Stochastic Calculus

Week 3

Topics:

- Towards Black-Scholes
- Stochastic Processes
- Brownian Motion
- Conditional Expectations
- Continuous-time Martingales

Towards Black Scholes

Suppose again that $S_{t+\delta t}$ equals $S_t u$ with probabililty p and $S_t d$ with probability $(1-p)$, where

$$\begin{aligned} u &= \exp(\sigma\sqrt{\delta t}), \qquad d = \exp(-\sigma\sqrt{\delta t}), \\ p &= \frac{1}{2}\left(1+\sqrt{\delta t}\frac{\mu}{\sigma}\right), \end{aligned}$$

where μ is some constant representing the *drift* in S_t. As before we have $\delta t = T/n$, and we look at the behaviour of the stock price as $n \to \infty$, and hence $\delta t \to 0$.

Let $Y_i = \log(S_{i\delta t}/S_{(i-1)\delta t})$ be the ith δt-period continuously compounded return. Then

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(Y_i) &= \sigma\sqrt{\delta t}(p-[1-p]) = \mu\delta t, \\ \mathrm{var}_{\mathbb{P}}(Y_i) &= \sigma^2\delta t(p+[1-p]) - \mu^2\delta t^2 \approx \sigma^2\delta t, \end{aligned}$$

where the approximation gets better as $\delta t \to 0$.

Now

$$\begin{aligned} \log S_T &= \log S_0 + \sum_{i=1}^{n} Y_i \\ &= \log S_0 + \mu T + \sigma\sqrt{T}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{Y_i - \mu\delta t}{\sigma\sqrt{\delta t}}\right). \end{aligned}$$

Each of the terms in parentheses has mean zero and variance (approximately) 1; furthermore they are independent. This implies a *Central Limit Theorem*:

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{Y_i-\mu\delta t}{\sigma\sqrt{\delta t}}\right)\xrightarrow{d} Z,$$

where $Z\sim N(0,1)$, the standard normal distribution:

$$\mathbb{P}\left(Z\le x\right)=\int_{-\infty}^{x}\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz=\int_{-\infty}^{x}\phi(z)dz,$$

with $\phi(\cdot)$ the standard normal *density* function. The symbol "$\xrightarrow{d}$" means that the distribution of the left-hand side converges to the distribution of the right-hand side.

All this implies, as $n\to\infty$,

$$\log S_T\xrightarrow{d}\log S_0+\mu T+\sigma\sqrt{T}Z\sim N(\log S_0+\mu T,\sigma^2T),$$

that is, stock-prices have a log-normal distribution.

So far distributions were under $\mathbb{P}$. It can be shown that

$$q=\frac{e^{r\delta t}-e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}}\approx\frac{1}{2}\left(1+\sqrt{\delta t}\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right).$$

Following the same derivations with μ replaced by $r-\frac{1}{2}\sigma^2$, it follows that under $\mathbb{Q}$,

$$\log S_T\xrightarrow{d}N(\log S_0+(r-\tfrac{1}{2}\sigma^2)T,\sigma^2T).$$

Remarks

- The distribution under $\mathbb{Q}$ implies the Black-Scholes formula for a European call option via

$$C_0=\mathbb{E}_{\mathbb{Q}}\left(e^{-rT}[S_T-K]^+\right).$$

- Using similar methods, we can show that under $\mathbb{P}$,

$$\log S_t-\log S_s\sim N(\mu(t-s),\sigma^2(t-s)),$$

  for all $0<s<t<T$. This defines $\log S_t$ to be a *Brownian motion process*, and S_t itself is a *geometric Brownian motion process*.

- When Z has a standard normal distribution, it can be derived that

$$\mathbb{E}\left(e^{\mu+\sigma Z}\right)=e^{\mu+\frac{1}{2}\sigma^2}.$$

  This implies that under $\mathbb{Q}$,

$$\mathbb{E}_{\mathbb{Q}}(e^{-rT}S_T)=S_0.$$

Stochastic Processes

- Random variable: $X : \Omega \to \mathbb{R}$, and hence $X(\omega), \omega \in \Omega$;
- Random vector: $\mathbf{X} : \Omega \to \mathbb{R}^k$, and hence $\mathbf{X}(\omega) = [X_1(\omega), \ldots, X_k(\omega)], \omega \in \Omega$;
- Stochastic process: $X : \Omega \times \mathbb{T} \to \mathbb{R}$, and hence $X_t(\omega), t \in \mathbb{T}, \omega \in \Omega$.

For fixed t: $X_t(\omega)$ is random variable. For fixed ω, $X_t(\omega)$ is a *sample path*, or *trajectory*. See Figure 1.2.1, p.24 of Mikosch.

Discrete time: $\mathbb{T} = \{0, 1, 2, \ldots\}$;
Continuous time: $\mathbb{T} = [0, T]$ or $\mathbb{T} = [0, \infty)$.

Fidis: finite-dimensional distributions of vectors $(X_{t_1}, \ldots, X_{t_n}), t_1 \ldots, t_n \in \mathbb{T}$.

Process is partly characterized by first two moments: $\mu_X(t) = \mathbb{E}(X_t)$, $\sigma_X^2(t) = \mathrm{var}(X_t)$ and $c_X(t, s) = \mathrm{cov}(X_t, X_s)$.

Some further classes:

- Gaussian processes have Gaussian (normal) fidis; they are completely characterized by $\mu_X(t)$ and $c_X(t, s)$;
- Stationary: $(X_{t_1}, \ldots, X_{t_n})$ has same distribution as $(X_{t_1+h}, \ldots, X_{t_n+h})$
- Stationary increments: $(X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ has same distribution as $(X_{t_2+h} - X_{t_1+h}, \ldots, X_{t_n+h} - X_{t_{n-1}+h})$;
- Independent increments: $(X_{t_2} - X_{t_1}), \ldots, (X_{t_n} - X_{t_{n-1}})$ are independent.

Brownian Motion/Wiener Process

Stochastic process $B_t, t \in [0, \infty)$, with

1. $B_0 = 0$;
2. Stationary & independent increments;
3. $B_t \sim N(0, t)$ for all t;
4. Continuous sample paths.

Alternatively: Gaussian process with $\mu_B(t) = 0$ and $c_B(t, s) = \min(t, s)$: e.g., when $s < t$,

$$\begin{aligned} c_B(t, s) &= \mathbb{E}(B_t B_s) = \mathbb{E}([B_s + (B_t - B_s)]B_s) \\ &= \mathbb{E}(B_s^2) + \mathbb{E}[(B_t - B_s)(B_s - B_0)] \\ &= s + 0, \end{aligned}$$

because of independent increments.

Further properties of sample paths:

- Continuity: by assumption, but intuitively: $B_{t+h} - B_t \sim N(0, h) \to 0$ as $h \to 0$;
- Nowhere differentiability: intuitively:
$$\frac{B_t - B_{t-h}}{h} \sim N\left(0, \frac{1}{h}\right), \quad \frac{B_{t+h} - B_t}{h} \sim N\left(0, \frac{1}{h}\right),$$
independently;
- Unbounded variation: for $0 = t_0 < \ldots < t_n = T$:
$$\sup \sum_{i=1}^{n} \left| B_{t_i} - B_{t_{i-1}} \right| = \infty.$$

All this implies that integrals like $\int_0^T f(t) dB(t)$ cannot be defined as $\int_0^T f(t) \frac{dB(t)}{dt} dt$. (Unlike Riemann-Stieltjes integrals such as $\int_{-\infty}^{\infty} x dF(x) = \int_{-\infty}^{\infty} x \frac{dF(x)}{dx} dx = \int_{-\infty}^{\infty} x f(x) dx$.)

Extensions:

- Brownian motion with drift: $X_t = X_0 + \mu t + \sigma B_t$, so $\mu_X(t) = X_0 + \mu t$ and $c_X(t, s) = \sigma^2 \min(t, s)$.
- Geometric Brownian motion $X_t = X_0 \exp(\mu t + \sigma B_t)$. In this case $\mu_X(t) = X_0 \exp(\mu t + \frac{1}{2}\sigma^2)$.

Representation of B_t as limit of sequence

Let Y_i be i.i.d. with mean zero and unit variance, and let

$$\tilde{S}_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Y_i,$$

where $[nt]$ is the largest integer smaller than or equal to nt. This process does not have continuous sample paths (step function). However, its fidis converge as $n \to \infty$ to the fidis of B_t on $[0, 1]$.

We can make it continuous by linear interpolation:

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Y_i + \frac{1}{\sqrt{n}} Y_{[nt]+1}(nt - [nt]).$$

Again, the fidis of S_n converge as $n \to \infty$ to those of B_t on $[0, 1]$.

Conditional Expectations

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Indicator function of $B \in \mathcal{F}$:

$$I_B(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in B, \\ \\ 0 & \text{if} \quad \omega \notin B. \end{cases}$$

Note that $\mathbb{E}(I_B) = 1 \cdot \mathbb{P}(B) + 0 \cdot [1 - \mathbb{P}(B)] = \mathbb{P}(B)$.

A discrete random variable X_n can be defined as follows

$$X_n(\omega) = \sum_{i=1}^{n} x_i I_{A_i}(\omega),$$

where $\{A_i\}$ is partition of Ω: $A_i \cap A_j = \varnothing$ and $\cup_{i=1}^n A_i = \Omega$.

Conditional expectation of X_n given event B with $\mathbb{P}(B) > 0$:

$$\begin{aligned} \mathbb{E}(X_n|B) &= \int_\Omega X_n(\omega) d\mathbb{P}(\omega|B) \\ &= \sum_{i=1}^{n} x_i \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} \\ &= \frac{1}{\mathbb{P}(B)} \int_\Omega X_n(\omega) I_B(\omega) d\mathbb{P}(\omega) \\ &= \frac{1}{\mathbb{P}(B)} \mathbb{E}(X_n I_B). \end{aligned}$$

Any random variable can be approximated arbitrarily well by X_n, choosing partitioning finer and finer. Hence

$$\mathbb{E}(X|B) = \lim_{n\to\infty} \mathbb{E}(X_n|B) = \frac{1}{\mathbb{P}(B)}\mathbb{E}(XI_B).$$

Conditional expectation of X given general σ-field $\mathcal{F}$: random variable satisfying:

1. $\mathbb{E}(X|\mathcal{F})$ is a function of $\mathcal{F}$;
2. $\mathbb{E}[\mathbb{E}(X|\mathcal{F})I_A] = \mathbb{E}(XI_A)$ for all A in $\mathcal{F}$.

Definition is not very constructive (doesn't tell us how to find the conditional expectation). Often in practice, $\mathcal{F} = \sigma(Y_1, \ldots, Y_i)$ or $\sigma(Y_s, s \leq t)$.
In that case the defining property is that $X - \mathbb{E}(X|\mathcal{F})$ is uncorrelated with any function of $(Y_1, \ldots, Y_i)$ or $(Y_s, s \leq t)$.

Example:
$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{F} = \{\varnothing, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, $\mathbb{P}(\omega_i) = 0.25$, and $X(\omega_1) = 2$, $X(\omega_2) = X(\omega_3) = 1$ and $X(\omega_4) = 0$. Then define

$$\mathbb{E}(X|\mathcal{F})(\omega) = \begin{cases} \frac{3}{2} & \text{if} \quad \omega \in \{\omega_1, \omega_2\}, \\ \\ \frac{1}{2} & \text{if} \quad \omega \in \{\omega_3, \omega_4\}. \end{cases}$$

This satisfies the defining properties of the conditional expectation (check!).

Rules for working with conditional expectations (same as discrete):

- $\mathbb{E}(X|\mathcal{F}) = X$ if X is $\mathcal{F}$-measurable
- $\mathbb{E}(XY|\mathcal{F}) = X\mathbb{E}(Y|\mathcal{F})$ if X is $\mathcal{F}$-measurable
- $\mathbb{E}(X|\mathcal{F}_0) = \mathbb{E}(X)$, where $\mathcal{F}_0 = \{\varnothing, \Omega\}$
- $\mathbb{E}(aX + bY|\mathcal{F}) = a\mathbb{E}(X|\mathcal{F}) + b\mathbb{E}(Y|\mathcal{F})$
- $\mathbb{E}[\mathbb{E}(X|\mathcal{F})] = \mathbb{E}(X)$
- $\mathbb{E}[\mathbb{E}(X|\mathcal{F})|\mathcal{G}] = \mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{F}] = \mathbb{E}(X|\mathcal{F})$ if $\mathcal{F} \subset \mathcal{G}$

Continuous-time Martingales

Filtration $\mathcal{F}_t$: collection of σ-fields with $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$.

Example:

$$\mathcal{F}_t = \sigma(B_s, 0 \le s \le t).$$

This σ-field contains all the possible events involving $B_s, s \le t$. It is also the σ-field generated by the random variables $(B_{t_1}, \ldots, B_{t_n})$, for arbitrary $t_1 \le \ldots \le t_n \le t$.

Martingale:$\{X_t, \mathcal{F}_t\}$ satisfying:

1. $\mathbb{E}(|X_t|) < \infty$;
2. X_t is $\mathcal{F}_t$-measurable (*adapted*);
3. $E(X_t|\mathcal{F}_s) = X_s$, $s < t$.

Examples:

- $X_t = \mathbb{E}(X|\mathcal{F}_t)$.
- Brownian motion: when $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$, then

$$\begin{aligned} \mathbb{E}(B_t|\mathcal{F}_s) &= \mathbb{E}(B_s + [B_t - B_s]|\mathcal{F}_s) \\ &= B_s + 0 \end{aligned}$$

  because of independent increments. Note that $\mathbb{E}(|B_t|) < \infty$ because $B_t \sim N(0, t)$;

- Geometric Brownian motion $X_t = X_0 \exp(\mu t + \sigma B_t)$:

$$\begin{aligned} \mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(X_s \exp(\mu[t-s] + \sigma[B_t - B_s])|\mathcal{F}_s) \\ &= X_s \exp(\mu[t-s])\mathbb{E}(\exp(\sigma[B_t - B_s])) \\ &= X_s \exp(\mu[t-s]) \exp(\tfrac{1}{2}\sigma^2(t-s)). \end{aligned}$$

  Hence this is a martingale only if $\mu = -\frac{1}{2}\sigma^2$.

- $X_t = B_t^2 - t$:

$$X_t = B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s) - t.$$

  Because $\mathbb{E}((B_t - B_s)^2|\mathcal{F}_s) = (t-s)$ and $\mathbb{E}(2B_s(B_t - B_s)|\mathcal{F}_s) = 2B_s\mathbb{E}((B_t - B_s)|\mathcal{F}_s) = 0$, we have

$$\mathbb{E}(X_t|\mathcal{F}_s) = B_s^2 + (t-s) - t = X_s.$$

- Martingale transform in discrete time: $\{B_{t_i}, i = 1, \ldots, n\}$, increments $\Delta_i B = B_{t_i} - B_{t_{i-1}}$, filtration $\mathcal{F}_i = \sigma(B_{t_1}, \ldots, B_{t_i})$ and previsible sequence $\{C_i\}$, such that C_i is in $\mathcal{F}_{i-1}$. Then the *martingale transform* $C \cdot \Delta B$ is

$$(C \cdot \Delta B)_k = \sum_{i=1}^{k} C_i \Delta_i B,$$

which is a discrete-time martingale with respect to $\mathcal{F}_i$. Interpretation: financial gain from a trading strategy; see exercise.

Exercises

1. Let (Z_1, Z_2) be a pair of independent standard normal random variables. Define the vector

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \mu_1 + \sigma_1 \left(\sqrt{1-\rho^2} Z_1 + \rho Z_2 \right) \\ \mu_2 + \sigma_2 Z_2 \end{pmatrix}.$$

Show that $\mathbf{X}$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\Sigma_{\mathbf{X}} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

2. Show that the Brownian motion process is 0.5-self-similar, as follows. Let B_t be a Brownian motion, and define, for $T > 0$,

$$X_t = T^{-1/2} B_{tT}$$

Show that X_t satisfies the defining properties of a Brownian motion.

3. Let $Y_i, i = 1, \ldots, n$ be independent random variables with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 0.5$. Let $S_n(t)$ be a continuous process on $t \in [0, 1]$, defined by

$$S_n\left(\frac{i}{n}\right) = \frac{1}{\sqrt{n}} \sum_{j=1}^{i} Y_j.$$

Show that as $n \to \infty$, this process has the same finite-dimensional distributions as a Brownian motion proces B_t at the points $t_i = i/n$.

4. Consider a multiplicative tree in which the stock price S satifies $S_i = S_{i-1}u$ or $S_i = S_{i-1}d$, both with probability 0.5. Let $u = \exp(n^{-1/2})$ and $d = 1/u$, and define the continuous process $X_n(t)$ on $t \in [0, 1]$, such that

$$X_n\left(\frac{i}{n}\right) = S_i.$$

Show that as $n \to \infty$, this process has the same finite-dimensional distributions as a geometric Brownian motion at the points $t_i = i/n$. What are the values of μ and σ^2?

5. Let S_t and B_t be a continuous-time stock price and bond processes, such that the discounted stock price process $B_t^{-1}S_t$ is a martingale under $\mathbb{Q}$. Let $V_t = \phi_t S_t + \psi_t B_t$ be the value of a self-financing portfolio, where the weights ϕ_t and ψ_t only change at discrete points in time $t_i, i = 0, 1, \ldots, n$. More precisely, the weights are constant over de intervals $[t_{i-1}, t_i)$, and are previsible in the sense that their value in the period $[t_{i-1}, t_i)$ depends only on $\{S_t, B_t, t \leq t_{i-1}\}$. Show that the self-financing property implies, with $t \in [t_m \leq t < t_{m+1})$,

$$\begin{aligned} V_t &= V_0 + \sum_{i=1}^{m} \phi_{t_{i-1}}(S_{t_i} - S_{t_{i-1}}) + \sum_{i=1}^{m} \psi_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) \\ &\quad + \phi_{t_m}(S_t - S_{t_m}) + \psi_{t_m}(B_t - B_{t_m}). \end{aligned}$$

Show also that under $\mathbb{Q}$, $B_t^{-1}V_t$ is a martingale with respect to $\mathcal{F}_t = \sigma(S_s, s \leq t)$.