Assignment 2 for Stochastic Calculus

The objective is to price a so-called click-fund, which is a type of barrier option on a stock market index. Let $S_t, t \in [0,T]$ denote the stock index price, and let $s_0, s_1, \ldots, s_m$ denote an increasing sequence of numbers (the click values). Let $S_{\max} = \max_{t\in[0,T]} S_t$, and finally let

$$K_T = \max_{0\le i\le m} \{s_i : s_i \le S_{\max}\}.$$

That is, K_T is the highest clickvalue that S_t has reached between time 0 and time T. Then the payoff, at time T, of this derivative is

$$X_T = \max(S_T, K_T).$$

Assume that S_t follows a geometric Brownian motion, that the interest rate is constant, and that there are no dividend payments (the latter is not really necessary), so that

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\tilde{W}_t, \\ S_t &= S_0 \exp\left(\left[r - \tfrac{1}{2}\sigma^2\right] t + \sigma \tilde{W}_t\right), \end{aligned}$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. The value at time 0 of the derivative is

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}(X_T|\mathcal{F}_0).$$

The assignment is to find this value for a particular choice of the parameters:

- $T = 5$, and there are two click-values, $s_0 = S_0$ and $s_1 = 1.5S_0$. Thus, the original value is guaranteed, and if at some time between 0 and T the stock price reaches $1.5S_0$, then that value is guaranteed.
- The interest rate is $r = 0.04$, and the volatility is $\sigma = 0.20$. For S_0 you may take the value 100, so that V_0 is interpretable as a percentage of the value of the index.

You are asked to value this option in two different ways:

1. Analytical. The payoff structure may be replicated using a combination of *up-and-in-puts*, which are put options, with strike K, that come into existence if S_t reaches a barrier H (see Hull, 1999, p.463). The price at time 0 of such an up-and-in-put is given by

$$P_{ui}(H,K) = K\left(\frac{H}{S_0}\right)^{2\lambda-2} e^{-rT}\Phi(-d_2^*) - \left(\frac{H}{S_0}\right)^{2\lambda} S_0\Phi(-d_1^*),$$

$$\lambda = \frac{r+\frac{1}{2}\sigma^2}{\sigma^2}, \qquad d_1^* = \frac{\ln(H^2/(S_0K)) + (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2^* = d_1^* - \sigma\sqrt{T}.$$

   Note that this is the usual put price if $H = S_0$.

   (a) Consider a portfolio V_0 consisting of:

   - 1 stock, with value $S_0 = s_0$;
   - 1 put with strike $K = s_0$, with value $P_{ui}(s_0, s_0)$;
   - a combination of 1 (long) up-and-in put with $H = K = s_1$, and -1 (short) up-and-in put with $H = s_1, K = s_0$, with combined value $[P_{ui}(s_1,s_1) - P_{ui}(s_1,s_0)]$.

   Show that this portfolio has the same payoff as the click-fund.

   (b) Obtain the value of the two clicks $P_{ui}(s_0,s_0)$ and $[P_{ui}(s_1,s_1) - P_{ui}(s_1,s_0)]$, and hence the price V_0.

2. Monte Carlo simulation. If we have N realizations $\{S_t^i, t \in [0,T]\}, i = 1, 2, \ldots, N$, of the process under $\mathbb{Q}$, then we can calculate the corresponding K_T^i and hence X_T^i for each of those realizations. The average discounted payoff over those realizations:

$$\hat{V}_0 = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} X_T^i$$

is an approximation of V_0: by the law of large numbers, $\hat{V}_0 \to V_0$ as $N \to \infty$. Hence all we need to do is let the computer simulate these realizations; this is called the Monte Carlo method. Obtain in this way an estimate $\hat{V}_0$, and compare it to the analytical price V_0; explain the difference.
Some hints on simulating S_t^i:

(a) You will first need a discretized Brownian motion, calculated at times $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. This is accomplished by

$$\tilde{W}_{t_i} = \tilde{W}_{t_{i-1}} + \sqrt{t_i - t_{i-1}}\varepsilon_i, \quad \tilde{W}_0 = 0,$$

where ε_i are independent $N(0,1)$ random variables. After that, it is easy to calculate $S_{t_i} = S_0 \exp\left(\left[r - \frac{1}{2}\sigma^2\right] t_i + \sigma \tilde{W}_{t_i}\right)$. It is probably reasonable to take $t_i = i/260$, where the idea is that there are 260 trading days in a year and you are simulating the daily (end-of-day) stock price.

(b) In Excel, ε_i is generated by the statement `NORMINV(RAND(),0,1)`, i.e., the inverse normal cumulative distribution function of a uniform random variable. In EViews, it is `NRND`.

(c) You may decide on the number N of replications. In practice, we would want at least $N = 1000$ to get a reasonable approximation. It is easy to perform these calculations in a program where you can use loops (one loop for the N iterations, another for the n steps in the geometric Brownian motion). This may be done in EViews, Microsoft Visual Basic (in combination with Excel), or any other programming language that you prefer, such as Pascal, C/C++, Gauss, Matlab, etc.

(d) Is you do not have sufficient programming experience, the following suggestion may be helpful. Suppose that you have filled an Excel sheet with one realization of $S_t^i, t \in [0,T]$, and calculated the corresponding X_T^i. Then you may repeat the procedure very simply as follows: copy the cell with X_T^i, and then "Paste Special" it in another cell, where you use the option to only paste the Value of the cell in the other cell. You will see that each time you paste like this in another cell, all the random numbers are redrawn. So this is a simple way to fill a number of cells with realizations of X_T.
Instead of doing this by hand 1000 times, you can use a so-called **macro** in Excel. The file ITERATE.BAS (available at the website) contains the macro "Iterate", which copies the contents of cell F99 of Worksheet 1 (you should change this into the cell that contains X_T^i), and then pastes its value it into cell A1 through A1000 of Sheet 2. You can import this macro into your Excel workbook, via Tools, Macro, etc.