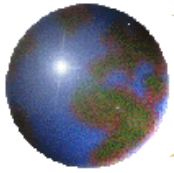


## *Chapter 27*

# Martingales and Measures



# *Derivatives Dependent on a Single Underlying Variable*

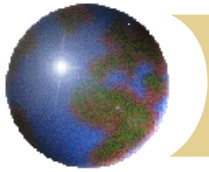
Consider a variable,  $\theta$ , (not necessarily the price of a traded security) that follows the process

$$\frac{d\theta}{\theta} = mdt + sdz$$

Imagine two derivatives dependent only on  $\theta$  and  $t$  with prices  $f_1$  and  $f_2$ . Suppose

$$\frac{df_1}{f_1} = \mu_1 dt + \sigma_1 dz$$

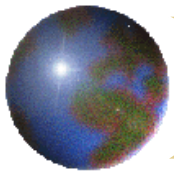
$$\frac{df_2}{f_2} = \mu_2 dt + \sigma_2 dz$$



# *Forming a Riskless Portfolio*

We can set up a riskless portfolio  $\Pi$ , consisting of  
+  $\sigma_2 f_2$  of the 1st derivative and  
-  $\sigma_1 f_1$  of the 2nd derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$
$$\Delta\Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2) \Delta t$$



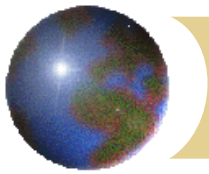
# Market Price of Risk

$$\Delta\Pi = r \Pi\Delta t$$

$$\mu_1\sigma_2 - \mu_2\sigma_1 = r\sigma_2 - r\sigma_1$$

$$\text{or } \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- ✚ This shows that  $(\mu - r)/\sigma$  is the same for all derivatives dependent only on the same underlying variable,  $\theta$ , and  $t$ .
- ✚ We refer to  $(\mu - r)/\sigma$  as the market price of risk for  $\theta$  and denote it by  $\lambda$



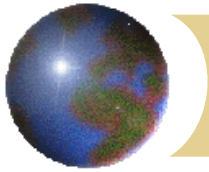
# *Differential Equation for $f$*

Using Ito's lemma to obtain expressions for  $\mu$  and  $\sigma$  in terms of  $m$  and  $s$ . The equation

$$\mu - \lambda\sigma = r$$

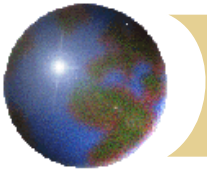
becomes

$$\frac{\partial f}{\partial t} + \theta \frac{\partial f}{\partial \theta} (m - \lambda s) + \frac{1}{2} s^2 \theta^2 \frac{\partial^2 f}{\partial \theta^2} = rf$$



# *Risk-Neutral Valuation*

- ✦ This analogy shows that we can value  $f$  in a risk-neutral world providing the drift rate of  $\theta$  is reduced from  $m$  to  $m - \lambda s$
- ✦ *Note:* When  $\theta$  is not the price of an investment asset, the risk-neutral valuation argument does not necessarily tell us anything about what would happen with  $\theta$  in a risk-neutral world .

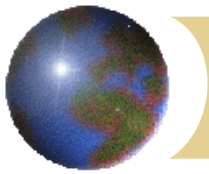


## *Extension of the Analysis to Several Underlying Variables*

$$\frac{d\theta_i}{\theta_i} = m_i dt + s_i dz_i$$

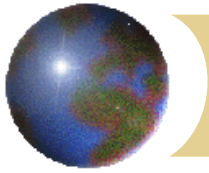
$$\frac{df}{f} = \mu dt + \sum_{i=1}^n \sigma_i dz_i$$

$$\mu - r = \sum_{i=1}^n \lambda_i \sigma_i$$



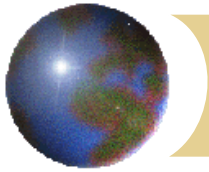
## *Traditional Risk-Neutral Valuation with Several Underlying Variables*

- ✚ A derivative can always be valued as if the world is risk neutral, provided that the expected growth rate of each underlying variable is assumed to be  $m_i - \lambda_i s_i$  rather than  $m_i$ .
- ✚ The volatility of the variables and the coefficient of the correlation between variables are not changed. (CIR, 1985).



## *How to measure $\lambda$ ?*

For a nontraded securities (i.e. commodity), we can use its future market information to measure  $\lambda$ .



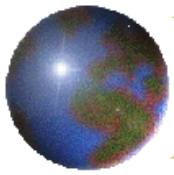
# *Martingales*

- ✚ A martingale is a stochastic process with zero drift

$$d\theta = \sigma dz$$

- ✚ A martingale has the property that its expected future value equals its value today

$$E(\theta_T) = \theta_0$$



# *Alternative Worlds*

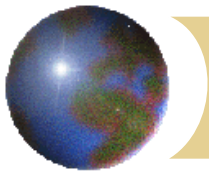
In the traditional risk - neutral world

$$df = rfdt + \sigma dz$$

In a world where the market price of risk

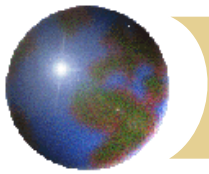
is  $\lambda^*$

$$df = (r + \lambda^* \sigma) fdt + \sigma dz$$



## *A Key Result*

If we set  $\lambda^*$  equal to the volatility of a security  $g$ , then Ito's lemma shows that  $f/g$  is a martingale for all derivative security prices  $f$  ( $f$  and  $g$  are assumed to provide no income during the period under consideration)

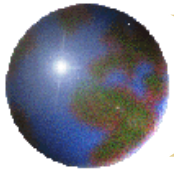


# *Forward Risk Neutrality*

We refer to a world where the market price of risk is the volatility of  $g$  as a world that is forward risk neutral with respect to  $g$ .

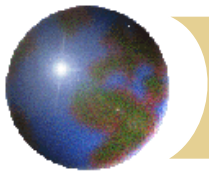
If  $E_g$  denotes a world that is FRN wrt  $g$

$$\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)$$



# *Alternative Choices for the Numeraire Security*

- ⊕ Money Market Account
- ⊕ Zero-coupon bond price
- ⊕ Annuity factor



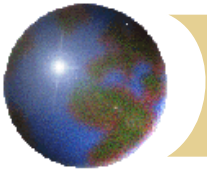
# *Money Market Account as the Numeraire*

⊕ The money market account is an account that starts at \$1 and is always invested at the short-term risk-free interest rate

⊕ The process for the value of the account is

$$dg = r g dt$$

⊕ This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world



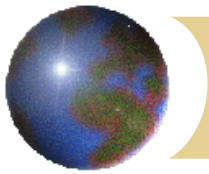
Since  $g_0 = 1$  and  $g_T = e^{\int_0^T r dt}$ , the equation

$$\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)$$

becomes

$$f_0 = \hat{E} \left[ e^{-\int_0^T r dt} f_T \right] = \hat{E}(e^{-\bar{r}T} f_T)$$

where  $\hat{E}$  denotes expectations in the traditional risk - neutral world



# *Zero-Coupon Bond Maturing at time T as Numeraire*

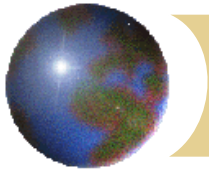
The equation

$$\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)$$

becomes

$$f_0 = P(0, T) E_T [f_T]$$

where  $P(0, T)$  is the zero - coupon bond price and  $E_T$  denotes expectations in a world that is FRN wrt the bond price



# *Forward Prices*

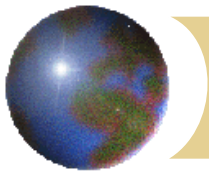
Consider an variable  $S$  that is not an interest rate. A forward contract on  $S$  with maturity  $T$  is defined as a contract that pays off  $S_T - K$  at time  $T$ . Define  $f$  as the value of this forward contract. We have

$$f_0 = P(0, T)[E_T(S_T) - K]$$

$f_0$  equals 0 if  $F=K$ ,

So,  $F = E_T(S_T)$

$F$  is the forward price.



# Interest Rates

$$\frac{1}{[1 + (T_2 - T_1)R(t, T_1, T_2)]} = \frac{P(t, T_2)}{P(t, T_1)}$$

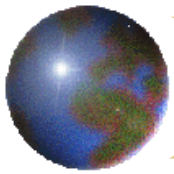
$$R(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left[ \frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)} \right]$$

$$\text{Setting : } f = \frac{1}{T_2 - T_1} [P(t, T_1) - P(t, T_2)]$$

$$\text{and : } g = P(t, T_2)$$

$$\text{then : } R(0, T_1, T_2) = E_2[R(T_1, T_1, T_2)]$$

In a world that is FRN wrt  $P(0, T_2)$  the expected value of an interest rate lasting between times  $T_1$  and  $T_2$  is the forward interest rate



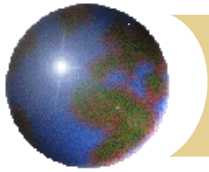
# *Annuity Factor as the Numeraire*

The equation

$$\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)$$

becomes

$$f_0 = A(0) E_A \left[ \frac{f_T}{A(T)} \right]$$



# *Annuity Factors and Swap Rates*

Suppose that  $s(t)$  is the swap rate corresponding to the annuity factor  $A$ .

Then:

$$s(t) = E_A[s(T)]$$

# Extension to Several Independent Factors

In a risk - neutral world

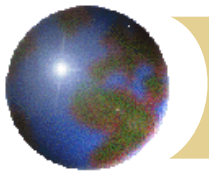
$$df(t) = r(t)f(t)dt + \sum_{i=1}^m \sigma_{fi}(t)f(t)dz_i$$

$$dg(t) = r(t)g(t)dt + \sum_{i=1}^m \sigma_{gi}(t)g(t)dz_i$$

For other worlds that are internally consistent

$$df(t) = \left[ r(t) + \sum_{i=1}^m \lambda_i \sigma_{fi}(t) \right] f(t) dt + \sum_{i=1}^m \sigma_{fi}(t) f(t) dz_i$$

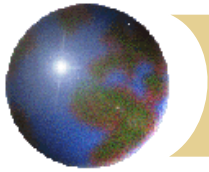
$$dg(t) = \left[ r(t) + \sum_{i=1}^m \lambda_i \sigma_{gi}(t) \right] g(t) dt + \sum_{i=1}^m \sigma_{gi}(t) g(t) dz_i$$



# *Extension to Several Independent Factors continued*

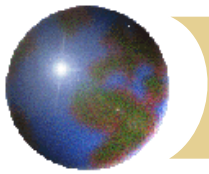
We define a world that is FRN wrt  $g$  as world where  $\lambda_i = \sigma_{gi}$

As in the one-factor case,  $f/g$  is a martingale and the rest of the results hold.



# *Applications*

- ✚ Valuation of a European call option when interest rates are stochastic
- ✚ Valuation of an option to exchange one asset for another



## *Valuation of a European call option when interest rates are stochastic*

Define:  $P(0,T) = e^{-RT}$

$$c = P(0,T)E_T[\max(S_T - X, 0)] = e^{-RT} E_T[\max(S_T - X, 0)]$$

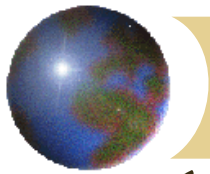
- Assume  $S_T$  is lognormal then:

$$E_T[\max(S_T - X, 0)] = E_T(S_T)N(d_1) - XN(d_2)$$

$$E_T(S_T) = S_0 e^{RT}$$

$$c = S_0 N(d_1) - X e^{-RT} N(d_2)$$

- The result is the same as BS except  $r$  replaced by  $R$ .



## Valuation of an option to exchange one asset( $U$ ) for another( $V$ )

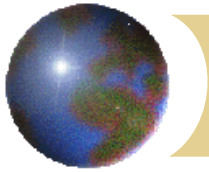
- Choose  $U$  as the numeraire, and set  $f$  as the value of the option so that  $f_T = \max(V_T - U_T, 0)$ , so,

$$f_0 = U_0 E_U \left[ \frac{\max(V_T - U_T, 0)}{U_T} \right] = U_0 E_U \left[ \max\left(\frac{V_T}{U_T} - 1, 0\right) \right]$$

$$= U_0 \left[ E_U \left( \frac{V_T}{U_T} \right) N(d_1) - N(d_2) \right]$$

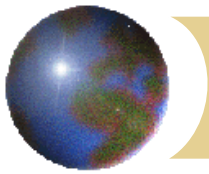
$$\therefore \frac{V_0}{U_0} = E_U \left( \frac{V_T}{U_T} \right)$$

$$\therefore f_0 = V_0 N(d_1) - U_0 N(d_2)$$



## *Change of Numeraire*

When we change the numeraire security from  $g$  to  $h$ , the drift of a variable  $v$  increases by  $\rho\sigma_v\sigma_q$  where  $\sigma_v$  is the volatility of  $v$ ,  $q = h/g$ ,  $\sigma_q$  is the volatility of  $q$ , and  $\rho$  is the correlation between  $v$  and  $q$



⊕ 当记帐单位从 $g$ 变为 $h$ 时， $v$ 的偏移率增加了

$$\alpha_v = \sum_{i=1}^n (\sigma_{h,i} - \sigma_{g,i}) \sigma_{v,i}$$

定义 $q = h/g$ , 从ITO引理可知,  $\sigma_{q,i} = \sigma_{h,i} - \sigma_{g,i}$ , 故

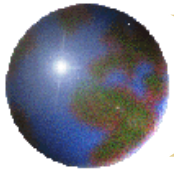
$$\alpha_v = \sum_{i=1}^n \sigma_{q,i} \sigma_{v,i}$$

$$\because dv = \dots + \sum \sigma_{v,i} v \varepsilon_i \sqrt{dt}, \quad dq = \dots + \sum \sigma_{q,i} q \varepsilon_i \sqrt{dt}$$

由于 $dz_i$ 不相关, 且从 $\rho$ 的定义有:

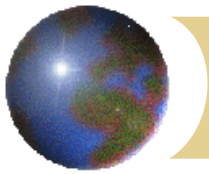
$$\rho v q \sigma_v \sigma_q dt = E(dvdq) - E(dv)(dq)$$

$$\therefore \rho \sigma_v \sigma_q = \sum_{i=1}^n \sigma_{q,i} \sigma_{v,i}$$



# Quantos

- ❖ Quantos are derivatives where the payoff is defined using variables measured in one currency and paid in another currency
- ❖ Example: contract providing a payoff of  $S_T - K$  dollars (\$) where  $S$  is the Nikkei stock index (a yen number)

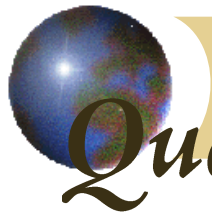


## *Quantos continued*

When we move from the traditional risk neutral world in currency  $Y$  to the traditional risk neutral world in currency  $X$ , the growth rate of a variable  $V$  increases by

$$\rho\sigma_V\sigma_S$$

where  $\sigma_V$  is the volatility of  $V$ ,  $\sigma_S$  is the volatility of the exchange rate (units of  $Y$  per unit of  $X$ ), and  $\rho$  is the coefficient of correlation between the two

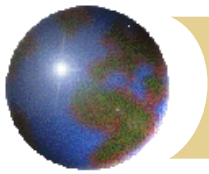


# *Quantos continued*

When we move from a forward risk neutral world in currency  $Y$  to a forward risk neutral world in currency  $X$  (both being wrt to zero - coupon bonds maturing at time  $T$ ), the growth rate of a variable  $V$  increases by

$$\rho\sigma_F\sigma_G$$

where  $\sigma_F$  is the volatility of the forward value of  $V$ ,  $\sigma_G$  is the volatility of the forward exchange rate (units of  $Y$  per unit of  $X$ ), and  $\rho$  is the coefficient of correlation between the two



## *Siegel's Paradox*

An exchange rate  $S$  (units of currency  $Y$  per unit of currency  $X$ ) follows the risk - neutral process

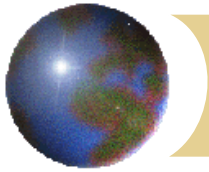
$$dS = [r_Y - r_X]Sdt + \sigma_S Sdz$$

This implies from Ito's lemma that

$$d(1/S) = [r_X - r_Y + \sigma_S^2](1/S)dt - \sigma_S(1/S)dz$$

Given that the process for  $S$  has a drift rate of  $r_Y - r_X$ , we expect the process for  $1/S$  to have a drift of  $r_X - r_Y$ .

What is going on here?



## *Siegel's Paradox(2)*

- ✚ In the process of  $dS$ , the numeraire is the money market account in currency  $Y$ . In the second equation, the numeraire is also the money market account in currency  $Y$ .
- ✚ To change the numeraire from  $Y$  to  $X$ , the growth rate of  $1/S$  increase by  $\rho\sigma_V\sigma_S$  where  $V=1/S$  and  $\rho$  is the correlation between  $S$  and  $1/S$ . In this case,  $\rho=-1$ , and  $\sigma_V=\sigma_S$ . It follows that the change of numeraire causes the growth rate of  $1/S$  to increase  $-\sigma_S^2$ .