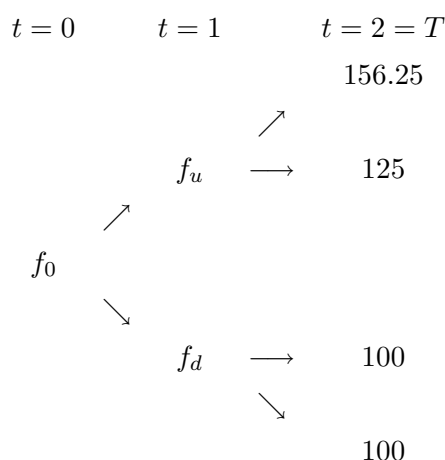


## Solutions to Midterm exam Stochastic Calculus, 1999-2000

1. (a) Constant volatility over the tree means constant  $u = \exp(\sigma\sqrt{\delta t})$  and  $d = \exp(-\sigma\sqrt{\delta t})$ . From the first step we note that  $u = 125/100 = 1.25$  and  $d = 1/u = 0.8$ , and this leads to  $S_{0uu} = 125 \times 1.25 = 156.25$ .
- (b)  $q = (1 + R - d)/(u - d) = (1.07 - 0.8)/(1.25 - 0.8) = 0.6$ .
- (c) Tree should be non-recombining because the path up-down ( $S_1 = 125, S_2 = 100$ ) gives a maximum value of 125, whereas the path down-up ( $S_1 = 80, S_2 = 100$ ) leads to a maximum value of 100. Thus the payoff does not only depend on the value of the stock price at time  $T$ , but also on the path. The payoffs are:



- (d)  $f_u = (156.25 \times q + 125 \times (1 - q))/(1 + R) = (156.25 \times 0.6 + 125 \times 0.4)/1.07 = 134.346$ . Similarly,  $f_d = (100 \times q + 100 \times (1 - q))/1.07 = 93.458$ , and  $f_0 = (134.346 \times q + 93.458 \times (1 - q))/1.07 = 110.272$ .
- (e) At time  $t = 0$ , buy one unit of the derivative and go one short in the stock; this costs  $F_0^* - S_0 = 0$ . At time  $t = 1$ , don't change anything. At time  $t = 2$ , the payoff is  $156.25 - 156.25 = 0$  (up-up),  $125 - 100 = 25$  (up-down),  $100 - 100 = 0$  (down-up) or  $100 - 64 = 36$  (down-down). So this is an arbitrage opportunity.

Alternatively, one may create an arbitrage as follows:

- at time  $t = 0$  buy one unit of the derivative and sell  $\phi_0 = (f_u - f_d)/(s_u - s_d) = 0.9086$  shares; borrow the remainder  $\psi_0 = 100 - 0.909S_0 = 9.14$ .
- if the stock goes up at time  $t = 1$ , change the short position in the stock to  $\phi_u = (f_{uu} - f_{ud})/(s_{uu} - s_{ud}) = 5/9$ , and borrow the required amount of money  $(\phi_0 - \phi_u)S_1 = 44.13$ . This changes to bank account balance to  $-\psi_0(1 + R) - 44.13 = -53.91 = -\psi_u(1 + R)$ .
- if the stock goes down at time  $t = 1$ , change the short position in the stock to  $\phi_d = (f_{du} - f_{dd})/(s_{du} - s_{dd}) = 0$ , and borrow the required amount of money  $(\phi_0 - \phi_d)S_1 = 72.69$ . This changes to bank account balance to  $-\psi_0(1 + R) - 72.69 = -82.47 = -\psi_d(1 + R)$ .

- The payoff in the up-up state is  $f_{uu} - \phi_u s_{uu} - \psi_u (1+R)^2 = 11.76$ ; in the up-down state, it is  $f_{ud} - \phi_u s_{ud} - \psi_u (1+R)^2 = 11.76$ ; in the down-up state, it is  $f_{du} - \phi_d s_{du} - \psi_d (1+R)^2 = 11.76$ , and in the down-down state it is again  $f_{dd} - \phi_d s_{dd} - \psi_d (1+R)^2 = 11.76$ .

Thus this strategy leads to a certain payoff of 11.76, which is exactly the difference between  $f_0$  and  $F_0^*$ , with accumulated interest:  $(f_0 - F_0^*)(1+R)^2 = 11.76$ .

2. (a)  $\log P_t = f(t, P_t)$ , with derivatives  $\partial f / \partial t = 0$ ,  $\partial f / \partial P_t = 1/P_t$ , and  $\partial^2 f / \partial P_t^2 = -1/P_t^2$ . Hence

$$\begin{aligned} d \log P_t &= \frac{1}{P_t} dP_t - \frac{1}{2P_t^2} \sigma_t^2 dt \\ &= r_t dt + \sigma(T-t) dB_t - \frac{1}{2} \sigma^2 (T-t)^2 dt \\ &= \left( r_t - \frac{1}{2} \sigma^2 (T-t)^2 \right) dt + \sigma(T-t) dB_t. \end{aligned}$$

- (b) Integrating the SDE yields

$$\log P_t = \log P_0 + \int_0^t \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds + \int_0^t \sigma(T-s) dB_s, \quad (1)$$

which indeed expresses  $\log P_t$  in terms of  $P_0$  and  $(r_s, B_s, s \leq t)$ . The expression can be rewritten using

$$\int_0^t -\frac{1}{2} \sigma^2 (T-s)^2 ds = \left[ \frac{1}{6} \sigma^2 (T-s)^3 \right]_0^t = \frac{1}{6} \sigma^2 [(T-t)^3 - T^3],$$

and, using integration by parts,

$$\int_0^t \sigma(T-s) dB_s = \sigma(T-t) B_t - \sigma \int_0^t B_s d(T-s) = \sigma(T-t) B_t + \sigma \int_0^t B_s ds.$$

Thus, (1) leads to

$$\log P_t = \log P_0 + \int_0^t r_s ds + \frac{1}{6} \sigma^2 [(T-t)^3 - T^3] + \sigma(T-t) B_t + \sigma \int_0^t B_s ds. \quad (2)$$

- (c)  $P_T = 1$  implies  $\log P_T = 0$ , and hence, via (1),

$$0 = \log P_0 + \int_0^T \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds + \int_0^T \sigma(T-s) dB_s,$$

and solving this for  $\log P_0$  and substituting this back into (1) gives

$$\begin{aligned} \log P_t &= - \int_0^T \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds - \int_0^T \sigma(T-s) dB_s \\ &\quad + \int_0^t \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds + \int_0^t \sigma(T-s) dB_s \\ &= - \int_t^T \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds - \int_t^T \sigma(T-s) dB_s, \end{aligned}$$

and hence

$$P_t = \exp \left( - \int_t^T \left( r_s - \frac{1}{2} \sigma^2 (T-s)^2 \right) ds - \int_t^T \sigma(T-s) dB_s \right), \quad (3)$$

which is of the required form. Analogously to (2), this can be further rewritten as

$$P_t = \exp \left( - \int_t^T r_s ds + \frac{1}{6} \sigma^2 (T-t)^3 + \sigma (T-t) B_t - \sigma \int_t^T B_s ds \right).$$

- (d) Itô's Lemma applied to  $R_t = f(t, \log P_t) = -\log P_t / (T-t)$  gives  $\partial f / \partial t = -\log P_t / (T-t)^2 = R_t / (T-t)$ ,  $\partial f / \partial \log P_t = -1 / (T-t)$  and  $\partial^2 f / \partial \log P_t^2 = 0$ , so

$$\begin{aligned} dR_t &= \frac{1}{T-t} R_t dt - \frac{1}{T-t} d \log P_t \\ &= \frac{1}{T-t} \left[ (R_t - r_t + \frac{1}{2} \sigma^2 (T-t)^2) dt - \sigma (T-t) dB_t \right] \\ &= \left( \frac{R_t - r_t}{T-t} + \frac{1}{2} \sigma^2 (T-t) \right) dt - \sigma dB_t. \end{aligned}$$

The volatility of  $R_t$  becomes  $-\sigma$ . The negative sign reflects that positive increments in  $B_t$ , which will lead to an increase in the bond price  $P_t$ , will obviously decrease the yield to maturity. Unlike the volatility  $\sigma_t$  of  $P_t$ , the volatility of  $R_t$  is constant, i.e., not a function of  $(T-t)$ . The fact that  $P_t$  should converge to  $P_T = 1$  as  $t \rightarrow T$ , requires the bond price volatility to approach zero as  $t \rightarrow T$ . In the present model the bond price volatility is proportional to  $(T-t)$ , which implies that the yield-to-maturity will exactly have a constant volatility.

- (e) One possibility would be to use the original SDE for  $P_t$ , which leads to the approximation

$$P_{t_i}^n = P_{t_{i-1}}^n + r_{t_{i-1}} P_{t_{i-1}}^n [t_i - t_{i-1}] + \sigma (T - t_{i-1}) P_{t_{i-1}}^n [B_{t_i} - B_{t_{i-1}}].$$

Here the starting value should be chosen such that  $P_T^n = 1$ ! Alternatively, one could use the exponent of

$$\log P_{t_i}^n = \log P_{t_{i-1}}^n + (r_{t_{i-1}} - \frac{1}{2} \sigma^2 (T - t_{i-1})^2) [t_i - t_{i-1}] + \sigma (T - t_{i-1}) [B_{t_i} - B_{t_{i-1}}],$$

again using the boundary condition  $\log P_T = 0$ .

- (f) Note that  $d \log \beta_t = r_t dt$ , which implies  $d\beta_t^{-1} = -r_t \beta_t^{-1} dt$ , which has no ‘‘Brownian motion term’’. This implies that

$$dP_t \beta_t^{-1} = \beta_t^{-1} dP_t + P_t d\beta_t^{-1},$$

see e.g. Baxter and Rennie, p.62, exercise 3.6. Hence

$$\begin{aligned} dZ_t &= \beta_t^{-1} dP_t - r_t P_t \beta_t^{-1} dt \\ &= \beta_t^{-1} (dP_t - r_t P_t dt) \\ &= \beta_t^{-1} \sigma (T-t) P_t dB_t \\ &= \sigma (T-t) Z_t dB_t. \end{aligned}$$

This is an Itô process with a zero drift term, and hence a martingale.