Stochastic Calculus

Week 5

**Topics:**

- Stochastic Differential Equations
- Weak and Strong Solutions
- Euler Approximations

## Stochastic Differential Equations

An ordinary (linear, first-order) differential equation (ODE) may take the form

$$dx(t) = a(t, x(t))dt,$$

where $a(\cdot, \cdot)$ is a known function, but $x(t)$ is unknown. The *solution* of this equation is a *function* $x(t)$ that satisfies the equation. In general there may be infinitely many such functions, which is why we need a *boundary condition*, such as $x(0) = x_0$, a known value.

If $a$ would be a function of $t$ only, then the problem reduces to evaluating the integral $x(t) = x_0 + \int_0^t a(s)ds$. The difficult part however is that $x(t)$ enters the function $a$, so that we obtain $x(t) = x_0 + \int_0^t a(s, x(s))ds$, which is not very helpful because it expresses $x(t)$ in terms of itself.

There are no general principles that always will yield a solution of an ODE. Sometimes we may guess a solution and check that it satisfies the equation, and sometimes we can use a clever substitution or transformation that will lead to a solution. For limited classes of ODE's a known solution exists.

**Example**: In the Black-Scholes model with fixed interest rates, the cash bond price $\beta(t)$ satisfies the ODE

$$d\beta(t) = r\beta(t)dt,$$

and we usually impose the boundary condition $\beta(0) = \beta_0 = 1$. It is well known that the solution to this equation is $\beta(t) = \beta_0 \exp(rt) = \exp(rt)$.
We could have found this by using $d \log \beta(t) = (1/\beta(t))d\beta(t) = rdt$, so that $\log \beta(t) = \log \beta_0 + rt = rt$. What happens when we replace $r$ by $r_t$?

A stochastic differential equation (SDE) takes the form

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t,$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are known functions, and $B_t$ is a Brownian motion. Thus $X_t$ is an Itô process, where the drift $\mu_t = a(t, X_t)$ and volatility $\sigma_t = b(t, X_t)$ are functions of the level of the process itself.

As before, we will need a boundary condition, which may take the form $X_0 = Y$ for some random variable $Y$ (could also be a fixed number $x_0$).

A solution of an SDE, if it exists, is a stochastic process, called a *diffusion*. We distinguish:

- A *strong* solution, which is an explicit function $f$ such that $X_t = f(t, B_s, s \le t)$;
- a *weak* solution, which is a process $\tilde{X}_t$ which has the same distribution as $X_t$, but is not expressed as a function of $(B_s, s \le t)$.

Conditions for existence and uniqueness of solution: see Mikosch, p.138.

Two examples of SDE with known strong solution:

- **Geometric Brownian motion**: The SDE

  $$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

  with boundary condition $X_0$, has the solution

  $$X_t = X_0 \exp\left(\left[\mu - \tfrac{1}{2}\sigma^2\right] t + \sigma B_t\right).$$

  Follows from Itô's Lemma applied to $\log X_t$.

- **Ornstein-Uhlenbeck process**: The SDE

  $$dY_t = -\gamma(Y_t - \mu)dt + \sigma dB_t,$$

  with boundary condition $Y_0$, has the solution

  $$Y_t = \mu + e^{-\gamma t}[Y_0 - \mu] + \sigma \int_0^t e^{-\gamma(t-s)} dB_s.$$

  Follows from Itô's Lemma applied to $X_t = e^{\gamma t} Y_t$. When $\gamma > 0$ and $Y_0 \sim N(\mu, \sigma^2/(2\gamma))$, independently of $B_t$, then $Y_t$ is stationary and *mean-reverting* with mean $\mu$ and variance $\sigma^2/(2\gamma)$.
  (*Note*: Mikosch uses $c = -\gamma$.)

## Euler Approximations

When no known explicit solution exists, then we may approximate it by a numerical solution, replacing differentials by differences. This is similar to numerical integration.

The best-known example of such methods is the Euler approximation. Take the usual grid $0 = t_0 < \ldots < t_n = T$. Then

$$X_{t_i} = X_{t_{i-1}} + \int_{t_{i-1}}^{t_i} a(s, X_s) ds + \int_{t_{i-1}}^{t_i} b(s, X_s) dB_s.$$

If the mesh of the partition is small enough, such that $a$ and $b$ are approximately constant over the intervals, then we may approximate $X_t$ by $X_t^n$, defined by

$$X_{t_i}^n = X_{t_{i-1}}^n + a(t_{i-1}, X_{t_{i-1}}^n)[t_i - t_{i-1}] + b(t_{i-1}, X_{t_{i-1}}^n)[B_{t_i} - B_{t_{i-1}}].$$

The approximation becomes better as $n \to \infty$. However, the process is not "kept on track"; errors may accumulate. Quality measures for numerical approximations:

- Strong: $\mathbb{E}\left(|X_T^n - X_T|\right)$;
- Weak: $|\mathbb{E}(f(X_T^n)) - \mathbb{E}(f(X_T))|$.

## Exercises

1. Consider the SDE

$$dX_t = \mu dt + \sigma \sqrt{X_t} dB_t,$$

a so-called *square-root* process.

(a) Derive the SDE for $Y_t = \sqrt{X_t}$.

(b) A condition for the existence of a positive solution for $Y_t$ or $X_t$ is that $\mu \geq \frac{1}{2}\sigma^2$; can you think of a reason why such a condition might be necessary?

(c) Suppose that $\mu = \frac{1}{4}\sigma^2$, so that the condition in the previous question is not satisfied. What would be the solution (if it existed) for $Y_t$ and hence $X_t$? Does this solution make any sense?

2. Suppose that we wish to find a weak numerical solution to the SDE for the geometric Brownian motion $S_t$. We can think of three methods:

- A binomial tree, with suitable choice of $u$, $d$, and $p$;
- An Euler approximation for $S_t$;
- The exponent of an Euler approximation for $\log S_t$;

Argue that the third method will give the best result.