Stochastic Calculus

Week 7

The Black-Scholes-Merton model

Assumptions/ingredients:

- No arbitrage opportunities;
- No transaction costs;
- Stock pays no dividends;
- Riskless borrowing and lending at the same continuously compounded rate r;
- Stock price process $X_t = S_t$: geometric Brownian motion

$$\begin{aligned} dX_t &= cX_t dt + \sigma X_t dB_t, \\ \Leftrightarrow \qquad X_t &= X_0 \exp[(c - \tfrac{1}{2}\sigma^2)t + \sigma B_t]. \end{aligned}$$

  Note: cX_t may be replaced by more general $\mu(t, X_t)$, including possible trend-reversion;

- Cash bond/bank account β_t:

$$\beta_t = \beta_0 e^{rt} \qquad \Leftrightarrow \qquad d\beta_t = r\beta_t dt.$$

- European call option:

$$\begin{aligned} V_T &= [X_T - K]^+ = \max(X_T - K, 0) \\ V_t &= f(t, X_t) = u(T - t, X_t). \end{aligned}$$

- Replicating, self-financing strategy, with $(a_t, b_t) = (\phi_t, \psi_t)$, adapted to B_t:

$$\begin{aligned} V_t &= a_t X_t + b_t \beta_t, \\ dV_t &= a_t dX_t + b_t d\beta_t, \\ V_T &= [X_T - K]^+. \end{aligned}$$

We use the same symbol for the value of the call option, and of the self-financing replicating strategy. This is the no-arbitrage condition: if the call and the strategy have the same terminal value, they should also have the same value at any time $t < T$.

Problem: to find the function $u(\cdot,\cdot)$ and hence V_t, as well as the portfolio weights a_t and b_t.

The solution is to obtain the SDE for $V_t = u(T-t, X_t)$, and equate this to $dV_t = a_t dX_t + b_t d\beta_t$.

In order to apply Itô's lemma, define:

$$\begin{aligned} f_1(t,x) &= \frac{\partial f(t,x)}{\partial t} = \frac{\partial u(T-t,x)}{\partial(T-t)}\frac{d(T-t)}{dt} = -u_1(T-t,x), \\ f_2(t,x) &= \frac{\partial f(t,x)}{\partial x} = u_2(T-t,x), \\ f_{22}(t,x) &= \frac{\partial^2 f(t,x)}{\partial x^2} = u_{22}(T-t,x). \end{aligned}$$

Then:

$$\begin{aligned} dV_t &= [-u_1(T-t,X_t) + \tfrac{1}{2}u_{22}(T-t,X_t)\sigma^2 X_t^2]dt \\ &\quad + u_2(T-t,X_t)dX_t. \end{aligned} \tag{1}$$

For the replicating strategy, we have

$$\begin{aligned} dV_t &= b_t r \beta_t dt + a_t dX_t \\ &= r(V_t - a_t X_t)dt + a_t dX_t. \end{aligned} \tag{2}$$

Equating the dX_t terms of the two processes (1) and (2) gives us the hedge ratio or "delta":

$$a_t = u_2(T-t, X_t) = \frac{\partial V_t}{\partial X_t}.$$

We then equate the dt terms, using this result, and writing $\tau = T - t$ for the first argument of $u(\cdot,\cdot)$, and x for the second:

$$-u_1(\tau,x) + \tfrac{1}{2}u_{22}(\tau,x)\sigma^2 x^2 = ru(\tau,x) - ru_2(\tau,x)x.$$

This is a so-called *partial differential equation* (PDE) for the function $u(\tau,x)$, with *boundary condition* $u(0,x) = [x-K]^+$. If we solve this PDE, i.e., obtain the function $u(\tau,x)$, then we know $V_t = u(T-t,X_t)$, as well as the hedge ratio $a_t = u_2(T-t,X_t)$.

Remark 1: An ordinary differential equation can be written in general as

$$g\left[f(t), \frac{df(t)}{dt}, \frac{d^2f(t)}{dt^2}, \ldots\right] = 0.$$

Its solution, given some boundary condition $f(0) = f_0$, is the function $f(t)$.

On the other hand, a PDE can be written as

$$g\left[u(\tau,x), \frac{\partial u(\tau,x)}{\partial \tau}, \frac{\partial u(\tau,x)}{\partial x}, \frac{\partial^2 u(\tau,x)}{\partial x^2}, \ldots\right] = 0.$$

The boundary condition in this case is the function $u(0,x) = u_0(x)$, and the solution is the function $u(\tau,x)$.

Note that we have transformed the stochastic problem into a deterministic problem.

In general, it is hard to find the solution of such a PDE. However, it turns out that this particular PDE is very close to the “heat equation”, known from physics, for which the solution is known.

Remark 2: The same PDE can be derived from no-aribitrage arguments as follows. Consider a portfolio with value Π_t consisting of a short position in one call option, and a long position of a_t shares. Hence $\Pi_t = -u(T-t, X_t) + a_t X_t$. Then Itô’s lemma gives

$$\begin{aligned} d\Pi_t &= \left[u_1(T-t,X_t) - \tfrac{1}{2}u_{22}(T-t,X_t)\sigma^2 X_t^2\right] dt \\ &\quad + \left[-u_2(T-t,X_t) + a_t\right] dX_t. \end{aligned}$$

When we choose $a_t = u_2(T-t, X_t)$, then the portfolio becomes riskless (no Brownian motion term), and hence should earn the risk-free interest rate. Hence

$$\begin{aligned} & u_1(T-t,X_t) - \tfrac{1}{2}u_{22}(T-t,X_t)\sigma^2 X_t^2 \\ =\;& r\Pi_t \\ =\;& r[-u(T-t,X_t) + u_2(T-t,X_t)X_t]. \end{aligned}$$

Evaluating in $T - t = \tau$ and $X_t = x$ gives the same PDE as before.

One can check that our PDE is solved by

$$\begin{aligned} u(\tau,x) &= x\Phi(g(\tau,x)) - Ke^{-r\tau}\Phi(h(\tau,x)), \\ \Phi(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy, \\ g(\tau,x) &= \frac{\ln(x/K) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ h(\tau,x) &= g(\tau,x) - \sigma\sqrt{\tau}. \end{aligned}$$

Evaluating this in $\tau = T - t$ and $x = X_t$ gives the famous Black-Scholes formula:

$$\begin{aligned} V_t &= X_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \\ d_1 &= \frac{\ln(X_t/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t} = \frac{\ln(X_t/K) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Table of notation

	Mikosch	Baxt.&Ren.	Hull
Brownian motion	B_t	W_t	z
Stock price	X_t	S_t	S
Bond	β_t	B_t	e^{rt}
Drift	c	μ	μ
Call option value	V_t	V_t	c
Delta	a_t	ϕ_t	Δ
Normal distr. fn.	$\Phi(x)$	$\Phi(x)$	$N(x)$

Greeks

The Greeks (or Greek letters) are derivatives of the financial derivative price with respect to its arguments. They measure the sensitivity of the option value w.r.t. changes in these arguments. We consider:

- Delta, $\Delta_t = \dfrac{\partial V_t}{\partial X_t}$. In the Black-Scholes model, it is

  $$\begin{aligned}\Delta_t &= \Phi(d_1) + X_t \frac{\partial \Phi(d_1)}{\partial X_t} - K e^{-r(T-t)} \frac{\partial \Phi(d_2)}{\partial X_t} \\ &= \Phi(d_1) + X_t \phi(d_1) \frac{1}{X_t \sigma \sqrt{T-t}} \\ &\quad - K e^{-r(T-t)} \phi(d_2) \frac{1}{X_t \sigma \sqrt{T-t}} \\ &= \Phi(d_1).\end{aligned}$$

  where $\phi(x) = d\Phi(x)/dx$, the standard normal density function. Note that $0 < \Phi(d_1) < 1$.

- Gamma, $\Gamma_t = \dfrac{\partial^2 V_t}{\partial X_t^2}$. This measures the 'non-linearity' in the option price. In the Black-Scholes case, it is

  $$\Gamma_t = \frac{\phi(d_1)}{X_t \sigma \sqrt{T-t}}.$$

- Theta, $\Theta_t = \dfrac{\partial V_t}{\partial t}$. This measures the sensitivity of time-to-maturity. In BS,

  $$\Theta_t = -\frac{X_t \sigma \phi(d_1)}{2\sqrt{T-t}} - K r e^{-r(T-t)} \Phi(d_2).$$

- Rho, $P_t = \rho_t = \dfrac{\partial V_t}{\partial r}$, and Vega, $\mathcal{V}_t = \dfrac{\partial V_t}{\partial \sigma}$. These measure sentivities which are assumed constant in the Black-Scholes model, but are varying in practice. In BS,

  $$\begin{aligned}P_t &= (T-t) K e^{-r(T-t)} \Phi(d_2), \\ \mathcal{V}_t &= X_t \sqrt{T-t} \phi(d_1).\end{aligned}$$

The 'risk-free' portfolio Π_t is called 'delta-neutral'. Because continuous hedging is not always possible, and because some of the 'constants' are actually non-constant, other types of hedging become important. For example, one may construct a portfolio which is 'gamma-neutral'.

Deviations from Black-Scholes assumptions

- Stocks may pay dividend; this can be fixed.
- Trading may not occur continuously, and involves transaction costs. Gamma hedging is designed to cope with this. Note that hedge ratios and prices may be obtained by ‘minimum risk hedging’ (minimizing the sum of squared deviations between option price and replicating portfolio), instead of no-arbitrage principle.
- Interest rates may vary, may be stochastic. Model and call price may be adapted to this situation, assuming a model for the interest rates (and hence term structure). In practice, this seems to have small effect.
- Returns may not be normally distributed, but may have fatter tails. The model can be extended to allow for *jumps*, but this creates a problem for hedging. Also, stochastic volatility models imply fatter tails of the marginal distribution of returns.
- Relatedly, the volatility may be varying. In practice, the implied volatility is time-varying. Moreover, we observe the *volatility smile*, i.e., the volatility is higher for deep in- or out-of-the-money options. This may also be caused by non-normality. The model may be extended by *GARCH* or *stochastic volatility*.

Exercises

1. Show that exactly the same PDE is obtained if the stock price process is generalized to $dX_t = \mu(t, X_t)dt + \sigma X_t dB_t$. Thus, we may replace the drift cX_t by any $\mu(t, X_t)$ and obtain the same option value.
2. Show that de Black-Scholes formula indeed solves the PDE.
3. Check the Greeks. Also, derive the Greeks for a put option, using the put-call parity.

Gamma hedging

Consider the case where a trader has a short position in a call option with value C_t on an underlying stock with price S_t. She or he may hedge this position by constructing a Delta-neutral portfolio of -1 option, $\Delta_t^c = \partial C_t / \partial S_t$ shares, and the balance of these two may be put in a bank account. Hence

$$\Pi_t^c = -C_t + \Delta_t^c S_t + b_t^c e^{rt} = 0,$$

so that $b_t^c = e^{-rt}(C_t - \Delta_t^c S_t)$. The portfolio is self-financing, so

$$\begin{aligned} d\Pi_t^c &= -dC_t + \Delta_t^c S_t + b_t^c de^{rt} \\ &= -dC_t + \Delta_t^c S_t + r(C_t - \Delta_t^c S_t)dt. \end{aligned}$$

Itô’s Lemma implies $dC_t = \Theta_t^c dt + \Delta_t^c dS_t + \frac{1}{2}\Gamma_t^c (dS_t)^2$, where $\Theta_t^c = \partial C_t / \partial t$ and $\Gamma_t^c = \partial^2 C_t / \partial S_t^2$. Hence

$$\begin{aligned} d\Pi_t^c &= -\Theta_t^c dt - \tfrac{1}{2}\Gamma_t^c (dS_t)^2 + r(C_t - \Delta_t^c S_t)dt \\ &= -\left[\Theta_t^c + \tfrac{1}{2}\Gamma_t^c \sigma^2 S_t^2 - rC_t + r\Delta_t^c S_t\right] dt \\ &\quad -\tfrac{1}{2}\Gamma_t^c \left\{(dS_t)^2 - \sigma^2 S_t^2 dt\right\}. \end{aligned}$$

The Black-Scholes PDE implies that the term in square brackets is zero. Furthermore, the term in curly brackets is also zero. Therefore, $d\Pi_t^c = 0$, which is not surprising since $\Pi_t^c = 0$ and the portfolio is self-financing. The important point is that the hedge is indeed succesful.

Now suppose that we only change the portfolio at discrete time points t_i. If we rebalance at these time points using the appropriate Δ_t^c and b_t^c, then the change in the portfolio is approximately (via the *Millstein* approximation, see Mikosch, p.164):

$$\begin{aligned} \Pi_{t_{t+1}}^c - \Pi_{t_i}^c &\approx -\left[\Theta_{t_i}^c + \tfrac{1}{2}\Gamma_{t_i}^c \sigma^2 S_{t_i}^2 - rC_{t_i} + r\Delta_{t_i}^c S_{t_i}\right](t_{i+1} - t_i) \\ &\quad -\tfrac{1}{2}\Gamma_{t_i}^c \left\{(S_{t_{i+1}} - S_{t_i})^2 - \sigma^2 S_{t_i}^2 (t_{i+1} - t_i)\right\}. \qquad (3) \end{aligned}$$

The term in square brackets is still zero, but now the term in curly brackets may deviate substantially from zero, because the squared return over $[t_i, t_{i+1}]$ may deviate from its expectation $\sigma^2(t_{i+1} - t_i)$. This will cause serious problems when a large shock hits the stock market in the period $[t_i, t_{i+1}]$. Note that the strategy is no longer self-financing.

This risk can be reduced if another derivative V_t is available on the same stock, and expiring at the same date or later. Let $dV_t = \Theta_t^v dt + \Delta_t^v dS_t + \frac{1}{2}\Gamma_t^v (dS_t)^2$, and construct a similar self-financing Delta-neutral portfolio $\Pi_t^v = -V_t + \Delta_t^v S_t + b_t^v e^{rt} = 0$. This again satisfies (3) with c replaced by v. Next, let

$$\begin{aligned}\Pi_t &= \Pi_t^c - \frac{\Gamma_t^c}{\Gamma_t^v}\Pi_t^v \\ &= -C_t + \left[\Delta_t^c - \frac{\Gamma_t^c}{\Gamma_t^v}\Delta_t^v\right] S_t + \frac{\Gamma_t^c}{\Gamma_t^v} V_t + \left[b_t^c - \frac{\Gamma_t^c}{\Gamma_t^v} b_t^v\right] e^{rt}.\end{aligned}$$

Using the approximation (3) for this portfolio, we will see that its increments are approximately zero. Thus we that we have eliminated this source of risk. The portfolio Π_t is called *Gamma-neutral*: its second partial derivative w.r.t. S_t is zero. Note that V_t should have sufficient curvature ($\Gamma_t^v \neq 0$), otherwise this won't work.

Simple example: use, for V_t, a put option with same strike and maturity. Then $\Delta_t^v = \Delta_t^c - 1$ and $\Gamma_t^v = \Gamma_t^c$. Thus

$$\Pi_t = -C_t + V_t + S_t - e^{-r(T-t)}K = 0,$$

which is just the put-call parity.

Finite difference methods

Consider again the Black-Scholes PDE

$$-u_1(\tau, x) + \frac{1}{2}u_{22}(\tau, x)\sigma^2 x^2 = ru(\tau, x) - ru_2(\tau, x)x.$$

where $\tau = T - t$ and $x = S$. The solution is known for the boundary condition $u(0, S) = [S - K]^+$, but not for all other boundary conditions and hence payoffs. In that case we can solve the PDe numerically, using so-called finite difference methods.

We contruct a time-grid $0 = t_0 < \ldots < t_n = T$, with $t_i = t_{i-1} + \delta t$, which of course corresponds to $0 = \tau_n < \ldots < \tau_0 = T$, with $\delta\tau = -\delta t$. Next, we use a price grid $0 = S_0 < \ldots < S_m = S_{\max}$, with $S_i = S_{i-1} + \delta S$.

We want to find $u(\tau_i, S_i)$ from the partial differential equation and the boundary condition. For this purpose, we use the approximations

$$\begin{aligned}u_1(\tau_i, S_j) &= \frac{u(\tau_{i+1}, S_j) - u(\tau_i, S_j)}{\delta\tau}, \\ u_2(\tau_i, S_j) &= \frac{u(\tau_i, S_{j+1}) - u(\tau_i, S_{j-1})}{2\,\delta S}, \\ u_{22}(\tau_i, S_j) &= \frac{u(\tau_i, S_{j+1}) - 2u(\tau_i, S_j) + u(\tau_i, S_{j-1})}{(\delta S)^2}.\end{aligned}$$

Substituting these in the PDE, and using $S_j = j\,\delta S$, we can recover the other values on the grid.

The method becomes simpler (‘explicit’) when we replace τ_i by τ_{i+1} in the right-hand side expressions for u_2 and u_{22}. In that case we simply get the backward recursion

$$u(\tau_i, S_j) = a_j u(\tau_{i+1}, S_{j-1}) + b_j u(\tau_{i+1}, S_j) + c_j u(\tau_{i+1}, S_{j+1}),$$

where

$$\begin{aligned} a_j &= \frac{1}{1 + r\,\delta t}\left(-\tfrac{1}{2} r\, j\,\delta t + \tfrac{1}{2}\sigma^2 j^2 \delta t\right), \\ b_j &= \frac{1}{1 + r\,\delta t}\left(1 - \sigma^2 j^2 \delta t\right), \\ c_j &= \frac{1}{1 + r\,\delta t}\left(+\tfrac{1}{2} r\, j\,\delta t + \tfrac{1}{2}\sigma^2 j^2 \delta t\right), \end{aligned}$$

Some adjustment has to be made along the edges ($S = S_0$ or $S = S_{\max}$), but often the values of the derivatives are known for these cases (deep in- or out-of-the-money).