Stochastic Calculus in Finance, 1999-2000

Website: http://www.fee.uva.nl/ke/course/StCalc

Topics:

- Discrete-time methods (binomial trees)
- Stochastic processes: Brownian motion, martingales
- Stochastic calculus: Itô's lemma, SDE's
- Black-Scholes-Merton model
- Equivalent martingale measures, risk-neutral valuation
- Various applications
- Term-structure models

Week 1:

- Introduction to financial derivatives
- Discrete-time methods: binomial branches & trees
- The Arbitrage Theorem
- Risk-neutral valuation

Financial derivative: Financial product with value depending on the value of another, *underlying* asset.

Examples:

- *Forward* contract on a stock: a contract written at time $t = 0$, to buy a stock at time $t = T$ at the strike price K.
- *European stock option*: a contract written at time $t = 0$, giving the right but not the obligation to buy a stock at time $t = T$ at strike price K. The option costs C now.
- Interest rate swap: contract between two parties: A pays B a floating rate (e.g., 3-month LIBOR), and B pays A a fixed rate R, both on the same principal amount X.

In each case the problem is to *value* or *price* the instrument, either at time $t = 0$, or later (if the derivative is traded). Thus we wish to find K, C and R, respectively.

A related question is how to *hedge* the risk of the derivative, or how the derivative is used to hedge the underlying asset's risk. The future development of the asset is uncertain, *random*. This is where probabilities, expectations, etc. come into play.

Binomial branch

- Two time points: $t = 0$ (“now”) and $t = T = 1$ (“tomorrow”).
- Two assets:
  - Cash bond / bank account, with value B_t, earning an interest rate R;
  - Stock, the price S_t of which may go up or down. At time $t = 0$ we do not know what is going to happen, only the probability: $\mathbb{P}(up) = 1 - \mathbb{P}(down) = p$.

Example: $R = 0.1$, $B_0 = 1$, $S_0 = 1$, $S_1(up) = 1.25$, $S_1(down) = 0.8$, $p = 0.8$:

$$\begin{tikzcd}
 & t = 0 & t = 1 \\
\text{Cash bond:} & B_0 = 1 \arrow[r] & B_1 = (1 + R) = 1.1 \\
 & & S_1(up) = 1.25 \\
\text{Stock:} & S_0 = 1 \arrow[ur] \arrow[dr] & \\
 & & S_1(down) = 0.8
\end{tikzcd}$$

Remarks:

1. R measures the time value of money (determined by market preferences): $1\$$ now is worth $(1 + R)\$$ at time $t = 1$. Assumption: limitless lending and borrowing at the same rate!
2. The present value of the stock at time $t = 0$ is $1.25/1.1$ in “good times”, and $0.8/1.1$ in “bad times”. The expected present value therefore is
$$\frac{1.25}{1.1}p + \frac{0.8}{1.1}(1 - p) = \frac{1.25}{1.1}0.8 + \frac{0.8}{1.1}0.2 = \frac{1.16}{1.1} > 1.$$
The difference $1.16/1.1 - 1$ can be thought of as the price of risk.

   In other words, the expected return is
$$0.25 \times 0.8 - 0.2 \times 0.2 = 0.16 > R = 0.1;$$
the difference $0.16 - R$ is a risk premium (determined by market preferences).

Consider now a forward contract, with value F_t, and strike price K. Since nothing is paid at time $t = 0$, have $F_0 = 0$, and the payoff structure is:

$$\begin{tikzcd}
 & F_1(up) = S_1(up) - K = 1.25 - K \\
F_0 = 0 \arrow[ur] \arrow[dr] & \\
 & F_1(down) = S_1(down) - K = 0.8 - K
\end{tikzcd}$$

Which value should the strike price K have?

Answer 1: The expected present value of the payoff should equal $F_0 = 0$, so

$$\begin{aligned}
F_0 &= \frac{1}{1+R}\left([S_1(up) - K]p + [S_1(down) - K](1-p)\right) \\
&= \frac{(1.25 - K) \times 0.8 + (0.8 - K) \times 0.2}{1.1} \\
&= \frac{1.16 - K}{1.1} = 0,
\end{aligned}$$

and hence $K = 1.16 = \mathbb{E}(S_1)$.

Answer 2: Create the following portfolio: buy 1 share now, and borrow its price $S_0 = 1$. This costs zero at time $t = 0$. At time $t = 1$, sell the share at S_1 and pay back the loan with interest, $S_0(1 + R)$. Hence the payoff of this portfolio is

$$\begin{tikzcd}
 & V_1(up) = S_1(up) - S_0(1+R) \\
V_0 = 0 \arrow[ur] \arrow[dr] & \\
 & V_1(down) = S_1(down) - S_0(1+R)
\end{tikzcd}$$

This is exactly the same payoff as the forward if $K = S_0(1 + R)$. If $K > S_0(1 + R)$, an *arbitrage opportunity* arises: sell the forward and create the above portfolio: this costs zero now and has a payoff of $K - S_0(1 + R)$ with probability 1. If $K < S_0(1 + R)$, then this can be reversed (buy the forward, sell the portfolio), giving the same riskless profit. Since such arbitrage opportunities are assumed not to exist (or persist) in an efficient market, the *no-arbitrage price* is $K = S_0(1 + R) = 1.1$.

Arbitrage

Consider a market with three assets: cash bond B_t, stock S_t, and derivative F_t. The price vector P (at time $t = 0$) and payoff matrix D (at time $t = 1$) are:

$$P = \begin{pmatrix} B_0 \\ S_0 \\ F_0 \end{pmatrix}, \qquad D = \begin{array}{c} \begin{array}{cc} \quad up \quad & \quad down \quad \end{array} \\ \begin{pmatrix} B_0(1+R) & B_0(1+R) \\ s_u & s_d \\ f_u & f_d \end{pmatrix} \end{array} .$$

Again, $\mathbb{P}(up) = 1 - \mathbb{P}(down) = p$, with $0 < p < 1$. The fact that the payoff of F depends only on whether the stock goes up or down (and no other uncertain factors) defines it to be a derivative.

Notation: R is the one-period interest rate. We also use the continuously compounded rate r, defined by $e^{r\,\delta t} = (1+R)$ or $r = \ln(1+R)/\delta t$, where δt is the time step (taken as 1 here). We also define the real numbers $u = s_u/S_0$ and $d = s_d/S_0$. Note that $u > (1+R) > d$ (why?).

A *portfolio* is defined by a vector of weights $\theta = (\psi, \phi, \chi)$. It costs

$$\theta \cdot P = \psi B_0 + \phi S_0 + \chi F_0,$$

and its payoff is (with $B_1 = B_0(1+R)$):

$$\theta \cdot D = \Big(\; [\psi B_1 + \phi s_u + \chi f_u] \quad [\psi B_1 + \phi s_d + \chi f_d] \; \Big) .$$

An *arbitrage* is a portfolio such that

$$\begin{array}{rlcl} \text{either} & \theta \cdot P < 0 & \text{and} & \theta \cdot D \geq 0, \\ \text{or} & \theta \cdot P \leq 0 & \text{and} & \theta \cdot D > 0. \end{array}$$

In words: with an arbitrage one either receives a positive amount today and a non-negative amount tomorrow, or a non-negative amount today and a positive amount tomorrow (both with probability 1).

Suppose that we know r, S_0 and the payoff matrix. Can we obtain from this the no-arbitrage price F_0? To do so, we construct a portfolio $\theta = (\psi, \phi, -1)$ such that $\theta \cdot D = (0, 0)$; the absence of arbitrage requires $\theta \cdot P = 0$. This choice of (ψ, ϕ) defines a portfolio of only the stock and the bond that *replicates* the payoff of the derivative, and therefore should have define the value and hence price F_0.

Define the value of this replicating portfolio as $V_t = \phi S_t + \psi B_t$, such that

$$\begin{aligned} V_1(up) &= \phi s_u + \psi B_0(1+R) = f_u, \\ V_1(down) &= \phi s_d + \psi B_0(1+R) = f_d. \end{aligned}$$

Solving for ϕ and ψ yields

$$\begin{aligned} \phi &= \frac{f_u - f_d}{s_u - s_d}, \qquad \text{(\textit{hedge ratio}, } \Delta\text{)}, \\ \psi &= \frac{1}{B_0(1+R)}\left(f_u - \frac{f_u - f_d}{s_u - s_d} s_u\right). \end{aligned}$$

Using $s_u = S_0 u$ and $s_d = S_0 d$, we find

$$\begin{aligned} V_0 &= \phi S_0 + \psi B_0 \\ &= \frac{f_u - f_d}{u-d} + \frac{1}{1+R}\left(f_u - \frac{f_u - f_d}{u-d} u\right) \\ &= f_u\left(\frac{1}{u-d} + \frac{1}{1+R}\left[1 - \frac{u}{u-d}\right]\right) \\ &\quad + f_d\left(\frac{-1}{u-d} + \frac{1}{1+R}\frac{u}{u-d}\right) \\ &= \frac{1}{1+R}\left(f_u \frac{1+R-d}{u-d} + f_d \frac{u-(1+R)}{u-d}\right) \\ &= \frac{1}{1+R}\left(f_u q + f_d[1-q]\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{1+R} F_1\right] = F_0. \end{aligned}$$

Interpretation: $q = (1+R-d)/(u-d)$ is a number between 0 and 1, because $u > (1+R) > d$. It may be interpreted as the “risk-neutral” probability of the stock going up, corresponding to the risk-neutral probability measure: $\mathbb{Q}(up) = 1 - \mathbb{Q}(down) = q$. This means that if q were the true probability, then there would be no risk-premium:

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\frac{S_1 - S_0}{S_0}\right] &= (u-1)q + (d-1)(1-q) \\ &= q(u-d) + (d-1) \\ &= R. \end{aligned}$$

We do not assume that $q = p$. The actual probability p is irrelevant for the value F_0 of the derivative!

Alternative but equivalent derivation: construct riskfree portfolio from ϕ shares and -1 derivative: $X_t = \phi S_t - F_t$, such that $X_1(up) = \phi s_u - f_u = \phi s_d - f_d = X_1(down)$, which again yields $\phi = (f_u - f_d)/(s_u - s_d)$. This risk-free portfolio should earn the riskfree rate R, so

$$\begin{aligned} X_1 &= X_0(1+R) \\ \Leftrightarrow \qquad \frac{f_u - f_d}{u-d} u - f_u &= \left(\frac{f_u - f_d}{u-d} - F_0\right)(1+R), \end{aligned}$$

which yields the same solution.

The Arbitrage Theorem: the absence of arbitrage implies and is implied by the existence of a *state-price vector* $\pi = (\pi_u, \pi_d) > 0$, such that $P = D\pi$.

Interpretation of π: take $(f_u, f_d) = (1, 0)$, then $F_0 = \pi_u$; and similarly $F_0 = \pi_d$ if $(f_u, f_d) = (0, 1)$. Hence π_u is the price of an insurance policy that pays 1 in the "up" state and nothing otherwise, and π_d is the same for the "down" state.

From the first row of the equation $P = D\pi$ we have $B_0 = B_0(1+R)(\pi_u + \pi_d)$, so if we define $q = (1+R)\pi_u$, then $(1+R)\pi_d = (1-q)$, and $0 < q < 1$. Hence the Arbitrage Theorem says that the absence of arbitrage is equivalent to the existence of a risk-neutral probability q such that

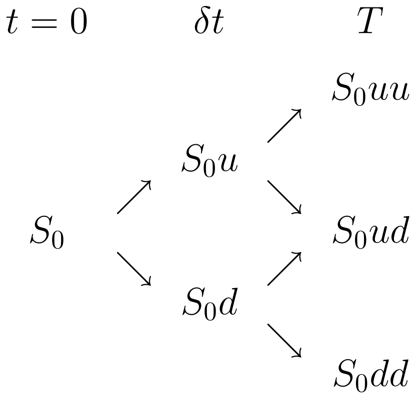
$$P = D \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \frac{1}{1+R} D \begin{pmatrix} q \\ 1-q \end{pmatrix} = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{1+R} D\right].$$

The risk-neutral probability and state-price vector are *unique* only if the market is *complete*. If there were three or more states, we could not find a unique q, and hence no unique no-arbitrage price. Also we cannot perfectly hedge anymore in that case.

Multiperiod models: Binomial trees

The binomial branch model only allows pricing at the start of the period, not during the holding period of the derivative. Furthermore, it only allows for two states at time $t = T$, which is quite unrealistic in practice. A solution is obtained by extending the branch to a *tree* (or *lattice*).

Simplest example: two periods:



where $T = 2\delta t$. Furthermore, $B_t = B_0 e^{r\,\delta t}$ for $t = 0, \delta t, T$, and further

$$\begin{aligned} \mathbb{P}(u) &= 1 - \mathbb{P}(d) = p, \\ \mathbb{P}(uu|u) &= 1 - \mathbb{P}(ud|u) = p, \\ \mathbb{P}(du|d) &= 1 - \mathbb{P}(dd|d) = p, \end{aligned}$$

which implies

$$\mathbb{P}(uu) = p^2, \quad \mathbb{P}(ud) = 2p(1-p), \quad \mathbb{P}(dd) = (1-p)^2.$$

Consider then a derivative F_t with

$$\begin{tikzcd}
 & & f_{uu} \\
 & f_u \arrow[ur] \arrow[dr] & \\
F_0 \arrow[ur] \arrow[dr] & & f_{ud} \\
 & f_d \arrow[ur] \arrow[dr] & \\
 & & f_{dd}
\end{tikzcd}$$

Now only f_{uu}, f_{ud} and f_{dd} are known, and we wish to obtain F_0, f_u and f_d.

The idea to start at the end, and work backwards. At time $t = \delta t$, we know if the stock has gone up or down. If it has gone up, then only the branch from f_u to f_{uu} or f_{ud} is relevant. This means that we can price f_u (and simlarly f_d) just as before:

$$\begin{aligned}
f_u &= e^{-r\,\delta t}\left[f_{uu}q_u + f_{ud}(1-q_u)\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-r\,\delta t}F_T | S_1 = S_0 u\right], \\
f_d &= e^{-r\,\delta t}\left[f_{ud}q_d + f_{dd}(1-q_d)\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-r\,\delta t}F_T | S_1 = S_0 d\right].
\end{aligned}$$

Note that q_u and q_d may be different, but in this case they are the same, since r, u and d are the same at each step.

The values f_u and f_d are the market prices (under the no-arbitrage condition), so the derivative can be sold at this price at time t, depending on whether the stock goes up and down. Thus at time $t = 0$ we know that the two possible payoffs in the next period are f_u and f_u, and thus

$$\begin{aligned}
F_0 &= e^{-r\,\delta t}\left[f_u q + f_d(1-q)\right] \\
&= e^{-rT}\left[f_{uu}q^2 + f_{ud}[q(1-q) + (1-q)q] + f_{dd}(1-q)^2\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}F_T\right].
\end{aligned}$$

Exercises

1. Determine the value of a European option at time $t = 0$, with $T = \delta t = 1$, $R = 0.05$, $S_0 = 100$, $u = 1/d = 1.2$, and $K = 90$. Determine also the replicating portfolio.
2. Refine the calculations in the previous exercise by considering a two-period binomial tree, with $\delta t = 0.5$, $r = \ln(1+R)$, and $u = 1/d = \sqrt{1.2}$, such that two up-movements will lead to $S_T(uu) = S_0 1.2$, just like one up-movement in the previous exercise.
3. Make exercises 2.1 and 2.2 from Baxter and Rennie.

Hedging on the tree

Once we know the derivatives' values at each node, we also know the hedge ratios. For example, in the two-period model at time $t = \delta t$ and when the first movement was up, then $\phi = (f_{uu} - f_{ud})/(S_0uu - S_0ud)$; the amount of cash bonds (borrowing) is simply determined as the rest: $\psi B_{\delta t} = (f_u - \phi S_0 u)$.

This leads to the following investment strategy consisting of stocks and bonds. Let F_t denote the value of the derivative, and $V_t = \phi_t S_t + \psi_t B_t$, where $B_t = e^{rt}$:

$$\begin{aligned} \phi_t &= \frac{F_{t+\delta t}(up) - F_{t+\delta t}(down)}{S_{t+\delta t}(up) - S_{t+\delta t}(down)}, \\ \psi_t &= B_t^{-1}(F_t - \phi_t S_t). \end{aligned}$$

This implies that the strategy is *replicating*: $V_t = F_t$. Furthermore, it is *self-financing*:

$$\begin{aligned} V_{t+\delta t} &= \phi_t S_{t+\delta t} + \psi_t B_{t+\delta t} \\ &= \phi_{t+\delta t} S_{t+\delta t} + \psi_{t+\delta t} B_{t+\delta t} \end{aligned}$$

which can also be written as

$$V_{t+\delta t} - V_t = \phi_t (S_{t+\delta t} - S_t) + \psi_t (B_{t+\delta t} - B_t).$$

How to use a tree in practice

We measure t in years. We know S_0, T, and the function $F(\cdot)$ which defines the payoff of the derivative via $F_T = F(S_T)$. For example, with a European call option, $F(S_T) = \max(S_T - K, 0) = [S_T - K]^+$.

Then we have to decide on the number n of steps, and hence $\delta t = T/n$. Trade-off between computational burden and accuracy.

Next, $r = \ln(1 + R_0)$, where R_0 is the current value of a suitable interest rate (yearly basis). E.g., if $T = 0.25$ (3 months), then R_0 might be the current 3-month LIBOR.

Finally, u and d have to be set. For this we need the *volatility* σ, which may be defined as

$$\sigma^2 \delta t = \mathrm{var}\left[\ln\left(\frac{S_{t+\delta t}}{S_t}\right)\right].$$

If δt is sufficiently small, then this may be accomplished by $u = e^{\sigma\sqrt{\delta t}}$ and $d = 1/u = e^{-\sigma\sqrt{\delta t}}$. σ may be obtained in different ways, most simply from the *historical stock price volatility*.