Solutions to Stochastic Calculus Final Exam, 1998-1999

1. (a) The tradable stock earns continuous dividend; reinvesting this dividend yields a portfolio with value $\tilde{S}_t$, which evolves as

$$d\tilde{S}_t = (\mu + \delta)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t,$$

with $\tilde{S}_0 = S_0$; this has as a solution

$$\tilde{S}_t = S_0 \exp\left(\left[\mu + \delta - \tfrac{1}{2}\sigma^2\right] t + \sigma W_t\right).$$

The discounted value equals

$$\begin{aligned} Z_t = B_t^{-1}\tilde{S}_t = e^{-rt}\tilde{S}_t &= S_0 \exp\left(\left[\mu + \delta - \tfrac{1}{2}\sigma^2 - r\right] t + \sigma W_t\right) \\ &= S_0 \exp\left(-\tfrac{1}{2}\sigma^2 t + \sigma \tilde{W}_t\right), \end{aligned}$$

where $\tilde{W}_t = W_t + \gamma t$, with $\gamma = (\mu + \delta - r)/\sigma$. If W_t is a Brownian motion under $\mathbb{P}$, then there is a measure $\mathbb{Q}$ under which $\tilde{W}_t$ is a Brownian motion, and under that measure Z_t is a martingale. We thus find

$$\begin{aligned} S_t &= S_0 \exp\left(\left[\mu - \tfrac{1}{2}\sigma^2\right] t + \sigma W_t\right) \\ &= S_0 \exp\left(\left[\mu - \tfrac{1}{2}\sigma^2\right] t + \sigma \tilde{W}_t - \sigma\gamma t\right) \\ &= S_0 \exp\left(\left[r - \delta - \tfrac{1}{2}\sigma^2\right] t + \sigma \tilde{W}_t\right). \end{aligned}$$

(b) It is easily checked that $Z = -\tilde{W}_T/\sqrt{T}$, and since $\tilde{W}_T \sim N(0,T)$ under $\mathbb{Q}$, this implies $Z \sim N(0,1)$ under $\mathbb{Q}$. Similarly, $Z = -W_T/\sqrt{T} - \gamma T/\sqrt{T} = -W_T/\sqrt{T} - \gamma\sqrt{T} \sim N(-\gamma\sqrt{T}, 1) = N(\sigma^{-1}[r - \mu - \delta]\sqrt{T}, 1)$ under $\mathbb{P}$.

(c) We immediately find

$$\begin{aligned} c_2 &= \frac{\ln(S_0/2S_0) + \left[r - \delta - \frac{1}{2}\sigma^2\right] T}{\sigma\sqrt{T}} = \frac{\ln(\frac{1}{2}) + \left[r - \delta - \frac{1}{2}\sigma^2\right] T}{\sigma\sqrt{T}}, \\ d_2 &= \frac{\ln(S_0/S_0) + \left[r - \delta - \frac{1}{2}\sigma^2\right] T}{\sigma\sqrt{T}} = \frac{\left[r - \delta - \frac{1}{2}\sigma^2\right] T}{\sigma\sqrt{T}}. \end{aligned}$$

(d) Using risk-neutral valuation implies, with a payoff function $X_T = X_T(Z)$, that

$$\begin{aligned} V_0 &= \mathbb{E}_{\mathbb{Q}}[e^{-rT} X_T | \mathcal{F}_0] \\ &= \int_{-\infty}^{\infty} e^{-rT} X_T(z)\phi(z) dz. \end{aligned}$$

The payoff is

$$X_T(Z) = \begin{cases} 2S_0 & \text{if} \quad Z \le c_2 \\ S_T & \text{if} \quad c_2 \le Z \le d_2 \\ S_0 & \text{if} \quad Z \ge d_2 \end{cases}$$

Using

$$e^{-rT} S_T = S_0 \exp\left(-\left[\delta + \tfrac{1}{2}\sigma^2\right] T + \sigma \tilde{W}_T\right) = S_0 \exp\left(-\left[\delta + \tfrac{1}{2}\sigma^2\right] T - \sigma\sqrt{T} Z\right),$$

we find that V_0 equals

$$
\begin{aligned}
& \int_{-\infty}^{c_2} e^{-rT} 2S_0 \phi(z)dz + \int_{c_2}^{d_2} S_0 \exp\left(-\left[\delta + \tfrac{1}{2}\sigma^2\right]T - \sigma\sqrt{T}z\right)\phi(z)dz + \int_{d_2}^{\infty} e^{-rT} S_0 \phi(z)dz \\
= \;& e^{-rT} 2S_0 \Phi(c_2) + \int_{c_2}^{d_2} S_0 \exp\left(-\left[\delta + \tfrac{1}{2}\sigma^2\right]T - \sigma\sqrt{T}z\right)\phi(z)dz + e^{-rT} S_0[1 - \Phi(d_2)] \\
= \;& e^{-rT} 2S_0 \Phi(c_2) + \int_{c_2}^{d_2} S_0 \exp\left(-\left[\delta + \tfrac{1}{2}\sigma^2\right]T - \sigma\sqrt{T}z\right)\phi(z)dz + e^{-rT} S_0 \Phi(-d_2).
\end{aligned}
$$

(e) The information so far is

$$
\begin{aligned}
V_0 &= e^{-rT} 2S_0\Phi(c_2) - e^{-\delta T} S_0 \Phi(c_1) + e^{-\delta T} S_0 \Phi(d_1) + e^{-rT} S_0[1 - \Phi(d_2)] \\
&= e^{-rT} S_0 + \left\{ e^{-\delta T} S_0 \Phi(d_1) - e^{-rT} K_2 \Phi(d_2) \right\} - \left\{ e^{-\delta T} S_0 \Phi(c_1) - e^{-rT} K_1 \Phi(c_2) \right\},
\end{aligned}
$$

where $K_1 = S_0$ and $K_2 = 2S_0$. One can recognize this as

$$V_0 = e^{-rT} S_0 + C_1 - C_2,$$

where C_i is the value at time 0 of a call option struck at K_i and expiring at time T, $i = 1, 2$. This suggests $\psi = e^{-rT} S_0$, $\phi_1 = 1$ and $\phi_2 = -1$. The payoff of this portfolio is

$$
\begin{aligned}
V_T &= S_0 + [S_T - S_0]^+ - [S_T - 2S_0]^+ \\
&= \min\left\{\max(S_T, S_0), 2S_0\right\},
\end{aligned}
$$

which is indeed the required payoff. The values for ψ, ϕ_1 and ϕ_2 could also have been guessed by making a graph of the payoff structure, and recognizing this as the sum of a constant S_0, the payoff of C_1 and minus the payoff of C_2.

2. (a) The model for r_t implies $r_s = r_t + \sigma(\tilde{W}_s - \tilde{W}_t)$, so that

$$
\begin{aligned}
\log\left(B_T^{-1} B_T\right) &= \log\left(\exp\left[-\int_0^T r_s ds\right] \exp\left[\int_0^t r_s ds\right]\right) \\
&= -\int_t^T r_s ds \\
&= -\int_t^T r_t ds - \sigma \int_t^T (\tilde{W}_s - \tilde{W}_t) ds \\
&= -r_t(T - t) - \sigma \int_t^T (\tilde{W}_s - \tilde{W}_t) ds.
\end{aligned}
$$

(b) If $X \sim N(\mu, \sigma^2)$ then $\mathbb{E}[\exp(X)] = \exp(\mu + \frac{1}{2}\sigma^2)$. Applying this here yields

$$
\begin{aligned}
P(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[\exp(\log(B_T^{-1} B_t)) | \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\exp(\log(B_T^{-1} B_t))\right] \\
&= \exp\left(-r_t(T - t) + \tfrac{1}{6}\sigma^3 [T - t]^3\right),
\end{aligned}
$$

where the second equality holds because $\tilde{W}_s - \tilde{W}_t$ is independent of $\mathcal{F}_t$.

(c) It's easiest to work backwards:

$$
\begin{aligned}
& \frac{(1 + \delta\bar{r})}{P(t, t + \delta)} [K - P(t, t + \delta)]^+ \\
= \;& \left[\frac{1}{P(t, t + \delta]} - (1 + \delta\bar{r})\right]^+ \\
= \;& [(1 + \delta L(t)) - (1 + \delta\bar{r})]^+ \\
= \;& \delta[L(t) - \bar{r}]^+ \\
= \;& X
\end{aligned}
$$

(d) The idea is to use $\mathbb{E}_{\mathbb{Q}}[\cdot] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(\cdot|\mathcal{F}_t)]$. Therefore

$$\begin{aligned} V_0 &= (1+\delta\bar{r})\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(B_{t+\delta}^{-1}P(t,t+\delta)^{-1}[K-P(t,t+\delta)]^+|\mathcal{F}_t)] \\ &= (1+\delta\bar{r})\mathbb{E}_{\mathbb{Q}}[P(t,t+\delta)^{-1}[K-P(t,t+\delta)]^+\mathbb{E}_{\mathbb{Q}}(B_{t+\delta}^{-1}|\mathcal{F}_t)], \end{aligned}$$

because $P(t,t+\delta)$ and K are contained in the information set $\mathcal{F}_t$. Using the fact that $\mathbb{E}_{\mathbb{Q}}(B_{t+\delta}^{-1}|\mathcal{F}_t) = B_t^{-1}\mathbb{E}_{\mathbb{Q}}(B_{t+\delta}^{-1}B_t|\mathcal{F}_t) = B_t^{-1}P(t,t+\delta)$, this leads to the required result.

(e)

$$\begin{aligned} d[B_t^{-1}P(t,t+\delta)] &= B_t^{-1}dP(t,t+\delta) + P(t,t+\delta)dB_t^{-1} \\ &= B_t^{-1}dP(t,t+\delta) - r_tB_t^{-1}P(t,t+\delta)dt \end{aligned}$$

Writing $P(t,t+\delta) = f(t,r_t) = \exp\left(-r_t\delta + \frac{1}{6}\sigma^3\delta^3\right)$, we have $f_1 = 0$, $f_2 = -\delta P(t,t+\delta)$ and $f_{22} = \delta^2P(t,t+\delta)$, and thus

$$\begin{aligned} dP(t,t+\delta) &= -\delta P(t,t+\delta)dr_t + \tfrac{1}{2}\delta^2P(t,t+\delta)(dr_t^2) \\ &= P(t,t+\delta)[\tfrac{1}{2}\delta^2\sigma^2dt - \delta\sigma d\tilde{W}_t]. \end{aligned}$$

Therefore, letting $Z_t = B_t^{-1}P(t,t+\delta)$, we find

$$\begin{aligned} dZ_t &= Z_t[\tfrac{1}{2}\delta^2\sigma^2dt - \delta\sigma d\tilde{W}_t] - r_tZ_tdt \\ &= Z_t[(\tfrac{1}{2}\delta^2\sigma^2 - r_t)dt - \delta\sigma d\tilde{W}_t]. \end{aligned}$$

(f) The previous answer implies $\log Z_t = \log Z_0 - \int_0^t r_sds - \delta\sigma\tilde{W}_t = -r_0\delta + \frac{1}{6}\sigma^3\delta^3 - r_0t - \sigma\int_0^t \tilde{W}_sds - \delta\sigma\tilde{W}_t$. This is a linear function of the Brownian motion process $\tilde{W}_t$, and therefore normally distributed. This is clarified even further if on writes the stochastic part as

$$\begin{aligned} -\sigma\int_0^t \tilde{W}_sds - \delta\sigma\tilde{W}_t &= -\sigma\tilde{W}_tt + \sigma\int_0^t sd\tilde{W}_s - \delta\sigma\tilde{W}_t \\ &= \sigma\int_0^t (s-t-\delta)d\tilde{W}_s, \end{aligned}$$

which is a stochastic integral with non-stochastic quadratic variation, hence normally distributed (conditional on the initial interest rate r_0). For $B_t^{-1}K$, note that

$$\log(B_t^{-1}K) = -\log(1+\delta\bar{r}) - \int_0^t r_sds = -\log(1+\delta\bar{r}) - r_0t - \sigma\int_0^t \tilde{W}_sds,$$

and the same argument as before shows that this is a normally distributed random variable, again conditional on r_0.

3. (a) $\log(S_t) = f(t,S_t)$, with $f_1 = 0$, $f_2 = 1/S_t$ and $f_{22} = -1/S_t^2$. Therefore

$$\begin{aligned} d\log(S_t) &= \frac{1}{S_t}dS_t + \frac{1}{2S_t^2}(dS_t)^2 \\ &= rdt + \sigma(t,S_t)d\tilde{W}_t + \tfrac{1}{2}\sigma(t,S_t)^2dt \\ &= [r + \tfrac{1}{2}\sigma(t,S_t)^2]dt + \sigma(t,S_t)d\tilde{W}_t. \end{aligned}$$

(b) The general solution is

$$\log(S_t) = \log(S_0) + rt + \tfrac{1}{2}\int_0^t \sigma(s,S_s)^2ds + \int_0^t \sigma(s,S_s)d\tilde{W}_s.$$

If $\sigma(\cdot,\cdot)$ is a function of S_t, then the drift is stochastic, and so is the quadratic variation of the stochastic integral part. Hence there is no reason to assume that the distribution of $\log(S_t)$ is normal. However, if σ is a deterministic function of t only, then the drift becomes non-stochastic, as well as the quadratic variation. In that case

$$\log(S_t) \sim N\left(\left\{\log(S_0) + rt + \tfrac{1}{2}\int_0^t \sigma(s)^2 ds\right\}, \int_0^t \sigma(s)^2 ds\right).$$

(c) The basic idea of the Euler approximation is the following. We have

$$S_{t_{i+1}} = S_{t_i} + \int_{t_i}^{t_{i+1}} rS_t dt + \int_{t_i}^{t_{i+1}} \sigma(t, S_t) S_t d\tilde{W}_t.$$

For small $\delta = t_{i+1} - t_i$, we have

$$\begin{aligned}\int_{t_i}^{t_{i+1}} rS_t dt &\approx rS_{t_i}\delta \\ \int_{t_i}^{t_{i+1}} \sigma(t, S_t) S_t d\tilde{W}_t &\approx \sigma(t_i, S_{t_i}) S_{t_i}[\tilde{W}_{t_i+\delta} - \tilde{W}_{t_i}].\end{aligned}$$

This approximation holds in the sense that if $n \to \infty$ and $\delta \to 0$, the cumulated right-hand side will converge in mean square to the cumulated left-hand side. The Euler approximation in our case thus is

$$\begin{aligned}S_{t_{i+1}} &= S_{t_i} + rS_{t_i}\delta + \sigma(t_i, S_{t_i}) S_{t_i}[\tilde{W}_{t_i+\delta} - \tilde{W}_{t_i}] \\ &= S_{t_i}\left(1 + r\delta + \sigma(t_i, S_{t_i})[\tilde{W}_{t_i+\delta} - \tilde{W}_{t_i}]\right).\end{aligned}$$

(d) A general binomial tree would be

$$S_{t_{i+1}} = \begin{cases} S_{t_i} u(t_i, S_{t_i}) & \text{with probability } p(t_i, S_{t_i}) \\ S_{t_i} d(t_i, S_{t_i}) & \text{with probability } 1 - p(t_i, S_{t_i}) \end{cases}$$

Here $u(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $p(\cdot,\cdot)$ can in general be functions of t_i and S_{t_i}. The correspondence we wish to obtain is to match the first and second conditional moment of $S_{t_{i+1}}$ (we abbreviate $u_i = u(t_i, S_{t_i})$, etc.)

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(S_{t_{i+1}}|S_{t_i}) &= S_{t_i}[u_i p_i + d_i(1 - p_i)], \\ \mathbb{E}_{\mathbb{Q}}(S_{t_{i+1}}^2|S_{t_i}) &= S_{t_i}^2[u_i^2 p_i + d_i^2(1 - p_i)]\end{aligned}$$

(by this we also match the conditional variance). If we can find and explicit formula for the left-hand side in both equations, then we can solve them for the two unknowns p_i and u_i, choosing $d_i = 1/u_i$. When σ is only a function of time, then we will be able to find these two moments from the log-normality of S_t. In general, this has to be done case by case.
We can also approximate the first two conditional moments from the Euler approximation. In our case that would become

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(S_{t_{i+1}}|S_{t_i}) &= (1 + r\delta)S_{t_i}, \\ \text{var}_{\mathbb{Q}}(S_{t_{i+1}}|S_{t_i}) &= \sigma(t_i, S_{t_i})^2 S_{t_i}^2 \delta.\end{aligned}$$