Stochastic Calculus, Week 8

Equivalent Martingale Measures

Outline

1. Intuition via binomial approximation
2. Cameron-Martin-Girsanov Theorem
3. Application to derivative pricing:
   equivalent martingale measures/risk neutral measures,
   martingale representation theorem
4. Obtaining the Black-Scholes formula once more

Intuition via Binomial Approximation

Let $W_t^n(\omega), t \in [0,T]$ be defined by

$$W_t^n(\omega) = \sqrt{\delta t}\sum_{j=1}^{i} Y_j(\omega), \quad t = i\,\delta t, \quad \delta t = \frac{T}{n},$$

where $\omega = (\omega_1, \ldots, \omega_n)$ represents the combined outcome of n consecutive binary draws, where each draw may be u (up) or d (down); further,

$$Y_i(\omega) = \begin{cases} 1 & \text{if} \ \ \omega_i = u, \\ -1 & \text{if} \ \ \omega_i = d. \end{cases}$$

Let Ω be the set of all possible ω's, let $\mathcal{F}$ be the appropriate σ-field, and let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$, such that the draws are independent under both $\mathbb{P}$ and $\mathbb{Q}$, and

$$\begin{aligned} \mathbb{P}(\omega_i &= u) = p = \frac{1}{2}, \\ \mathbb{Q}(\omega_i &= u) = q = \frac{1}{2}\left(1 - \gamma\sqrt{\delta t}\right), \end{aligned}$$

for some constant γ.

As $n \to \infty$, $W_t^n(\omega) \xrightarrow{d} W_t(\omega)$ (convergence in distribution), where

$$W_t(\omega) = \begin{cases} \text{Standard Brownian motion under } \mathbb{P}, \\ \\ \text{Brownian motion with drift } -\gamma \text{ under } \mathbb{Q}. \end{cases}$$

The measures $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent*, which means that for any even $A \in \mathcal{F}$, we have $\mathbb{P}(A) > 0$ if and only if $\mathbb{Q}(A) > 0$. This requires that γ is such that $0 < q < 1$. If q would be either 0 or 1, the two measures would not be equivalent (why?).

Thus, by changing the measure from $\mathbb{P}$ to $\mathbb{Q}$, we change the drift in W_t from 0 to $-\gamma$.

We can also change the drift by adding γt to $W_t^n(\omega)$. Thus, define

$$\begin{aligned} \tilde{W}_t^n(\omega) &= W_t^n(\omega) + \gamma t \\ &= \sqrt{\delta t} \sum_{j=1}^{i} \tilde{Y}_j(\omega), \quad t = i\,\delta t, \quad \delta t = \frac{T}{n}, \\ \tilde{Y}_i(\omega) &= \begin{cases} 1 + \gamma\sqrt{\delta t} & \text{if} \quad \omega_i = u, \\ -1 + \gamma\sqrt{\delta t} & \text{if} \quad \omega_i = d. \end{cases} \end{aligned}$$

Then $\tilde{W}_t^n(\omega) \xrightarrow{d} \tilde{W}_t(\omega)$, where

$$\tilde{W}_t(\omega) = \begin{cases} \text{Brownian motion with drift } \gamma \text{ under } \mathbb{P}, \\ \\ \text{Standard Brownian motion under } \mathbb{Q}. \end{cases}$$

In summary, we can write down the following table with drifts:

	$\mathbb{P}$	$\mathbb{Q}$
W_t	0	$-\gamma$
$\tilde{W}_t$	γ	0

Cameron-Martin-Girsanov Theorem

Suppose that $W_t(\omega)$ is a $\mathbb{P}$-Brownian motion, let γ_t be adapted to this Brownian motion, and consider the Itô process

$$\begin{aligned} d\tilde{W}_t &= \gamma_t dt + dW_t, \quad \tilde{W}_0 = 0, \\ \tilde{W}_t &= \int_0^t \gamma_s ds + W_t. \end{aligned}$$

Question: does a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ exist, such that $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion?

Answer: (C-M-G theorem): Yes: $\mathbb{Q}$ is characterized by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right).$$

The derivative $d\mathbb{Q}/d\mathbb{P}$ is a so-called *Radon-Nikodym derivative*. In general this is a random variable, such that for any random variable X on $(\Omega, \mathcal{F})$:

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}\left(X\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

It may be best thought of as the ratio of densities of a the process under $\mathbb{Q}$ and $\mathbb{P}$, respectively. This is seen as follows. Suppose that $\gamma_t = \gamma$ (constant), so that

$$W_T \sim \begin{cases} N(0,T) & \text{with density } f_{\mathbb{P}}(w) \quad \text{under } \mathbb{P}, \\ N(-\gamma T, T) & \text{with density } f_{\mathbb{Q}}(w) \quad \text{under } \mathbb{Q}. \end{cases}$$

Then

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(W_T) &= \int_{-\infty}^{\infty} w f_{\mathbb{Q}}(w) dw \\ &= \int_{-\infty}^{\infty} w \frac{f_{\mathbb{Q}}(w)}{f_{\mathbb{P}}(w)} f_{\mathbb{P}}(w) dw \\ &= \mathbb{E}_{\mathbb{P}}\left(W_T \frac{f_{\mathbb{Q}}(W_T)}{f_{\mathbb{P}}(W_T)}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f_{\mathbb{Q}}(W_T)}{f_{\mathbb{P}}(W_T)} &= \frac{\frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(W_T + \gamma T)^2\right)}{\frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{1}{2T} W_T^2\right)} \\ &= \exp\left(-\frac{1}{2T}(W_T + \gamma T)^2 + \frac{1}{2T} W_T^2\right) \\ &= \exp\left(-\gamma W_T - \frac{1}{2}\gamma^2 T\right). \\ &= \exp\left(-\int_0^1 \gamma_t dW_t - \frac{1}{2}\int_0^1 \gamma_t^2 dt\right). \end{aligned}$$

Note that $\zeta_T = d\mathbb{Q}/d\mathbb{P}$ is a random variable. We can create a martingale ζ_t from that as usual: $\zeta_t = \mathbb{E}_{\mathbb{P}}(\zeta_T|\mathcal{F}_t)$, which is the *Radon-Nikodym process*. This satisfies:

$$\mathbb{E}_{\mathbb{Q}}(X_t|\mathcal{F}_s) = \zeta_s^{-1}\mathbb{E}_{\mathbb{P}}\left(\zeta_t X_t|\mathcal{F}_s\right).$$

The C-M-G theorem allows us to change the drift of an Itô process by changing the probability measure. Let W_t be a $\mathbb{P}$-Brownian motion, and X_t be an Itô process with drift μ_t and volatility σ_t under $\mathbb{P}$. Then

$$\begin{aligned} dX_t &= \mu_t dt + \sigma_t dW_t \\ &= \nu_t dt + (\mu_t - \nu_t)dt + \sigma_t dW_t \\ &= \nu_t dt + \sigma_t \left[\frac{\mu_t - \nu_t}{\sigma_t}dt + dW_t\right] \\ &= \nu_t dt + \sigma_t d\tilde{W}_t, \end{aligned}$$

where $\tilde{W}_t = \int_0^t \gamma_s ds + W_t$, with $\gamma_t = (\mu_t - \nu_t)/\sigma_t$. Thus $\tilde{W}_t$ is a Brownian motion under the measure $\mathbb{Q}$ defined by the C-M-G theorem, and hence X_t is an Itô process with drift ν_t and volatility σ_t under $\mathbb{Q}$. Note that we can only change the drift, not the volatility. If X_t would have a different volatility under $\mathbb{P}$ and $\mathbb{Q}$, then these two measures are no longer equivalent.

Application to derivative pricing

Consider a stock price S_t generated by a geometric Brownian motion under $\mathbb{P}$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with W_t a $\mathbb{P}$-Brownian motion.

Remark: the solution is $S_t = S_0 \exp([\mu - \frac{1}{2}\sigma^2]t + \sigma W_t)$. Baxter and Rennie start from $S_t = S_0 \exp(\mu t + \sigma W_t)$, so their μ corresponds to our $\mu - \frac{1}{2}\sigma^2$.

Consider also a cash bond with price $B_t = e^{rt}$, so that

$$dB_t = rB_t dt.$$

Define the discounted stock price $Z_t = B_t^{-1}S_t = e^{-rt}S_t$, so that, by Itô's lemma,

$$\begin{aligned} dZ_t &= e^{-rt}dS_t + S_t de^{-rt} \\ &= e^{-rt}[\mu S_t dt + \sigma S_t dW_t] - re^{-rt}S_t dt \\ &= (\mu - r)Z_t dt + \sigma Z_t dW_t. \end{aligned}$$

Finally, let X be some claim, determined at time T. For a European call option, $X = [S_T - K]^+$. We wish to find the no-arbitrage value of this claim at time $t < T$.

We proceed in the following steps:

1. Find the measure $\mathbb{Q}$ under which Z_t is a martingale:

$$\begin{aligned} dZ_t &= \sigma Z_t \left[\frac{(\mu - r)}{\sigma} dt + dW_t \right] \\ &= \sigma Z_t d\tilde{W}_t, \end{aligned}$$

   where $\tilde{W}_t = \int_0^t \gamma_s ds + W_t$, with $\gamma_t = (\mu - r)/\sigma$. This is the *market price of risk*: the excess expected return required as compensation for one additional unit of standard deviation.

   Thus Z_t is a martingale under $\mathbb{Q}$, defined from the C-M-G theorem for this choice of γ_t. Hence we often refer to $\mathbb{Q}$ as the *equivalent martingale measure*.

   The martingale property for Z_t under $\mathbb{Q}$ implies:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(S_t|\mathcal{F}_s) &= e^{rt}\mathbb{E}_{\mathbb{Q}}(Z_t|\mathcal{F}_s) \\ &= e^{rt} Z_s \\ &= e^{r(t-s)} S_s. \end{aligned}$$

   Thus, under $\mathbb{Q}$, the stock price has an average (continuously compounded) growth rate of r, the risk-free rate. Hence under this measure there is no compensation for risk. Therefore $\mathbb{Q}$ is also called the *risk-neutral measure*.

2. Define $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X|\mathcal{F}_t)$, which is a martingale with respect to $\mathcal{F}_t$ under $\mathbb{Q}$ (just like Z_t).

3. The martingale representation states that because both E_t and Z_t are $\mathbb{Q}$-martingales with respect to $\mathcal{F}_t$,

$$dE_t = \phi_t dZ_t,$$

   for some $\mathcal{F}_t$-adapted ϕ_t.

   Define $\psi_t = E_t - \phi_t Z_t$, and the strategy of holding ϕ_t stocks and ψ_t bonds, so that the value of the portfolio is

$$\begin{aligned} V_t &= \phi_t S_t + \psi_t B_t \\ &= \phi_t S_t + E_t B_t - \phi_t Z_t B_t \\ &= B_t E_t. \end{aligned}$$

   Since $E_T = B_T^{-1} X$, this implies that $V_T = X$, i.e., the strategy is replicating.

What remains to be shown is that it is also self-financing. This follows from $V_t = B_t E_t$, so

$$\begin{aligned} dV_t &= B_t dE_t + E_t dB_t \\ &= \phi_t B_t dZ_t + (\psi_t + \phi_t Z_t) dB_t \\ &= \phi_t d(B_t Z_t) + \psi_t dB_t \\ &= \phi_t dS_t + \psi_t dB_t. \end{aligned}$$

Thus the strategy is self-financing and replicates the claim; that means that, by no-arbitrage, V_t must equal the value of the claim at time t.

In other words, the value of the claim is $V_t = B_t E_t = e^{rt} E_t$, so

$$\boxed{V_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)}$$

This is the discounted expected payoff, with the expectation taken under the risk-neutral measure. This is a very general expression for the value of a claim, that will also hold for, e.g., interest rate derivatives (which are not driven by a geometric Brownian motion). It is the central result that underlies *risk-neutral valuation*.

Obtaining the Black-Scholes formula once more

For a European call option, $X = [S_T - K]^+$. Defining the time to expiration $\tau = T - t$, this means that its value is

$$V_t = e^{-r\tau} \mathbb{E}_{\mathbb{Q}} \left([S_T - K]^+ | \mathcal{F}_t \right).$$

Because $dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$, we have

$$\begin{aligned} S_T &= S_t \exp\left([r - \tfrac{1}{2}\sigma^2]\tau + \sigma[\tilde{W}_T - \tilde{W}_t] \right) \\ &= S_t \exp\left([r - \tfrac{1}{2}\sigma^2]\tau - \sigma\sqrt{\tau} Y \right), \end{aligned}$$

where $Y = -(\tilde{W}_T - \tilde{W}_t)/\sqrt{\tau} \sim N(0,1)$ under $\mathbb{Q}$. The option expires exactly at the money if

$$S_T = S_t \exp\left([r - \tfrac{1}{2}\sigma^2]\tau - \sigma\sqrt{\tau} Y \right) = K,$$

which can be rewritten as

$$Y = \frac{\ln(S_t/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_2.$$

This implies that

$$[S_T - K]^+ = \begin{cases} S_T - K & \text{if } \ Y \leq d_2, \\ 0 & \text{if } \ Y > d_2. \end{cases}$$

Thus

$$
\begin{aligned}
V_t &= e^{-r\tau}\int_{-\infty}^{d_2}[S_t\exp\left([r-\tfrac{1}{2}\sigma^2]\tau-\sigma\sqrt{\tau}y\right)-K]\phi(y)dy \\
&= \int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}S_t\exp\left(-\sigma\sqrt{\tau}y-\tfrac{1}{2}\sigma^2\tau-\tfrac{1}{2}y^2\right)dy \\
&\quad -e^{-r\tau}K\int_{-\infty}^{d_2}\phi(y)dy \\
&= S_t\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\left[y+\sigma\sqrt{\tau}\right]^2\right)dy-e^{-r\tau}K\Phi(d_2) \\
&= S_t\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}v^2\right)dv-e^{-r\tau}K\Phi(d_2) \\
&= S_t\Phi(d_1)-e^{-r\tau}K\Phi(d_2),
\end{aligned}
$$

where $v=y+\sigma\sqrt{\tau}$ and $d_1=d_2+\sigma\sqrt{\tau}$.

This is the Black-scholes formula. Note that $\Phi(d_2)$ may be interpreted as the risk-neutral probability that the option will expire in the money.

Exercises

1. Suppose that $X_t(\omega)$ is an Itô process with volatility $\sigma_t = \sigma X_t$ under $\mathbb{P}$, whereas it is an Itô process with volatility $\sigma_t = \sigma$ under $\mathbb{Q}$. Argue that $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent.
2. Show that
$$\zeta_t := \mathbb{E}_{\mathbb{P}}(\zeta_T|\mathcal{F}_t) = \exp\left(-\int_0^t \gamma_s dW_s - \tfrac{1}{2}\int_0^t \gamma_s^2 ds\right).$$
3. Use risk-neutral valuation to obtain the value of a claim
$$X = \begin{cases} 1 & \text{if} \quad S_T \geq K \\ 0 & \text{if} \quad S_T < K \end{cases}.$$
4. Use risk-neutral valuation to show that the value K of a forward contract is $e^{r(T-t)}S_t$.
5. Obtain, via risk-neutral valuation, the value of a straddle, which is a combination of a call and a put option with the same exercise price and expiration date. Derive also the Greeks for the straddle.