Stochastic Calculus, Week 10

Term-Structure Models & Interest Rate Derivatives

Topics:

1. Definitions and notation for the interest rate market
2. Term-structure models
3. Interest rate derivatives

Definitions and Notation

- *Zero-coupon bond*, or discount bond: asset which pays, with certainty, 1 at time T. Price at time t is $P(t,T)$.
  Pull-to-par: $\lim_{t\to T} P(t,T) = P(T,T) = 1$.
  With fixed rate r: $P(t,T) = \exp(-r[T-t])$.
- *Yield-to-maturity*: average continuously compounded return on zero-coupon bond, over remaining lifetime:
  $$\begin{aligned} R(t,T) &= \frac{\log P(T,T) - \log P(t,T)}{T-t} \\ &= -\frac{\log P(t,T)}{T-t}. \end{aligned}$$
  With fixed rate, $R(t,T) = r$.
- *Yield curve*, term structure: plot of $R(t,T)$ against $(T-t)$. With fixed rate, yield curve is flat line.
- Instantaneous rate, *short rate*:
  $$r_t = \lim_{T\downarrow t} R(t,T) = -\left.\frac{\partial \log P(t,T)}{\partial T}\right|_{T=t}.$$

- Forward contract: agreement at time $t < T$, to buy an S-bond at time $T < S$ at price K. Replicated by portfolio of +1 S-bond and $-K$ T-bonds. Value at time t should be zero, so $K = P(t,S)/P(t,T)$. Return (yield) over period $[T,S]$ is therefore

$$\frac{\log P(S,S) - \log K}{S-T} = -\frac{\log P(t,S) - \log P(t,T)}{S-T}.$$

Note that this return is known at time t! The limit, as S approaches T, of this return is the *instantaneous forward rate*,

$$\begin{aligned} f(t,T) &= \lim_{S \downarrow T} -\frac{\log P(t,S) - \log P(t,T)}{S-T} \\ &= -\frac{\partial \log P(t,T)}{\partial T}. \end{aligned}$$

Note that $f(t,t) = r_t$. Also,

$$f(t,T) = R(t,T) + (T-t)\frac{\partial R(t,T)}{\partial T}.$$

The term structure may also be described by $f(t,T)$ as a function of $(T-t)$.

- The definitions of $R(t,T)$ and $f(t,T)$ in terms of $P(t,t)$ may be inverted to obtain

$$\begin{aligned} P(t,T) &= \exp\left(-(T-t)R(t,T)\right) \\ &= \exp\left(-\int_t^T f(t,s)ds\right). \end{aligned}$$

In general, $P(t,T)$ can *not* be obtained from r_t alone!

Term structure models

We only consider single-factor models (one Brownian motion). Ingredients:

- A cash bond B_t paying the short rate (r_t): $dB_t = r_t B_t dt$, so $B_t = \exp\left(\int_0^t r_s ds\right)$.
- An Itô process for the zero-coupon bonds:

$$\begin{aligned} dP(t,T) &= P(t,T)\left[\mu_P(t,T)dt + \Sigma(t,T)dW_t\right] \\ &= P(t,T)\left[r_t dt + \Sigma(t,T)d\tilde{W}_t\right], \end{aligned}$$

  with W_t a $\mathbb{P}$-Brownian motion and $\tilde{W}_t$ a $\mathbb{Q}$-Brownian motion. Under $\mathbb{Q}$, $P(t,T)/B_t$ is a martingale.

- The SDE for $P(t,T)$ implies, via Itô's lemma, SDE's for $R(t,T)$, $f(t,T)$ and r_t. Heath-Jarrow-Morton start with Itô processes for $f(t,T)$:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW_t \\ &= -\sigma(t,T)\Sigma(t,T)dt + \sigma(t,T)d\tilde{W}_t, \end{aligned}$$

  where

$$\sigma(t,T) = -\frac{\partial\Sigma(t,T)}{\partial T}, \qquad \Sigma(t,T) = -\int_t^T \sigma(t,s)ds,$$

  and

$$\begin{aligned} \alpha(t,T) &= -\frac{\partial\mu_P(t,T)}{\partial T} - \sigma(t,T)\Sigma(t,T), \\ \mu_P(t,T) &= r_t - \int_t^T \alpha(t,s)ds + \frac{1}{2}\Sigma(t,T)^2. \end{aligned}$$

- The market price of risk does not depend on T:

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds,$$

  with

$$\gamma_t = \frac{\mu_P(t,T) - r_t}{\Sigma(t,T)} = \frac{1}{2}\Sigma(t,T) - \frac{1}{\Sigma(t,T)}\int_t^T \alpha(t,s)ds.$$

- Note that under risk-neutrality, everything depends on the term structure of volatilities $\sigma(t,T)$.

Short-rate models

Usually the single factor is associated with the short rate r_t. Short-rate models are usually formulated in "risk-neutral form", i.e., as a SDE

$$dr_t = \rho(r_t, t)dt + \nu(r_t, t)d\tilde{W}_t,$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. (Baxter and Rennie denote this by W_t in Section 5.4).

Let $B_t = \exp\left(\int_0^t r_s ds\right)$ be the cash bond price. Under the risk-neutral measure, $P(t,T)/B_t$ is a martingale, so that

$$\begin{aligned} P(t,T) &= B_t \mathbb{E}_{\mathbb{Q}}\left[B_T^{-1} P(T,T) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[B_t\, B_T^{-1} \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

Therefore, a single-factor (risk-neutral) model for r_t implies $P(t,T)$, and hence the term structure, as a function of r_t.

We distinguish:

- *Endogenous* term structure models: here the current term structure follows from the SDE for r_t, which may be different from the actual term structure at time t. Well-known examples are the Vasicek model:

  $$dr_t = (\theta - \alpha r_t)dt + \sigma d\tilde{W}_t,$$

  and the Cox-Ingersoll-Ross model

  $$dr_t = (\theta - \alpha r_t)dt + \sigma \sqrt{r_t} d\tilde{W}_t.$$

  These are also called *equilibrium models*.

- *Exogenous* term structure models: here the drift $\rho(r_t, t)$ is adjusted such that the current term structure is fitted exactly. The simplest example is the Ho-Lee model:

  $$dr_t = \theta_t dt + \sigma d\tilde{W}_t,$$

  where θ_t is such that $f(0,T) = r_0 - \frac{1}{2}\sigma^2 T^2 + \int_0^T \theta_s ds$ matches the initial term structure.
  These are also called *no-arbitrage models*.

Interest rate derivatives

We assume again the general single-factor model

$$dP(t,T) = r_t P(t,T)dt + \Sigma(t,T)P(t,T)d\tilde{W}_t,$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. We consider:

- Forward contracts
- Coupon-bearing bonds
- Floating rate bonds
- Bond options
- Swaps
- Swaptions
- Caps, floors, collars
- Stock options with stochastic interest rates

Some of these turn out to have a price that depend only on $P(t,T)$, independent of model assumptions (i.e., independent of $\Sigma(t,T)$).

Forward contracts

The forward price of a zero-coupon T_2-bond, to be delivered at T_1, is simply $F(t,T_1,T_2) = P(t,T_2)/P(t,T_1)$.

Coupon-bearing bonds

Suppose that a bond has a principal of 1 dollar, and pays coupons at times $T_i = i\delta, i = 1,\ldots,n$, at rate k; at time T_n the principal amount is repaid. This means that the payoff is $k\delta$ at time $T_i, i < n$, and $1 + k\delta$ at time T_n. Since the no-arbitrage value at time t, of a certain payment of x at time T_i is

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^{T_i} r_s ds\right\} x \,\middle|\, \mathcal{F}_t\right) = P(t,T_i)x,$$

the bond value at time t between T_j and T_{j+1}becomes

$$V_t = \sum_{i=j+1}^{n} k\delta P(t,T_i) + P(t,T_n).$$

Since we expect that $V_0 = 1$ (the par value), we find that the coupon rate should be

$$k = \frac{1 - P(t,T_n)}{\sum_{i=1}^{n} \delta P(t,T_i)}.$$

Floating-rate bonds

Suppose that the coupon rate it not fixed, but equal to the LIBOR rate at the previous payment time T_{i-1}. The δ-period LIBOR rate $L(t)$ is defined by $P(t, t+\delta)(1+\delta L(t)) = 1$, or

$$L(t) = \frac{1}{\delta}\left(\frac{1}{P(t, t+\delta)} - 1\right).$$

Hence the payoff at time T_i is $1/P(T_{i-1}, T_i) - 1$ for $i < n$, and the payoff at time T_n is $1/P(T_{n-1}, T_n)$.

Then the no-arbitrage value V_0 of this bond should be 1. The replicating strategy is:

- At time 0, buy $1/P(0, T_1)$ zero coupon bonds maturing at T_1 (costing 1);
- At time $T_1 = \delta$, sell the T_1-bonds (yielding $1/P(0, T_1)$), and buy $1/P(T_1, T_2)$ of T_2-bonds (costing 1). Hence the payoff is $1/P(0, T_1) - 1$;
- At time T_i, sell the current position of $1/P(T_{i-1}, T_i)$ of T_i-bonds, and buy $1/P(T_i, T_{i+1})$ of T_{i+1}-bonds.

This gives exactly the right payoff, and costs 1.

Bond options

A European call option on a zero-coupon T-bond, struck at k with exercise time S, is worth

$$C_t = B_t \mathbb{E}_{\mathbb{Q}}\left(B_S^{-1}[P(S,T) - k]^+ \middle| \mathcal{F}_t\right).$$

When the short rate r_t is a Gaussian process, then both B_t and $P(t,T)$ will have a log-normal distribution. This can be used to obtain a Black-Scholes-type formula

$$C_t = P(t,S)\left\{F(t,S,T)\Phi(d_1) - k\Phi(d_2)\right\},$$

where $F(t,S,T)$ is the forward price, and

$$d_{1,2} = \frac{\log(F(t,S,T)/k)}{\bar{\sigma}\sqrt{S-t}} \pm \frac{1}{2}\bar{\sigma}\sqrt{S-t},$$

where $\bar{\sigma}^2$ is the conditional variance (conditional on $\mathcal{F}_t$) of $\log(P(S,T)/P(t,T))/\sqrt{S-t}$, i.e.,

$$\bar{\sigma}^2 = \frac{1}{S-t}\int_t^S \Sigma(u,T)^2 du.$$

For example, in the Ho-Lee model, $\bar{\sigma} = \sigma(T-S)$.

Options on coupon-bearing bonds will only have a closed-form price in single-factor models where r_t is a diffusion process. See Baxter & Rennie, p.170, or Hull, pp.568-570.

Swaps

A fixed-versus variable interest rate swap (on a principal of 1) has payoff, at times T_i, equal to $\delta[L(T_{i-1}) - k]$, where k is the fixed swap rate. This means that the swap can be replicated by a portfolio of a short position in a fixed coupon bond, and a long position of a floating coupon bond. Since the swap should have intial value of zero, we have

$$1 - \sum_{i=1}^{n} k\delta P(t, T_i) - P(t, T_n) = 0,$$

or

$$k = \frac{1 - P(t, T_n)}{\sum_{i=1}^{n} \delta P(t, T_i)},$$

which is also the coupon rate that sets the bond value equal to par.

Swaptions

A swaption is an option to enter a swap. Its value is the same as the value of a call option on a fixed-coupon bearing bond, struck at 1.

Caps, floors, collars

For a cap, the payoff at T_i is $\delta[L(T_{i-1}) - k]^+$. The payoff of one caplet can be replicated as follows: Buy $(1 + k\delta)$ put options on a T_i-bond, struck at $1/(1+k\delta)$ with exercise time T_{i-1}.

At time T_{i-1}, this yields $(1 + k\delta)[1/(1 + k\delta) - P(T_{i-1}, T_i)]^+ = [1 - (1 + k\delta)P(T_{i-1}, T_i)]^+$. Then put this money in the bank, to yield the LIBOR rate $L(T_{i-1})$, so that the payoff at time T_i becomes $[1 + \delta L(T_{i-1}) - (1 + k\delta)]^+ = \delta[L(T_{i-1}) - k]^+$. Thus the value of a cap can be derived from the value of the put option.

Similarly, the payoff of a floor is $\delta[k - L(T_{i-1})]^+$ at time T_i, the value of which can be derived from a call option. A swap is essentially the sum of a cap and a floor, at the same rate k; this results in a put-call parity. A collar is also a combination of a cap and a floor, but at different rates (cap rate bigger than floor rate).

Stock options with stochastic interest rates

Suppose we wish to price a European call option on a stock with price S_t, and that the interest rate is stochastic. In particular, let

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma S_t d\tilde{W}_{1t} \\ dP(t,T) &= r_t P(t,T)dt + \Sigma(t,T)P(t,T)d\tilde{W}_{2t}, \end{aligned}$$

and furthermore define the cash bond as usual. Here $(\tilde{W}_{1t}, \tilde{W}_{2t})$ is a bivariate $\mathbb{Q}$-Brownian motion, with correlation ρ. If r_t is a Gaussian process, such that B_t and $P(t,T)$ are log-normal, then the price of a European call option becomes

$$C_t = P(t,T)\left\{F(t,T)\Phi(d_1) - k\Phi(d_2)\right\},$$

where $F(t,T) = P(t,T)^{-1}S_t$ is the forward stock price, and

$$d_{1,2} = \frac{\log(F(t,T)/k)}{\hat{\sigma}\sqrt{T-t}} \pm \frac{1}{2}\hat{\sigma}\sqrt{T-t},$$

where

$$\hat{\sigma}^2 = \frac{1}{T-t}\int_t^T \left[\sigma^2 + \Sigma(s,T)^2 - 2\rho\sigma\Sigma(s,T)\right] ds.$$

This result was derived by Merton (1973).