Chapter 13

The Black-Scholes-Merton Model

Why Geometric Brownian Motion?

- Geometric brownian motion

$$dS = \mu S dt + \sigma S dz$$

- With Ito lemma, we have

$$d \ln S = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dz$$

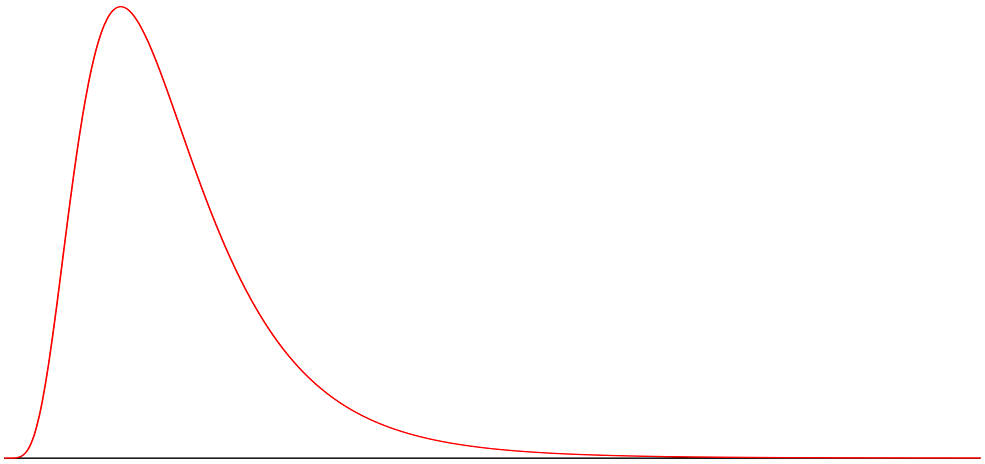
Lognormal Property of Stock Prices

1. S is positive
2. The logarithm of S_T is normal and S_T is lognormally distributed

$$\ln S_T - \ln S_0 \sim \varphi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

$$\ln S_T \sim \varphi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

The Lognormal Distribution



$$E(S_T) = S_0 \, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\,2} e^{2\mu T} (e^{\sigma^2 T} - 1)$$

3. The continuously-compounded return is normally distributed

$$\eta = \frac{\ln S_T - \ln S_0}{T} \sim \varphi\left[\left(\mu - \frac{\sigma^2}{2}\right),\ \frac{\sigma}{\sqrt{T}}\right]$$

Example

- Consider a stock with an initial price of $40, an expected return of 16% per annum, and a volatility of 20% per annum. Then the probability distribution of the stock price in 6 months' time is given by

$$\ln S_T \sim \phi\left(\ln 40 + \left(0.16 - \frac{0.2^2}{2}\right) \times 0.5, 0.2 \times \sqrt{0.5}\right)$$

$$\ln S_T \sim \phi(3.759, 0.141)$$

- The 95% confidence interval is

$$3.759 - 1.96 \times 0.141 < \ln S_T < 3.759 + 1.96 \times 0.141$$

$$e^{3.759-1.96\times 0.141} < S_T < e^{3.759+1.96\times 0.141}$$

Percentage Return

- In a short period of time of length Δt, the percentage return on the stock is normally distributed

$$\frac{\Delta S}{S} \sim \phi\left(\mu \Delta t, \sigma \sqrt{\Delta t}\right)$$

- But in the long run……

The Expected Return

- How to estimate μ ?
- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is $\mu - \sigma^2/2$, not μ
- This is because $$\ln\left(E\left(\frac{S_T}{S_0}\right)\right) \neq E\left(\ln\frac{S_T}{S_0}\right)$$

$$E\left[\ln\left(\frac{S_T}{S_0}\right)\right] = \left(\mu - \frac{\sigma^2}{2}\right) T$$

$$\ln\left[E\left(\frac{S_T}{S_0}\right)\right] = \mu T$$

μ and $\mu - \sigma^2/2$

- μ is the expected return in a very short time, Δt, expressed with a compounding frequency of Δt
- $\mu - \sigma^2/2$ is the expected return in a long period of time expressed with continuous compounding (or, to a good approximation, with a compounding frequency of Δt)

Suppose we have daily data for a period of several months

- μ is the average of the returns in each day [$=E(\Delta S/S)$]
- $\mu-\sigma^2/2$ is the expected return over the whole period covered by the data measured with continuous compounding (or daily compounding, which is almost the same)

Mutual Fund Returns

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The return that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ
- The geometric mean of 12.4% is analogous to $\mu - \sigma^2/2$

The Volatility σ

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in a short time period time Δt is approximately $\sigma\sqrt{\Delta t}$
- If a stock price is $50 and its volatility is 25% per year, what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years (e.g. for weekly data $\tau = 1/52$)
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s , of the u_i´ s
4. The historical volatility estimate is:

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued
- It is assumed that there are 252 trading days in one year for most assets

The Concepts Underlying Black-Scholes-Merton

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

Assumptions of BS Formula

- The short-term interest rate is known and is constant through time.
- The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of stock prices is lognormal. The variance rate of the return on the stock is constant.
- The sock pays no dividends.
- The option is "European".
- There are no transaction costs.
- It's possible to borrow money to buy stocks.
- There are no penalties to short selling.

The Derivation of the Black-Scholes Differential Equation

$$\Delta S = \mu S \, \Delta t + \sigma S \, \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \, \Delta z$$

We set up a portfolio consisting of

$$-1: \text{ derivative}$$

$$+\frac{\partial f}{\partial S}: \text{ shares}$$

This gets rid of the dependence on Δz.

The value of the portfolio, Π, is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The return on the portfolio must be the risk-free rate. Hence

$$\Delta\Pi = r\ \Pi\Delta t$$

$$-\Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t$$

$$\because \Pi(t) = \Pi(0)e^{rt}$$

$$\ln\Pi(t) = \ln\Pi(0) + rt$$

$$\frac{d\Pi}{\Pi} = rdt$$

We substitute for Δf and ΔS in this equation to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is

$$f = S - K \text{ when } t = T$$

- The solution to the equation is

$$f = S - Ke^{-r(T-t)}$$

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the expected return from the stock price is the risk-free rate

2. Calculate the expected payoff from the option

3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is
$$S_0 e^{rT} - K$$
- Present value of expected payoff is
$$e^{-rT}[S_0 e^{rT} - K] = S_0 - K e^{-rT}$$

Valuing an European call with Risk-Neutral Valuation

- 由于

$$c = e^{-r(T-t)}\hat{E}\left[\max(S_T - X, 0)\right]$$

和

$$\ln S_T \sim \ \phi\left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right]$$

- 令

$$W = \frac{\ln S_T - m}{s}$$

- 其中

$$m = \hat{E}(\ln S_T) = \ln S + (r - \frac{\sigma^2}{2})(T - t)$$

$$s = \sqrt{Var(\ln S_T)} = \sigma\sqrt{T - t}$$

- 显然

$$W \sim N(0, 1)$$

- 即随机变量 W 的密度函数 $h(W)$ 为

$$h(W) = \frac{1}{\sqrt{2\pi}} e^{-\frac{W^2}{2}}$$

$$\hat{E}[\max(S_T - X, 0)]$$

$$= \int_{-\infty}^{\infty} \max(S_T - X, 0) h(S_T) dS_T = \int_{X}^{\infty} S_T h(S_T) dS_T - \int_{X}^{\infty} X h(S_T) dS_T$$

$$= \int_{\ln X}^{\infty} e^{\ln S_T} h(\ln S_T) d(\ln S_T) - \int_{\ln X}^{\infty} X h(\ln S_T) d(\ln S_T) = \int_{\frac{\ln X - m}{s}}^{\infty} e^{sW+m} h(W) dW - \int_{\frac{\ln X - m}{s}}^{\infty} X h(W) dW$$

$$= \int_{\frac{\ln X - m}{s}}^{\infty} e^{sW+m} \frac{1}{\sqrt{2\pi}} e^{-\frac{W^2}{2}} dW - \int_{\frac{\ln X - m}{s}}^{\infty} X h(W) dW$$

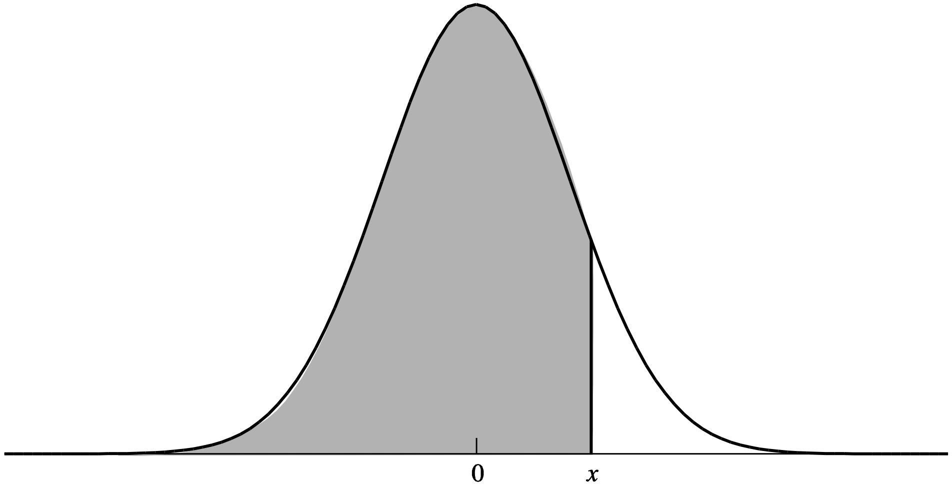
$$= \int_{\frac{\ln X - m}{s}}^{\infty} e^{\frac{s^2}{2}+m} \frac{1}{\sqrt{2\pi}} e^{-\frac{(W-s)^2}{2}} dW - XN\left(\frac{m - \ln X}{s}\right) *$$

$$= \int_{\frac{\ln X - m}{s} - s}^{\infty} e^{\frac{s^2}{2}+m} h(W) dW - XN\left(\frac{\ln \frac{S}{X} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$= \hat{E}(S_T) N\left(\frac{\ln \frac{S}{X} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) - XN\left(\frac{\ln \frac{S}{X} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

The N(x) Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x



- See tables at the end of the book

The Black-Scholes-Merton Formulas

$$c = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$p = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Understanding the B-S-M Formula

- $N(d_2)$ 是在风险中性世界中 S_T 大于 X 的概率，即欧式看涨期权被执行的概率，因此 $Xe^{-r(T-t)}N(d_2)$ 可以看成预期执行期权所需支付的现值。
- 而

$$e^{r(T-t)}SN(d_1) = \hat{E}(S_T)N(d_1)$$

则是在风险中性世界里，一个如果 $S_T > X$ 就等于 S_T 否则就等于 0 的一个变量的期望值， $SN(d_1)$ 则是这个值的贴现值，可以看成期权持有者预期执行期权所得收入的现值。

- 因此整个看涨期权定价公式就是在风险中性世界里期权未来期望回报的现值。

- 我们可以用股票和负债复制期权。
  - 可以证明，$N(d_1) = \frac{\partial f}{\partial S}$，它是构造无风险组合 Π 时的 Δ，是复制投资组合中股票的数量，$SN(d_1)$ 就是股票的市值
  - 而 $Xe^{-r(T-t)}N(d_2)$ 则是复制交易策略中负债的价值。
- 由于主要参数都是时变的，因此这种复制策略是动态复制策略，必须不断调整相关头寸数量。

- 从金融工程的角度来看，欧式看涨期权可以分拆成或有资产看涨期权（Asset-or-nothing Call Option）多头和 X 份或有现金看涨期权（Cash-or-nothing Call Option）空头之和。

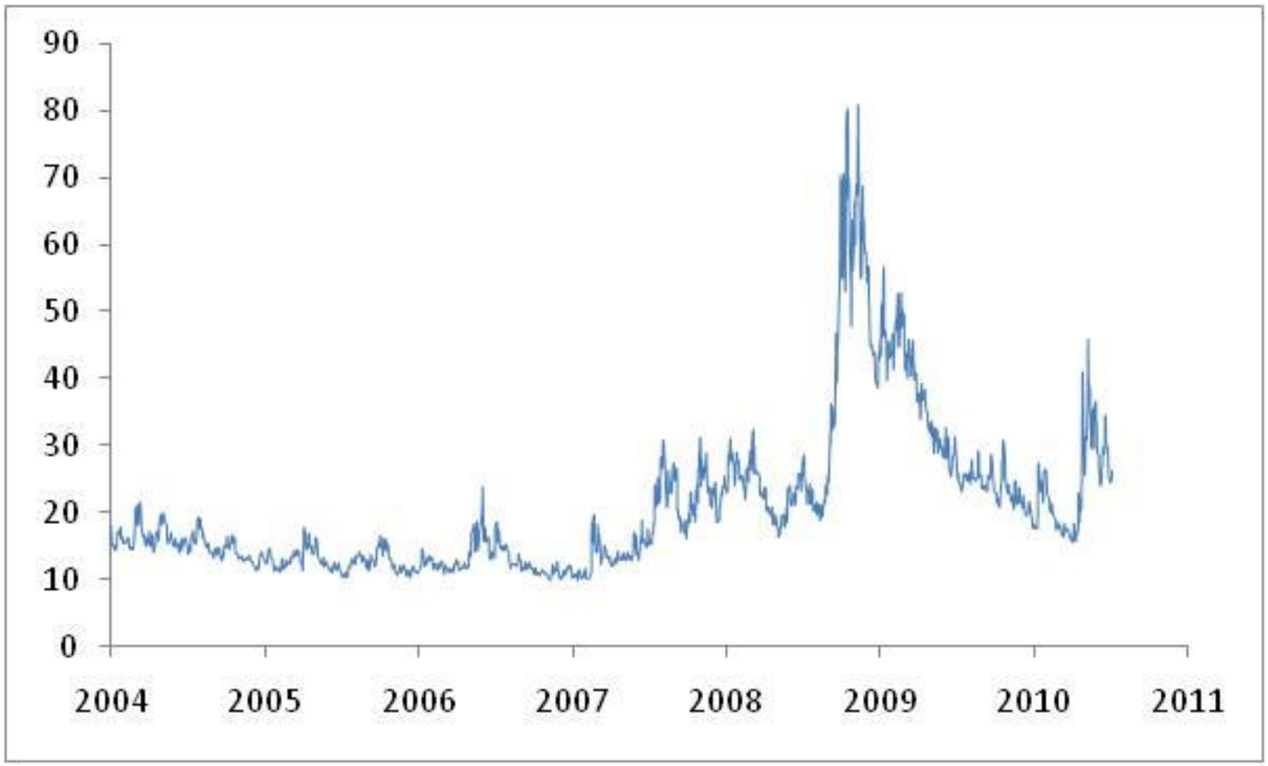
Properties of B-S-M Formula

- As S_0 becomes very large, c tends to $S_0 - Xe^{-rT}$ and p tends to zero
- As S_0 becomes very small, c tends to zero and p tends to $Xe^{-rT} - S_0$

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index



Chapter 24 explains how the index is calculated

An Issue of Warrants & Executive Stock Options

- When a regular call option is exercised the stock that is delivered must be purchased in the open market
- When a warrant or executive stock option is exercised new treasury stock is issued by the company
- If little or no benefits are foreseen by the market the stock price will reduce at the time the issue of is announced.
- There is no further dilution (See Business Snapshot 14.3.)

The Impact of Dilution

- After the options have been issued it is not necessary to take account of dilution when they are valued

- Before they are issued, we can calculate the cost of each option as $N/(N+M)$ times the price of a regular option with the same terms where N is the number of existing shares and M is the number of new shares that will be created if exercise takes place

The cost of the warrants issurance

- The share price immediately after exercise becomes

$$\frac{NS_T + MbX}{N + Mb}$$

- The payoff

$$\max\left(\frac{NS_T + MbX}{N + Mb} - X, 0\right) = \frac{N}{N + Mb}\max\left(S_T - X, 0\right)$$

Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The "dividend" should be the expected reduction in the stock price expected

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date
- Suppose dividend dates are at times t_1, t_2, …t_n. Early exercise is sometimes optimal at time t_i if the dividend at that time is greater than

$$X[1-e^{-r(t_{i+1}-t_i)}]$$

Black's Approximation for Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date