# Solutions to Final Exam Stochastic Calculus, March 2000

1. (a) General approach, in steps:
   - Write the derivative value $C_t$ as function $u(\tau, S_t)$ of time to expiration $\tau = T - t$ and the underlying price $S_t$. If the payoff at time $T$ is $X_T$, then $u(0, S_T) = X_T$.
   - Use Itô's Lemma to obtain
   $$dC_t = -u_1(\tau, S_t)dt + u_2(\tau, S_t)dS_t + \frac{1}{2}u_{22}(\tau, S_t)\sigma^2 S_t^2 dt,$$
   where $u_1 = \partial u/\partial\tau$, $u_2 = \partial u/\partial S_t$ and $u_{22} = \partial^2 u/(\partial S_t)^2$.
   - Construct a replicating self-financing portfolio of stock and cash bond with price $B_t = e^{rt}$: $V_t = \phi_t S_t + \psi_t B_t$, such that $V_T = X_T$ (replicating) and $dV_t = \phi_t dS_t + \psi_t dB_t$ (self-financing). Using $dB_t = rB_t dt$ and $\psi_t B_t = V_t - \phi_t S_t$, we find
   $$dV_t = \phi_t dS_t + \psi_t r B_t dt = \phi_t dS_t + r(V_t - \phi_t S_t)dt.$$
   - No arbitrage implies that $C_t = V_t$, because both are self-financing and have the same payoff. Equating the two differential equations implies $\phi_t = u_2(\tau, S_t)$, and
   $$-u_1(\tau, S_t) + \frac{1}{2}u_{22}(\tau, S_t)\sigma^2 S_t^2 = ru(\tau, S_t) - ru_2(\tau, S_t)S_t.$$
   - Together with the boundary condition $u(0, S_T) = X_T$, this uniquely determines the function $u$ and hence the option price.

   (b) Define $f_1 = \partial f/\partial t$, $f_2 = \partial f/\partial S_t$, $f_3 = \partial f/\partial \bar{S}_t$, and $f_{22} = \partial^2 f/(\partial S_t)^2$. The second derivative with respect to $\bar{S}_t$ will not be needed, since $d\bar{S}_t$ has no Brownian motion term, and hence $(d\bar{S}_t)^2 = 0$. We now have
   $$\begin{aligned} dC_t &= f_1(t, S_t, \bar{S}_t)dt + f_2(t, S_t, \bar{S}_t)dS_t + f_3(t, S_t, \bar{S}_t)d\bar{S}_t + \frac{1}{2}f_{22}(t, S_t, \bar{S}_t)\sigma^2 S_t^2 dt \\ &= \left[f_1(\cdot) + f_3(\cdot)\frac{1}{t}(S_t - \bar{S}_t) + \frac{1}{2}f_{22}(\cdot)\sigma^2 S_t^2\right]dt + f_2(\cdot)dS_t, \end{aligned}$$
   where $(\cdot)$ is notation for $(t, S_t, \bar{S}_t)$. Equating this with the SDE for the replicating portfolio, we find the PDE
   $$f_1(\cdot) + f_3(\cdot)\frac{1}{t}(S_t - \bar{S}_t) + \frac{1}{2}f_{22}(\cdot)\sigma^2 S_t^2 = rf(\cdot) - rf_2(\cdot)S_t,$$
   with boundary condition $f(T, S_T, \bar{S}_T) = [\bar{S}_T - K]^+$. Note: a replicating portfolio can be composed of $S_t$ and $B_t$ only, and not $\bar{S}_t$, because $\bar{S}_t$ has no additional risk factor; in other words, all risk in $C_t$ can be hedged with $S_t$.

   (c) The solution to the SDE is $S_t = S_0 \exp([r - \frac{1}{2}\sigma^2]t + \sigma\tilde{W}_t)$. This means that the stock return from time $s$ to time $t$ is
   $$\frac{S_t}{S_s} - 1 = e^{r(t-s)}\exp\left(\sigma[\tilde{W}_t - \tilde{W}_s] - \frac{1}{2}\sigma^2(t-s)\right) - 1.$$
   Since $\sigma[\tilde{W}_t - \tilde{W}_s] \sim N(0, \sigma^2(t-s))$ under $\mathbb{Q}$, it follows that $\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\sigma[\tilde{W}_t - \tilde{W}_s]\right)\right] = \frac{1}{2}\sigma^2(t-s)$, so that under $\mathbb{Q}$ the expected stock return is $e^{r(t-s)} - 1$, which is the cash bond return. Hence the additional risk in the stock is not compensated by a higher expected return under $\mathbb{Q}$, hence agents are risk-neutral under this measure. It is also called the equivalent martingale measure, since under $\mathbb{Q}$, the discounted stock price $Z_t = B_t^{-1}S_t$ is a martingale:
   $$dZ_t = B_t^{-1}dS_t + S_t dB_t^{-1} = B_t^{-1}S_t[rdt + \sigma d\tilde{W}_t] - rB_t^{-1}S_t dt = \sigma Z_t d\tilde{W}_t,$$
   an Itô process with no drift. The measure is equivalent since the volatility of $S_t$ or $Z_t$ is the same under $\mathbb{P}$ and $\mathbb{Q}$.

(d) The solution of $S_t$ implies $S_s = S_t \exp([r - \frac{1}{2}\sigma^2](s-t) + \sigma Y_s)$. Hence

$$\bar{S}_T = \frac{1}{T}\int_0^t S_s ds + \frac{1}{T}\int_t^T S_s ds = \frac{t}{T}\bar{S}_t + \frac{1}{T}S_t \int_t^T \exp([r - \frac{1}{2}\sigma^2](s-t) + \sigma Y_s)ds.$$

And since in general

$$C_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[X_T | \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[[\bar{S}_T - K]^+ \middle| \mathcal{F}_t\right],$$

this implies the given formula.

(e) Conditional on $\mathcal{F}_t$, $S_t$ and $\bar{S}_t$ are fixed, so the only randomness is in $Y_s$. Note that $Y_s, t \leq s \leq T$ is a Brownian motion starting at $t$, with $Y_t = 0$. Draw $M$ realizations $Y_s^{(i)}, i = 1, \ldots, M$, evaluated at disrete time points $t = s_0 < s_1 < \ldots < s_n = T$, and simulate $\bar{S}_T^{(i)}$ as

$$\bar{S}_T^{(i)} = \frac{t}{T}\bar{S}_t + \frac{1}{T}S_t \sum_{j=1}^{n} \exp([r - \frac{1}{2}\sigma^2](s_j - t) + \sigma Y_{s_j}^{(i)})(s_j - s_{j-1}),$$

where we have replaced the integral by the corresponding Riemann sum. This gives $X_T^{(i)} = [\bar{S}_T^{(i)} - K]^+$, and hence

$$\hat{C}_t = \frac{1}{M}\sum_{i=1}^{M} e^{-r(T-t)} X_T^{(i)}.$$

2. (a) By defnition, a zero-coupon bond satisfies $P(T,T) = 1$. The martingale property for $\exp\left(-\int_0^t r_s ds\right) P(t,T)$ implies

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) P(T,T) \middle| \mathcal{F}_t\right] = \exp\left(-\int_0^t r_s ds\right) P(t,T),$$

which may be rewritten as

$$\begin{aligned} P(t,T) &= \exp\left(\int_0^t r_s ds\right) \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(\int_0^t r_s ds\right) \exp\left(-\int_0^T r_s ds\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right]. \end{aligned}$$

(b) Itô's Lemma implies

$$\begin{aligned} dP(t,T) &= f_1(t,r_t)dt + f_2(t,r_t)dr_t + \frac{1}{2}f_{22}(t,r_t)(dr_t)^2 \\ &= f_1(t,r_t)dt + f_2(t,r_t)(\theta - \alpha r_t)dt + f_2(t,r_t)\sigma d\tilde{W}_t + \frac{1}{2}f_{22}(t,r_t)\sigma^2 dt \\ &= \left[f_1(t,r_t) + f_2(t,r_t)(\theta - \alpha r_t) + \frac{1}{2}f_{22}(t,r_t)\sigma^2\right] dt + f_2(t,r_t)\sigma d\tilde{W}_t. \end{aligned}$$

(c) Risk-neutrality holds if $Z_t = \exp\left(-\int_0^t r_s ds\right) P(t,T)$ is a martingale, i.e., an Itô process with a zero drift. If $\mu_P(t,T)P(t,T)$ is the drift of $P(t,T)$, then it follows that

$$\begin{aligned} dZ_t &= -r_t \exp\left(-\int_0^t r_s ds\right) P(t,T)dt + \exp\left(-\int_0^t r_s ds\right) dP(t,T) \\ &= -r_t Z_t dt + \exp\left(-\int_0^t r_s ds\right) P(t,T)[\mu_P(t,T)dt + \Sigma(t,T)d\tilde{W}_t] \\ &= \left[-r_t + \mu_P(t,T)\right] Z_t dt + Z_t \Sigma(t,T) d\tilde{W}_t, \end{aligned}$$

which is a martingale only if $\mu_P(t,T) = r_t$, hence the drift in $P(t,T)$ is $r_t P(t,T)dt$. Combining this with the answer to question (b) gives

$$f_1(t,r_t) + f_2(t,r_t)(\theta - \alpha r_t) + \frac{1}{2} f_{22}(t,r_t)\sigma^2 = r_t f(t,r_t),$$

which is indeed a partial differential equation.

3. (a) $X_t$ is an Itô process with a nonzero drift, so it is not a martingale.

   (b) Write $Y_t = f(t,X_t) = (1-t)^{-1}X_t$, which has partial derivatives $f_1(t,X_t) = (1-t)^{-2}X_t$, $f_2(t,X_t) = (1-t)^{-1}$ and $f_{22}(t,X_t) = 0$. Hence Itô's lemma implies

$$dY_t = \frac{1}{(1-t)^2}X_t dt + \frac{1}{(1-t)}dX_t = \frac{1}{1-t}dW_t.$$

   (c) $X_0 = 0$ implies $Y_0 = 0$, hence

$$Y_t = \int_0^t \frac{1}{1-s}dW_s, \qquad \Longrightarrow \qquad X_t = (1-t)\int_0^t \frac{1}{1-s}dW_s.$$

   (d) Since $Y_t$ is a stochastic integral with non-stochastic integrating function, it follows that

$$Y_t \sim N\left[0, \int_0^t \left(\frac{1}{1-s}\right)^2 ds\right].$$

   The variance can be further rewritten as

$$\int_0^t \frac{1}{(1-s)^2}ds = \left[\frac{1}{1-s}\right]_0^t = \frac{1}{1-t} - 1 = \frac{t}{1-t}.$$

   Furthermore, $Y_t$ has independent increments, since $Y_t - Y_s = \int_s^t \frac{1}{1-u}dW_u$ is independent of $Y_s = \int_0^s \frac{1}{1-u}dW_u$. Therefore,

$$\text{Cov}[Y_t,Y_s] = \text{Cov}[Y_s + (Y_t - Y_s), Y_s] = \text{Var}[Y_s] + \text{Cov}[Y_t - Y_s, Y_s] = \text{Var}[Y_s] = \frac{s}{1-s},$$

   for $s < t$. Using $X_t = (1-t)Y_t$, this implies

$$X_t \sim N\left[0, t(1-t)\right],$$

   and

$$\text{Cov}[X_t,X_s] = (1-t)(1-s)\,\text{Cov}[Y_t,Y_s] = (1-t)s,$$

   for $s < t$.

   (e) Write $B_t = W_t - t[W_t + (W_1 - W_t)] = (1-t)W_t - t(W_1 - W_t)$. Since $W_t$ and $(W_1 - W_t)$ are independent, and $W_t \sim N(0,t)$ and $(W_1 - W_t) \sim N(0,[1-t])$, it follows that

$$E[B_t] = 0, \qquad \text{Var}[B_t] = (1-t)^2 t + t^2(1-t) = t(1-t), \qquad B_t \sim N(0, t[1-t]).$$

   For the covariance between $B_t$ and $B_s$, $s < t$, we use

$$\begin{aligned} B_t &= (1-t)W_s + (1-t)[W_t - W_s] - t[W_1 - W_t], \\ B_s &= (1-s)W_s - s[W_t - W_s] - s[W_1 - W_t]. \end{aligned}$$

   The three increments are independent, normal, with mean zero and variances $s$, $(t-s)$ and $(1-t)$, respectively. This implies

$$\begin{aligned} \text{Cov}[B_t,B_s] = E[B_t B_s] &= (1-t)(1-s)s - (1-t)s(t-s) + ts(1-t) \\ &= (1-t)s[(1-s) - (t-s) + t] \\ &= (1-t)s. \end{aligned}$$

   Thus we find the same finite-dimensional distributions.

(f) The Euler approximation in general is

$$X_{t_{i+1}} = X_{t_i} - \frac{X_{t_i}}{1-t_i}(t_{i+1}-t_i) + (W_{t_{i+1}} - W_{t_i}).$$

If we take $t_{i+1} = 1$ and $t_i = 1 - \delta t$, this implies

$$\begin{aligned} X_1 &= X_{1-\delta t} - \frac{X_{1-\delta t}}{\delta t}\delta t + (W_1 - W_{1-\delta t}) \\ &= W_1 - W_{1-\delta t} \\ &\sim N(0, \delta t). \end{aligned}$$

Since the approximation gets better as $\delta t \to 0$, this must imply $X_1 = 0$.