ = 1,500, $\sigma$ = 20%, $R$ = 0.3, $r$ = 5%
- $t_1 = 0.5$, $t_2 = 1.5$
- Suppose $q_1 = 0.02$ and $q_2 = 0.03$
- $v_1 = 92.67$ and $v_2 = 130.65$
- $\text{CVA} = 0.02 \times 92.67 + 0.03 \times 130.65 = 5.77$
- Value after CVA =

$$(1600-1500)e^{-0.05\times 2} - 5.77 = 84.71$$

# Default Correlation

- The credit default correlation between two companies is a measure of their tendency to default at about the same time
- Default correlation is important in risk management when analyzing the benefits of credit risk diversification
- It is also important in the valuation of some credit derivatives, eg a first-to-default CDS and  CDO tranches.

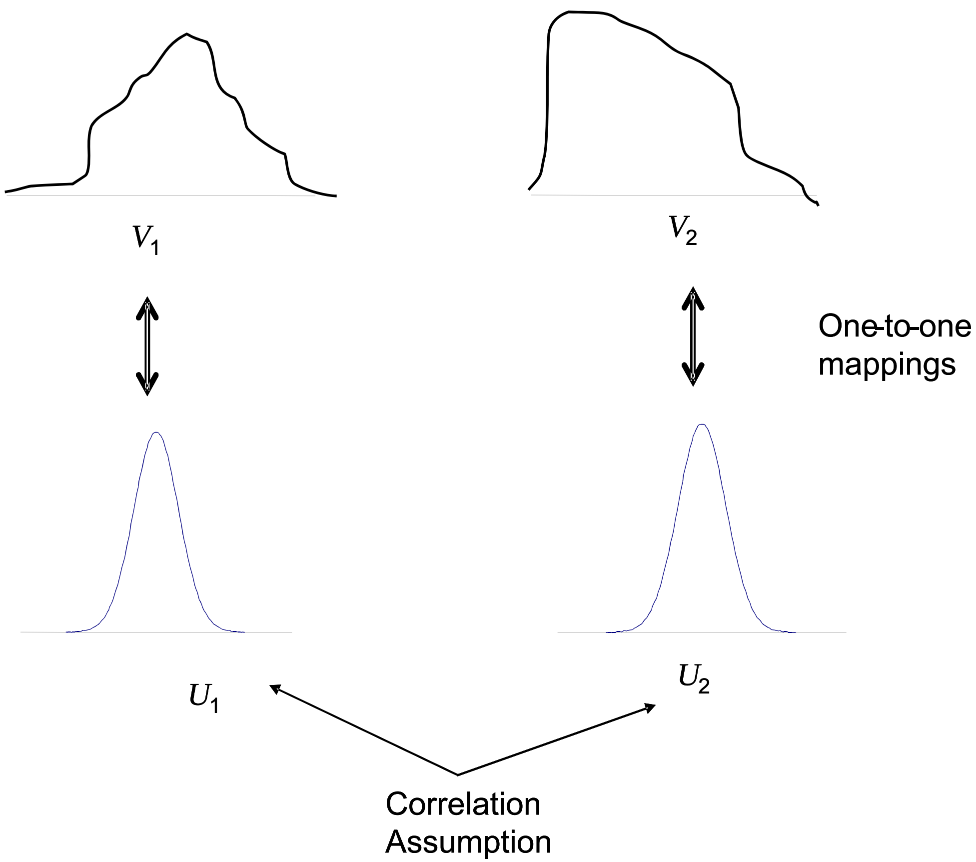
# Measurement

- There is no generally accepted measure of default correlation
- Default correlation is a more complex phenomenon than the correlation between two random variables

# Survival Time Correlation

- Define $t_i$ as the time to default for company $i$ and $Q_i(t_i)$ as the cumulative probability distribution for $t_i$
- The default correlation between companies $i$ and $j$ can be defined as the correlation between $t_i$ and $t_j$
- But this does not uniquely define the joint probability distribution of default times

# The Gaussian Copula Model



# Gaussian Copula Model (continued, page 562-563)

- Define a one-to-one correspondence between the time to default, $t_i$, of company $i$ and a variable $x_i$ by

$$Q_i(t_i) = N(x_i) \quad \text{or} \quad x_i = N^{-1}[Q(t_i)]$$

  where $N$ is the cumulative normal distribution function.
- This is a "percentile to percentile" transformation. The $p$ percentile point of the $Q_i$ distribution is transformed to the $p$ percentile point of the $x_i$ distribution. $x_i$ has a standard normal distribution
- We assume that the $x_i$ are multivariate normal. The default correlation measure, $\rho_{ij}$ between companies $i$ and $j$ is the correlation between $x_i$ and $x_j$

# Example of Use of Gaussian Copula ( page 563)

Suppose that we wish to simulate the defaults for $n$ companies . For each company the cumulative probabilities of default during the next 1, 2, 3, 4, and 5 years are 1%, 3%, 6%, 10%, and 15%, respectively

# Use of Gaussian Copula continued

- We sample from a multivariate normal distribution (with appropriate correlations) to get the $x_i$
- Critical values of $x_i$ are
  $N^{-1}(0.01) = -2.33$, $N^{-1}(0.03) = -1.88$,
  $N^{-1}(0.06) = -1.55$, $N^{-1}(0.10) = -1.28$,
  $N^{-1}(0.15) = -1.04$

# Use of Gaussian Copula continued

- When sample for a company is less than
  -2.33, the company defaults in the first year
- When sample is between -2.33 and -1.88, the company defaults in the second year
- When sample is between -1.88 and -1.55, the company defaults in the third year
- When sample is between -1,55 and -1.28, the company defaults in the fourth year
- When sample is between -1.28 and -1.04, the company defaults during the fifth year
- When sample is greater than -1.04, there is no default during the first five years

# A One-Factor Model for the Correlation Structure

$$x_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$

- The correlation between $x_i$ and $x_j$ is $a_i a_j$
- The $i$th company defaults by time $T$ when $x_i < N^{-1}[Q_i(T)]$ or

$$Z_i < \frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1 - a_i^2}}$$

- Conditional on $F$ the probability of this is

$$Q_i(T|F) = N\left\{\frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1 - a_i^2}}\right\}$$

# Credit VaR (page 564-565)

- Can be defined analogously to Market Risk VaR
- A $T$-year credit VaR with an $X$% confidence is the loss level that we are $X$% confident will not be exceeded over $T$ years

## Calculation from a Factor-Based Gaussian Copula Model (equation 24.10, page 565)

- Consider a large portfolio of loans, each of which has a probability of $Q(T)$ of defaulting by time $T$. Suppose that all pairwise copula correlations are $\rho$ so that all $a_i$'s are $\sqrt{\rho}$
- We are $X$% certain that $F$ is less than

$$N^{-1}(1-X) = -N^{-1}(X)$$

- It follows that the VaR is

$$V(X,T) = N\left\{\frac{N^{-1}[Q(T)]+\sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right\}$$

# Example (page 565)

- A bank has \$100 million of retail exposures
- 1-year probability of default averages 2% and the recovery rate averages 60%
- The copula correlation parameter is 0.1
- 99.9% worst case default rate is

$$V(0.999,1) = N\left(\frac{N^{-1}(0.02) + \sqrt{0.1}N^{-1}(0.999) \cdot}{\sqrt{1-0.1}}\right) = 0.128$$

- The one-year 99.9% credit VaR is therefore $100 \times 0.128 \times (1\text{-}0.6)$ or \$5.13 million

# CreditMetrics (page 565-566)

- Calculates credit VaR by considering possible rating transitions
- A Gaussian copula model is used to define the correlation between the ratings transitions of different companies