Chapter 27

Martingales and Measures

Derivatives Dependent on a Single Underlying Variable

Consider a variable, θ, (not necessarily the price of a traded security) that follows the process

$$\frac{d\theta}{\theta} = m\,dt + s\,dz$$

Imagine two derivatives dependent only on θ and t with prices f_1 and f_2. Suppose

$$\frac{df_1}{f_1} = \mu_1 dt + \sigma_1 dz$$

$$\frac{df_2}{f_2} = \mu_2 dt + \sigma_2 dz$$

Forming a Riskless Portfolio

We can set up a riskless portfolio Π, consisting of

$+\sigma_2 f_2$ of the 1st derivative and

$-\sigma_1 f_1$ of the 2nd derivative

$$\Pi = (\sigma_2 f_2)\, f_1 - (\sigma_1 f_1) f_2$$

$$\Delta\Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\Delta t$$

Market Price of Risk

$$\Delta\Pi = r\,\Pi\Delta t$$

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\,\sigma_2 - r\,\sigma_1$$

$$\text{or} \quad \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- This shows that $(\mu - r)/\sigma$ is the same for all derivatives dependent only on the same underlying variable, θ, and t.
- We refer to $(\mu - r)/\sigma$ as the market price of risk for θ and denote it by λ

Differential Equation for f

Using Ito's lemma to obtain expressions for μ and σ in terms of *m* and *s*. The equation

$$\mu - \lambda\sigma = r$$

becomes

$$\frac{\partial f}{\partial t} + \theta \frac{\partial f}{\partial \theta}(m - \lambda s) + \frac{1}{2} s^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} = rf$$

Risk-Neutral Valuation

- This analogy shows that we can value f in a risk-neutral world providing the drift rate of θ is reduced from m to $m - \lambda s$
- *Note: When* θ is not the price of an investment asset, the risk-neutral valuation argument does not necessarily tell us anything about what would happen with θ in a risk-neutral world .

Extension of the Analysis to Several Underlying Variables

$$\frac{d\theta_i}{\theta_i} = m_i dt + s_i dz_i$$

$$\frac{df}{f} = \mu\, dt + \sum_{i=1}^{n} \sigma_i\, dz_i$$

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Traditional Risk-Neutral Valuation with Several Underlying Variables

- A derivative can always be valued as if the would is risk neutral, provided that the expected growth rate of each underlying variable is assumed to be m_i-$\lambda_i s_i$ rather than m_i.
- The volatility of the variables and the coefficient of the correlation between variables are not changed. (CIR,1985).

How to measure λ?

For a nontraded securities(i.e.commodity),we can use its future market information to measure λ.

Martingales

- A martingale is a stochastic process with zero drfit

$$d\theta = \sigma dz$$

- A martingale has the property that its expected future value equals its value today

$$E(\theta_T) = \theta_0$$

Alternative Worlds

In the traditional risk - neutral world

$$df = rfdt + \sigma fdz$$

In a world where the market price of risk is λ^*

$$df = (r + \lambda^* \sigma) fdt + \sigma fdz$$

A Key Result

If we set λ^* equal to the volatility of a security g, then Ito s lemma shows that f/g is a martingale for all derivative security prices f

(f and g are assumed to provide no income during the period under consideration)

Forward Risk Neutrality

We refer to a world where the market price of risk is the volatility of g as a world that is forward risk neutral with respect to g.

If E_g denotes a world that is FRN wrt g

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

Aleternative Choices for the Numeraire Security g

- Money Market Account
- Zero-coupon bond price
- Annuity factor

Money Market Account as the Numeraire

- The money market account is an account that starts at \$1 and is always invested at the short-term risk-free interest rate
- The process for the value of the account is

$$dg = rg\,dt$$

- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world

Since $g_0 = 1$ and $g_T = e^{\int_0^T r\,dt}$, the equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = \hat{E}\left[e^{-\int_0^T r\,dt} f_T\right] = \hat{E}(e^{-\bar{r}T} f_T)$$

where $\hat{E}$ denotes expectations in the traditional risk - neutral world

Zero-Coupon Bond Maturing at time T as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = P(0,T)E_T[f_T]$$

where $P(0,T)$ is the zero-coupon bond price and E_T denotes expectations in a world that is FRN wrt the bond price

Forward Prices

Consider an variable S that is not an interest rate. A forward contract on S with maturity T is defined as a contract that pays off S_T-K at time T. Define f as the value of this forward contract. We have

$$f_0 = P(0,T)[E_T(S_T) - K]$$

f_0 equals 0 if F=K,

So, $F=E_T(S_T)$

F is the forward price.

Interest Rates

$$\frac{1}{[1+(T_2-T_1)R(t,T_1,T_2)]}=\frac{P(t,T_2)}{P(t,T_1)}$$

$$R(t,T_1,T_2)=\frac{1}{T_2-T_1}[\frac{P(t,T_1)-P(t,T_2)}{P(t,T_2)}]$$

$$Setting\ : f=\frac{1}{T_2-T_1}[P(t,T_1)-P(t,T_2)]$$

$$and : g=P(t,T_2)$$

$$then : R(0,T_1,T_2)=E_2[R(T_1,T_1,T_2)]$$

In a world that is FRN wrt $P(0,T_2)$ the expected value of an interest rate lasting between times T_1 and T_2 is the forward interest rate

Annuity Factor as the Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right]$$

Annuity Factors and Swap Rates

Suppose that $s(t)$ is the swap rate corresponding to the annuity factor A.

Then:

$$s(t)=E_A[s(T)]$$

Extension to Several Independent Factors

In a risk - neutral world

$$df(t) = r(t) f(t) dt + \sum_{i=1}^{m} \sigma_{fi}(t) f(t) dz_i$$

$$dg(t) = r(t) g(t) dt + \sum_{i=1}^{m} \sigma_{gi}(t) g(t) dz_i$$

For other worlds that are internally consistent

$$df(t) = \left[r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{fi}(t) \right] f(t) dt + \sum_{i=1}^{m} \sigma_{fi}(t) f(t) dz_i$$

$$dg(t) = \left[r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{gi}(t) \right] g(t) dt + \sum_{i=1}^{m} \sigma_{gi}(t) g(t) dz_i$$

Extension to Several Independent Factors continued

We define a world that is FRN wrt g

as world where $\lambda_i = \sigma_{gi}$

As in the one-factor case, f/g is a

martingale and the rest of the results hold.

Applications

- Valuation of a European call option when interest rates are stochastic
- Valuation of an option to exchange one asset for another

Valuation of a European call option when interest rates are stochastic

$$Define: P(0,T) = e^{-RT}$$

$$c = P(0,T)E_T[\max(S_T - X, 0)] = e^{-RT} E_T[\max(S_T - X, 0)]$$

- Assume S_T is lognormal then:

$$E_T[\max(S_T - X, 0)] = E_T(S_T)N(d_1) - XN(d_2)$$

$$E_T(S_T) = S_0 e^{RT}$$

$$c = S_0 N(d_1) - Xe^{-RT} N(d_2)$$

- The result is the same as BS except r replaced by R.

Valuation of an option to exchange one asset(U) for another(V)

- Choose U as the numeraire, and set f as the value of the option so that f_T=max(V_T-U_T,0), so,

$$f_0 = U_0 E_U [\frac{\max(V_T - U_T, 0)}{U_T}] = U_0 E_U [\max(\frac{V_T}{U_T} - 1, 0]$$

$$= U_0 [E_U (\frac{V_T}{U_T}) N(d_1) - N(d_2)]$$

$$\because \frac{V_0}{U_0} = E_U (\frac{V_T}{U_T})$$

$$\therefore f_0 = V_0 N(d_1) - U_0 N(d_2)$$

Change of Numeraire

When we change the numeraire security from g to h, the drift of a variable v increases by $\rho\sigma_v\sigma_q$ where σ_v is the volatility of $v, q = h/g, \sigma_q$ is the volatility of q, and ρ is the correlation between v and q

- 当记帐单位从g变为h时，V的偏移率增加了

$$\alpha_v = \sum_{i=1}^{n} \left(\sigma_{h,i} - \sigma_{g,i}\right)\sigma_{v,i}$$

定义$q = h/g$，从ITO引理可知，$\sigma_{q,i} = \sigma_{h,i} - \sigma_{g,i}$，故

$$\alpha_v = \sum_{i=1}^{n} \sigma_{q,i}\sigma_{v,i}$$

$$\because dv = \ldots + \sum \sigma_{v,i} v \varepsilon_i \sqrt{dt},\quad dq = \ldots + \sum \sigma_{q,i} q \varepsilon_i \sqrt{dt}$$

由于dz_i不相关，且从ρ的定义有：

$$\rho v q \sigma_v \sigma_q \mathrm{dt} = E(dvdq) - E(dv)(dq)$$

$$\therefore \rho \sigma_v \sigma_q = \sum_{i=1}^{n} \sigma_{q,i}\sigma_{v,i}$$

Quantos

- Quantos are derivatives where the payoff is defined using variables measured in one currency and paid in another currency
- Example: contract providing a payoff of $S_T - K$ dollars ($) where S is the Nikkei stock index (a yen number)

Quantos continued

When we move from the traditional risk neutral world in currency Y to the traditional risk neutral world in currency X, the growth rate of a variable V increases by

$$\rho\sigma_V\sigma_S$$

where σ_V is the volatility of V, σ_S is the volatility of the exchange rate (units of Y per unit of X), and ρ is the coefficient of correlation between the two

Quantos continued

When we move from a forward risk neutral world in currency Y to a forward risk neutral world in currency X (both being wrt to zero-coupon bonds maturing at time T), the growth rate of a variable V increases by

$$\rho\sigma_F\sigma_G$$

where σ_F is the volatility of the forward value of V, σ_G is the volatility of the forward exchange rate (units of Y per unit of X), and ρ is the coefficient of correlation between the two

Siegel's Paradox

An exchange rate S (units of currency Y per untit of currency X) follows the risk-neutral process

$$dS = [r_Y - r_X]Sdt + \sigma_S S dz$$

This implies from Ito's lemma that

$$d(1/S) = [r_X - r_Y + \sigma_S^2](1/S)dt - \sigma_S (1/S)dz$$

Given that the process for S has a drift rate of $r_Y - r_X$, we expect the process for $1/S$ to have a drift of $r_X - r_Y$.

What is going on here?

Siegel's Paradox(2)

- In the process of dS, the numeraire is the money market account in currency Y. In the second equation, the numeraire is also the money market account in currency Y.
- To change the numeraire from Y to X,the growth rate of 1/S increase by $\rho\sigma_V\sigma_S$ where V=1/S and ρ is the correlation between S and 1/S.In this case, ρ=-1, and $\sigma_v = \sigma_S$.It follows that the change of numeraire causes the growth rate of 1/S to increase $-\sigma_S^2$.