Single Factor Heath-Jarrow-Morton Term Structure Models Based on Markov Spot Interest Rate Dynamics

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Abstract

This paper considers the class of Heath-Jarrow-Morton term structure models where the spot interest rate is Markov and the term structure at time t is a function of time, maturity, and the spot interest rate at time t. A representation for this class of models is derived and I show that the functional forms of the forward rate volatility structure and the initial forward rate curve cannot be arbitrarily chosen. I provide necessary and sufficient conditions indicating which combinations of these functional forms are allowable. I also derive a partial differential equation representation of the term structure dynamics that does not require explicit modeling of both the market price of risk and the drift term for the spot interest rate process. Using the analysis presented in this paper, a class of intertemporal term structure models is derived.

I. Introduction

The term structure of interest rates is here defined as the relationship between default-free bonds of all maturities. This study considers the term structure and how it evolves over time; that is, the term structure dynamics. A large number of single factor models of the term structure dynamics have been constructed by characterizing the dynamics of a single proposed underlying stochastic economic variable (usually the spot interest rate), which is assumed to drive the dynamics of the term structure. The underlying stochastic economic variable, henceforth assumed to be the spot interest rate, is commonly modeled as a Markov process. By assuming the term structure at time t is a smooth function of time, the Markovian spot interest rate at time t, and maturity, a partial differential equation (PDE) representation of the term structure dynamics is obtained.[1] For convenience, such frameworks will be referred to as Markovian spot interest rate based paradigms.

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[1] The term "smooth" is used to indicate a term structure that is sufficiently differentiable with respect to each of its variables, which allows the PDE representation to be used.

Classic examples of such paradigms are Vasicek's (1977) framework and the model resulting from the Cox-Ingersoll-Ross (1985b) (hereafter CIR) general equilibrium framework. Both of these frameworks characterize the term structure dynamics by specifying (or determining) the drift and volatility of the spot interest rate process, and the market price of risk for the single source of uncertainty. The Hull-White (1990), (1993) framework is also a Markovian spot interest rate based paradigm. The difference in their approach is the use of different information to characterize the term structure dynamics, which avoids specifying the market price of risk and the drift term of the spot interest rate process. Their information set specifies the spot interest rate volatility and the functional form of the term structure itself, and they use the observed initial term structure and the volatility of the term structure at time zero as exogenous inputs.

The above Markovian spot interest rate based paradigms are fundamentally different from the Heath-Jarrow-Morton (1992) framework (hereafter referred to as HJM) in two respects. First, the HJM framework does not focus on the dynamics of a particular economic variable. Instead, the transitions over time of the whole term structure are modeled. This framework allows for the spot interest rate to be non-Markov. Second, even though the HJM framework is similar to the Vasicek (1977) setting, in the sense that a no-arbitrage condition in the bond market places a restriction on the intertemporal dynamics of the term structure, the HJM framework differs by the information set used to characterize the term structure dynamics. In particular, the HJM framework specifies the forward rate volatility structure (the volatility of each forward rate with fixed maturity date) and the initial forward rate curve. This information is sufficient to characterize the term structure dynamics without explicitly modeling utility dependent parameters (such as the market price of risk) and the drift term (the expected dynamic component) of forward rates. Since explicit modeling of utility dependent parameters is avoided, I will refer to such a framework as "preference free."[2,3]

The differences between the HJM and Markovian spot interest rate based paradigms raise the following issues. First, it is well known that some HJM-based models cannot arise in a Markovian spot interest rate based paradigm. The question then is, which HJM models can arise in such a paradigm? A closely related issue is what implicit structural assumptions are being made about the term structure and the volatility structure in Markovian spot interest rate based paradigms? Second, how can the structure imposed by a Markovian spot interest rate based paradigm be exploited in a HJM setting to provide a methodology for representing the term structure dynamics? An advantage of models constructed in a Markovian spot interest rate based paradigm is the simplification of numerical procedures for evaluating both the term structure and interest rate sensitive claims. This numerical simplification arises because i) the term structure at time t is a function of time t, maturity T, and the realization of the spot interest rate at time t, so the

[2]This does not imply that prices do not depend on preferences, it only indicates that I avoid modeling preferences directly. Information about preferences is embedded in the specified information. Therefore, I consider the framework to be preference free in the same sense that the Black and Scholes framework is preference free for pricing derivatives on stocks.

[3]The Hull and White (1990) framework can also be considered preference free, however, the construct of their framework is considerably different from the HJM framework. The Hull and White (1990) framework will be further discussed in Section IV.

only stochastic variable that needs to be considered is the spot interest rate, and ii) a Markov evolution of the spot interest rate can be modeled with a recombining lattice. The recombining lattice improves the computational speed of evaluating interest rate contingent claims using methods such as Nelson and Ramaswamy (1990) and Li, Ritchken, and Sankarasubramanian (1994). Finally, Markovian spot interest rate based paradigms have generally represented the term structure dynamics via a partial differential equation (PDE) that involves the market price of risk, drift, and volatility of the spot interest rate process. The preference-free frameworks of Hull and White (1990) and HJM do not require the specification of the market price of risk and drift component of the spot interest rate.[4] Consequently, can a general PDE representation of the term structure dynamics be provided that does not involve the market price of risk and the drift component of the spot interest rate process?

To examine the above issues, I work within the HJM framework where the term structure at time t is a function of time t, maturity T, and the Markovian spot interest rate at time t. From this analysis, I find that the functional forms of the forward rate volatility structure and the initial forward rate curve cannot be arbitrarily chosen, however, I provide necessary and sufficient conditions that they must satisfy to be allowable in such a framework.[5,6] These conditions indicate i) which HJM models are allowable in a Markovian spot interest rate based paradigm, and ii) the structural restrictions placed on the initial term structure and the volatility structure when working in a Markovian spot interest rate based paradigm. A representation for HJM term structure models in a Markovian spot interest rate based setting is also derived. This representation provides a method for testing this class of models and, as mentioned above, can be incorporated into existing methods for term structure derivative pricing. To answer the last issue, I derive a PDE representation of the term structure dynamics that does not involve the market price of risk and the drift component of the spot interest rate process. Such a representation may prove useful since a number of techniques exist for solving PDEs. Again, this can be used to test term structure models and be incorporated into existing methods for pricing term structure derivatives.

To illustrate the analysis presented in this paper, a class of forward rate volatility structures is considered, which has the property that the volatility of the forward

[4]Hull and White (1990) do focus on the above mentioned PDE and place a structure on i) the market price of risk, ii) the drift component of the spot interest rate process, and iii) the form of the entire term structure. After some manipulation, Hull and White eliminate the necessity of knowing the exact specification of the spot interest rate's drift component and the market price of risk.

[5]The volatility structures arising from the conditions derived in this paper cannot be considered a subset or a superset of the volatility structure class considered in Ritchken and Sankarasubramanian (1995). Their class of volatility structures allows for single state non-Markov behavior in the spot interest rate, which is not allowable in this paper. Unlike Ritchken and Sankarasubramanian, where the functional form of the initial term structure can be arbitrary, I derive a restriction on the functional form of the initial term structure that allows the use of volatility structures outside the class considered by Ritchken and Sankarasubramanian.

[6]Carverhill (1994) derives a necessary and sufficient condition for deterministic forward rate volatility structures to imply a Markovian spot interest rate process. Volatility structures in this paper are allowed to be stochastic via possible dependence on the Markovian spot interest rate. Carverhill's result is consistent with the volatility structure condition developed in this paper. This consistency is verified by the example presented in Section V where the class of volatility structures considered captures all deterministic volatility structures as a subset.

rate at time t for date T is proportional to the spot interest rate volatility at time t. This example captures all deterministic forward rate volatility structures and further captures a class of volatility structures dependent on the stochastic spot interest rate. In particular, the volatility structures (and models) by Vasicek (1977), CIR (1985b), Hull and White (1990), and the "affine class" of term structure models by Duffie (1992), are captured by this example. In addition, the volatility structure form considered in this example demonstrates which volatility structures considered by Ritchken and Sankarasubramanian (1995) are allowable in a Markovian spot interest rate based paradigm.

The remainder of this paper is organized as follows. Section II provides a brief summary of the HJM framework that defines the notation and framework adopted in this paper. Section III provides the analysis of the HJM framework within a Markovian spot interest rate based paradigm. Section IV provides a PDE representation of the term structure dynamics, which is preference free and does not involve the drift component of the spot interest rate. The embodiment of Hull and White (1990) and HJM are also discussed. Section V provides an example of a class of dynamic term structure models, and Section VI concludes.

II. Summary of the Heath-Jarrow-Morton Framework with a Single Stochastic Variable

For ease of exposition, this section summarizes the single stochastic factor HJM (1992) framework. The HJM framework models the evolution of the term structure by focusing on the instantaneous forward rate at time t for date T, denoted $f(t,T)$, which is defined through the relation,

$$\text{(1)} \qquad \text{Ln}\,[P(t,T)] \;=\; -\int_t^T f(t,v)dv,$$

where $P(t,T)$ is the price of a one dollar face value, default-free, zero coupon bond at time t that will mature at time T. Furthermore, the instantaneous forward rate at time t for date t is the instantaneous spot interest rate at time t, that is, $f(t,t) = r(t)$.

The HJM framework models the uncertain evolution of the term structure via the following stochastic differential equation,

$$\text{(2)} \qquad df(t,T) \;=\; \alpha(\omega,t,T)dt + \gamma(\omega,t,T)dz(t),$$

where $z(t)$ is a Wiener process and ω indicates the possible dependence on the history of the Wiener process. To ensure that no-arbitrage exists in the bond market, HJM (1992) show that the arbitrage-free condition can be represented in terms of the following restriction on the forward rate drift component,[7]

$$\alpha(\omega,t,T) \;=\; \gamma(\omega,t,T)\left[\int_t^T \gamma(\omega,t,v)dv + \lambda(\omega,t)\right],$$

[7]To derive the arbitrage-free condition, refer to HJM (1992) or, alternatively, reexpress the term structure dynamics in terms of bond prices via equation (1) and then apply the standard no-arbitrage condition for bond prices as in, for example, Vasicek (1977).

where $\lambda(\omega, t)$ is interpreted as the market price per unit of risk in the term structure's evolution. Substituting this condition into equation (2) results in the following arbitrage-free characterization of the term structure dynamics in terms of forward rates,

$$(3) \quad df(t,T) = \left\{ \gamma(\omega,t,T) \left[\int_t^T \gamma(\omega,t,v)dv + \lambda(\omega,t) \right] \right\} dt + \gamma(\omega,t,T)dz(t).$$

In the following section of this paper, Section III, the spot interest rate process plays a key role in determining the representation of the term structure dynamics. Since $r(t) = f(t,t)$, equation (3) implies the following structure for the spot interest rate process,[8]

$$(4) \quad dr(t) = \left[\gamma(\omega,t,t)\lambda(\omega,t) + \left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t} \right] dt + \gamma(\omega,t,t)dz(t).$$

This representation has the following intuitive interpretation: the instantaneous change at the origin in forward rates over maturity, $[\partial f(t,T)/\partial T]_{T=t}$, is equal to the expected instantaneous rate of change in the spot interest rate over time conditional on the information available up until time t (denoted $\mathcal{F}_t$), that is $E[dr(t)|\mathcal{F}_t]/dt$, plus the risk premium $-\gamma(\omega,t,t)\lambda(\omega,t)$, which is required for the uncertainty associated with spot interest rate movements.[9]

III. HJM Framework Subject to a Markovian Spot Interest Rate Paradigm

This section considers the class of HJM models that are allowable in a Markovian spot interest rate based paradigm. As a consequence, the following notation is adopted. The volatility structure is now only a function of the spot interest rate at time t (denoted by r), current time t, and the maturity of the forward rate T, consequently, denote $\gamma(\omega,t,T)$ by $\gamma(r,t,T)$. To emphasize that the forward rate for time T depends on the spot rate at time t and time t only, denote the forward rate $f(t,T)$ as $f(r,t,T)$. It is also assumed in this section that the forward rate curve $f(r,t,T)$ is a "smooth" function of r, t, and T.[10]

Within a Markovian spot interest rate based paradigm, the spot interest rate process given in equation (4) has the following structure,

$$(5) \quad \begin{aligned} dr &= \mu(r,t)dt + \sigma(r,t)dz(t), \\ \text{where} \quad r &= r(t) = f(r,t,t), \end{aligned}$$

[8]The derivation of this process appears in Appendix 1 where it is sufficient to assume the term structure $f(t,T)$ is continuous in maturity T and $\partial f(t,T)/\partial T$ exists for all T.

[9]To characterize risk-averse behavior, the risk premium must be positive, that is, $-\gamma(\omega,t,t)\lambda(\omega,t) > 0$.

[10]The term "smooth" is used here to mean the term structure $f(r,t,T)$ is continuous and the following partial derivatives exist: $\partial f(r,t,T)/\partial t \, \forall t$, $\partial^2 f(r,t,T)/\partial r^2 \, \forall r$, and $\partial f(r,t,T)/\partial T \, \forall T$. This is sufficient to i) allow Itô's lemma to be applied to $f(r,t,T)$ and ii) allow differentiation of any forward rate curve with respect to maturity.

$$\begin{aligned}\mu(r,t) &= \gamma(r,t,t)\lambda(r,t) + \left.\frac{\partial f(r,t,T)}{\partial T}\right|_{T=t},\\ \sigma(r,t) &= \gamma(r,t,t).\end{aligned}$$

Since it is assumed that the forward rate curve at time t is a smooth function of time t, maturity T, and the spot interest rate at time t, Itô's lemma applied to $f(r,t,T)$ yields the following stochastic process, which describes the evolution of forward rates,

$$(6)\quad df(r,t,T) = \left[\frac{\partial f(r,t,T)}{\partial r}\mu(r,t) + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2\right]dt + \frac{\partial f(r,t,T)}{\partial r}\sigma(r,t)dz(t).$$

However, the arbitrage-free dynamics of forward rates as given in the HJM framework (equation (3)) must represent the same dynamics as given in equation (6) above. Consequently, the term structure dynamics can be obtained by equating the drift and volatility coefficients across these two equations,

$$(7)\quad \gamma(r,t,T)\left[\int_t^T \gamma(r,t,v)dv + \lambda(r,t)\right] = \frac{\partial f(r,t,T)}{\partial r}\mu(r,t) + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2,$$

$$(8)\quad \gamma(r,t,T) = \frac{\partial f(r,t,T)}{\partial r}\sigma(r,t),$$

where $f(r,t,t) = r$ is a boundary condition.

The HJM framework characterizes the term structure dynamics by specifying the forward rate volatility structure and the initial term structure. This information is also sufficient to solve the system of equations (7) and (8) to provide a unique $f(r,t,T)$, however, the volatility structure and the initial forward rate curve cannot be arbitrarily chosen.[11] Necessary and sufficient conditions are now provided to determine i) which volatility structures are allowable in a Markovian spot interest rate based paradigm (Condition C1), and ii) the set of allowable initial term structures corresponding to a given volatility structure (Condition C2).[12]

Condition C1. A volatility structure $\gamma(r,t,T)$ is allowable in a Markovian spot interest rate based paradigm if there exists a pair of functions $\theta(r,t)$ and $h(t,T)$ that satisfy the following equation,

$$\gamma(r,t,T)\int_t^T \gamma(r,t,v)dv = \frac{\gamma(r,t,T)}{\gamma(r,t,t)}\theta(r,t) + \frac{\partial}{\partial t}\left[\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm\right] + h(t,T) + \frac{1}{2}\gamma(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,t,T)}{\gamma(r,t,t)}\right].$$

[11] See Appendix 2.

[12] Conditions C1 and C2 are determined in Appendix 2.

Condition C1 clearly limits the types of volatility structures allowable. For example, a seemingly reasonable extension to the HJM (1992) constant forward rate volatility structure model[13] ($\gamma(r,t,T) = \sigma$) is a volatility structure that is proportional to r^{β}, that is, $\gamma(r,t,T) = \sigma r^{\beta}$. From Condition C1, $\theta(r,t)$ must be of the form $\sigma^2 r^{2\beta}(T-t) - h(t,T)$, which is impossible since there is no choice for $h(t,T)$ such that $\sigma^2 r^{2\beta}(T-t) - h(t,T)$ is independent of T. Consequently, the volatility structure $\gamma(r,t,T) = \sigma r^{\beta}$ where $\beta \neq 0$ is not allowable in the Markovian spot interest rate based paradigm.

Condition C2. Given a volatility structure that satisfies Condition C1, the corresponding initial forward rate curve must be of the following form,

$$f(r,0,T) \quad = \quad \int_0^r \frac{\gamma(m,0,T)}{\gamma(m,0,0)} dm + k(T),$$

where $k(T) = -\int_0^T h(s,T)ds$ for any valid $h(t,T)$ in Condition C1.

Condition C2 demonstrates that the set of allowable initial term structures depends on the choices for $h(t,T)$ in Condition C1. The following theorem addresses this issue.[14]

Theorem 1. Let $\gamma(r,t,T)$ be a particular volatility structure satisfying Condition C1, and let $\xi(t,T)$ be a deterministic function of t and T.

i) If $\gamma(r,t,T)$ is not of the form $\xi(t,T)\gamma(r,t,t)$ then there is only one valid pair of functions $\theta(r,t)$ and $h(t,T)$ for Condition C1, hence, $k(T)$ in Condition C2 is completely defined by $\gamma(r,t,T)$.

ii) If $\gamma(r,t,T)$ is of the form $\xi(t,T)\gamma(r,t,t)$ then the set of valid pairs of functions $\theta(r,t)$ and $h(t,T)$ for Condition C1 is

$$\theta(r,t) \quad = \quad -\left[\frac{\partial}{\partial t}\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)} dm\right]\Bigg|_{T=t} - c(t),$$

$$h(t,T) \quad = \quad \xi(t,T)\left(c(t) - h_p(t,t)\right) + h_p(t,T),$$

where $h_p(t,T)$ represents any particular valid $h(t,T)$ for Condition C1 and $c(t)$ is an arbitrary function. Further, for Condition C2, any arbitrary function $k(T)$ where $k(0) = 0$ can be obtained by a unique choice of $c(t)$.

Theorem 1 provides two important results. First, for a given allowable volatility structure $\gamma(r,t,T)$ that is not of the form $\xi(t,T)\gamma(r,t,t)$, then the volatility structure uniquely determines the initial term structure via Condition C2. Consequently, fitting the initial term structure and choosing a volatility structure cannot be done independently. This result also implies that an allowable volatility structure not of the form $\xi(t,T)\gamma(r,t,t)$ is enough to uniquely solve for $f(r,t,T)$ in the system of equations (7) and (8). Second, for a given allowable volatility structure $\gamma(r,t,T)$

[13]The constant forward rate volatility structure model is allowable in a Markovian spot interest rate based paradigm; it is a special case of the class of models arising in Section V.

[14]The proof of Theorem 1 is in Appendix 2.

that is of the form $\xi(t,T)\gamma(r,t,t)$, $k(T)$ in Condition C2 can be chosen to correspond to any observed initial term structure that is once differentiable in maturity.

Assuming Conditions C1 and C2 are met, a representation of the term structure dynamics resulting from the solution to the system of equations (7) and (8), in terms of the forward rate volatility structure and the initial forward rate curve, is[15]

$$(9)\quad f(r,t,T) = -\int_0^t \frac{\gamma(r,s,T)}{\gamma(r,s,s)}\theta(r,s)ds - \frac{1}{2}\int_0^t \gamma(r,s,s)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,s,T)}{\gamma(r,s,s)}\right]ds + \int_0^t \gamma(r,s,T)\left[\int_s^T \gamma(r,s,v)dv\right]ds + f(r,0,T).$$

Theorem 1 indicates that if the volatility structure is not of the form $\xi(t,T)\gamma(r,t,t)$, then $\theta(r,t)$ and $f(r,0,T)$ can be determined from the specification of the volatility structure via Conditions C1 and C2 directly. Otherwise, if the volatility structure is of the form $\xi(t,T)\gamma(r,t,t)$, then choose $k(T)$ in Condition C2 by fitting the initial forward rate curve and then $\theta(r,t)$ can be determined from the following equation,[16]

$$(10)\quad \int_0^t \frac{\gamma(r,s,t)}{\gamma(r,s,s)}\theta(r,s)ds = f(r,0,t) - \frac{1}{2}\int_0^t \gamma(r,s,s)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,s,t)}{\gamma(r,s,s)}\right]ds + \int_0^t \gamma(r,s,t)\left[\int_s^t \gamma(r,s,v)dv\right]ds - r.$$

In the above representation, the function $\theta(r,t)$ can be interpreted as the drift of the spot interest rate in an equivalent risk-neutral economy, or alternatively, the slope of the forward rate curve at the origin,[17]

$$(11)\quad \theta(r,t) = \mu(r,t) - \lambda(r,t)\sigma(r,t) = \left.\frac{\partial f(r,t,T)}{\partial T}\right|_{T=t}$$

This is consistent with Vasicek's (1977) observation that in his framework, where the drift and volatility of spot interest rates are given, the market price of risk can

[15]This representation appears in Appendix 2, equation (A-4). Note, in this solution, r is the spot interest rate at time t.

[16]Equation (10) is obtain from equation (9) evaluated at $T = t$. Note, for a volatility structure of the form $\gamma(r,t,T) = \xi(t,T)\gamma(r,t,t)$, $\theta(r,s)$ and $f(r,0,T)$ are of the form,

$$\theta(r,s) = -\frac{\partial}{\partial s}\int_0^r \frac{\gamma(m,s,T)}{\gamma(m,s,s)}dm\Big|_{T=s} - c(s) \text{ and } f(r,0,T) = \int_0^r \frac{\gamma(m,0,T)}{\gamma(m,0,0)}dm + k(T),$$

where $c(s)$ is arbitrary and $k(T)$ is chosen to fit the observed initial term structure.

[17]By definition, $\theta(r,t) = \mu(r,t) - \lambda(r,t)\sigma(r,t)$ (see equation (A-1) in Appendix 2), which is the drift of the spot interest rate in an equivalent risk-neutral economy, and the interpretation of being the slope of the forward rate curve at the origin follows from equation (5).

be obtained by the empirical measurement of the "slope at the origin of the yield curves" (p. 184).[18]

Even though the above discussion demonstrates the restrictive nature of Markovian spot interest rate based paradigms, a wide variety of models is still available. Section V of this paper considers, as an example, the class of HJM models, subject to a Markovian spot interest rate based paradigm, which have a volatility structure of the form $\gamma(r, t, T) = \xi(t, T)\gamma(r, t, t)$. This example also serves to further the analysis of this section by determining the class of volatility structures that is able to fit any observed initial term structure.

IV. Preference-Free Partial Differential Equation Representations of the Term Structure Dynamics

In a Markovian spot interest rate based paradigm, the price of a one dollar face value, default-free, zero-coupon bond at time t that matures at time T depends on the spot rate at time t, time t and maturity T only. Consequently, such a bond is denoted $P(r, t, T)$. Generally, representations of the term structure dynamics in a Markovian spot interest rate based paradigm have been via the following bond price partial differential equation, which involves the market price of risk, drift, and volatility of the spot interest rate process ($\lambda(r, t)$, $\mu(r, t)$, and $\sigma(r, t)$, respectively),

$$\begin{aligned}\frac{\partial P(r,t,T)}{\partial r}\,[\mu(r,t) - \lambda(r,t)\sigma(r,t)] + \frac{\partial P(r,t,T)}{\partial t} \\ + \frac{1}{2}\frac{\partial^2 P(r,t,T)}{\partial r^2}\sigma(r,t)^2 - rP(r,t,T) &= 0,\end{aligned}$$

where $P(r, t, t) = 1$. The Vasicek (1977) and CIR (1985b) frameworks specify/determine the market price of risk, drift, and volatility of the spot interest rate process, which is enough to provide a unique solution $P(r, t, T)$ to the above PDE. It is interesting to note that to solve the above PDE, it is only necessary to specify $\mu(r, t) - \lambda(r, t)\sigma(r, t)$ and $\sigma(r, t)$ where, from equation (11), we know that $\mu(r, t) - \lambda(r, t)\sigma(r, t)$ can be interpreted as the slope of the forward rate curve at the origin. Therefore, this PDE can be considered a preference-free representation of the term structure dynamics when these two components are specified.

A PDE representation that is consistent with both the Hull and White (1990) and HJM frameworks, in the sense that the specification of the drift component of the spot interest rate and the market price of risk are not specified, is the following,[19]

$$(12)\qquad \frac{\partial^2 P(r,t,T)}{\partial r \partial T}\left[-\frac{\partial P(r,t,T)}{\partial t} - \frac{1}{2}\frac{\partial^2 P(r,t,T)}{\partial r^2}\sigma(r,t)^2 + rP(r,t,T)\right]$$

[18] Note that the yield $y(t, T)$ is defined as

$$y(t,T) = \frac{1}{T-t}\int_t^T f(t,v)dv, \text{ consequently, } \left.\frac{\partial y(t,T)}{\partial T}\right|_{T=t} = \frac{1}{2}\left.\frac{\partial f(t,T)}{\partial T}\right|_{T=t}$$

[19] The derivation of this partial differential equation appears in Appendix 3 where $P(r, t, T)$ is assumed to be sufficiently differentiable in each variable for all partial derivatives in equation (12) to exist.

$$+\frac{\partial P(r,t,T)}{\partial r}\left[\frac{\partial^2 P(r,t,T)}{\partial t \partial T}+\frac{1}{2}\frac{\partial^3 P(r,t,T)}{\partial r \partial r \partial T}\sigma(r,t)^2 - r\frac{\partial P(r,t,T)}{\partial T}\right] = 0,$$

where $P(r,T,T)=1$. To consider the HJM models from Section III, it is convenient to reexpress the above PDE in terms of forward rates as follows,[20]

$$\begin{aligned}
&\frac{\partial^2 f(r,t,T)}{\partial r \partial T}\left[-\frac{\partial f(r,t,T)}{\partial t}-\frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2\right] \\
(13)\qquad &-\frac{\partial f(r,t,T)}{\partial r}\left[\frac{\partial f(r,t,T)}{\partial r}\sigma(r,t)\right]^2 \\
&+\frac{\partial f(r,t,T)}{\partial r}\left[\frac{\partial^2 f(r,t,T)}{\partial t \partial T}+\frac{1}{2}\frac{\partial^3 f(r,t,T)}{\partial r \partial r \partial T}\sigma(r,t)^2\right] = 0,
\end{aligned}$$

where $f(r,t,t)=r$.

The Hull and White (1990) framework is obtained by specifying: i) the spot interest rate volatility to be of the form $\sigma(t)$ or $\sigma(t)\sqrt{r}$, and ii) $P(r,t,T)=A(t,T)\exp\{-B(t,T)r\}$ for some functions $A(t,T)$ and $B(t,T)$. The term structure dynamics can be expressed purely in terms of $A(0,T)$ and $B(0,T)$. $B(0,T)$ can be chosen to correspond to the initial forward rate volatility structure[21] and $A(0,T)$ is chosen such that $P(r,0,T)=A(0,T)\exp\{-B(0,T)r\}$ corresponds to the observed initial term structure. This is enough to solve PDE (12).

The HJM framework requires the specification of the forward rate volatility structure and the full observation of the initial forward rate curve. Consequently, this information is enough to solve PDE (13). Note that in the Markovian spot interest rate based paradigm (analyzed in Section III), the forward rate volatility structure $\gamma(r,t,T)$ is equal to $[\partial f(r,t,T)/\partial r]\sigma(r,t)$. Further, equation (5) states that $\gamma(r,t,t)=\sigma(r,t)$, which indicates the implicit specification of the spot interest rate volatility from the specification of the forward rate volatility structure. It is important to note that the functional forms of both the forward rate volatility structure and the initial term structure must be such that a solution to PDE (13) exists. The required restrictions on these functions are Conditions C1 and C2 presented in Section III.

V. A Preference-Free Dynamic Term Structure Example: A Volatility Structure of the Form $\gamma(r,t,T)=\xi(t,T)\gamma(r,t,t)$

This example provides a class of HJM models, subject to a Markovian spot interest rate based paradigm, where the chosen volatility structure is of the form $\gamma(r,t,T)=\xi(t,T)\gamma(r,t,t)$. Section III shows that this class of models is the only class that has the property of being able to fit any observed initial term structure given the volatility structure. Volatility structures of the form $\gamma(r,t,T)=$

[20]The derivation of this partial differential equation appears in Appendix 4 where $f(r,t,T)$ is assumed to be sufficiently differentiable with respect to each variable for all partial derivatives in equation (13) to exist.

[21]From equation (1), $\partial \mathrm{Ln}(P(r,t,T))/\partial T=-f(t,T)$, which implies $\partial B(t,T)/\partial T=-\partial f(r,t,T)/\partial r$, but from equations (5) and (8), $\partial f(r,t,T)/\partial r=\gamma(r,t,T)/\gamma(r,t,t)$. Note the boundary condition $P(r,t,t)=1$ implies $B(t,t)=0$ and, therefore, $B(t,T)=-\int_t^T \gamma(r,t,v)/\gamma(r,t,t)dv$. Hence, $B(0,T)$ is fully determined from the initial forward rate volatility structure.

$\xi(t,T)\gamma(r,t,t)$ capture many existing examples already presented in the term structure literature: for example, the volatility structures implied by the models of Vasicek (1977); CIR (1985b); Duffie's (1992) "affine class;" the constant and exponential decay forward rate volatility structures given in HJM (1992); and all the volatility structures allowable in the Hull and White (1990) framework.[22] This example also captures the class of volatility structures considered in Ritchken and Sankarasubramanian (1995) that are allowable in a Markovian spot interest rate based paradigm.[23] It is important to also note that some volatility structures generated in this example are not allowable in the Ritchken and Sankarasubramanian (1995) framework.

First, to determine the allowable form for $\theta(r,t)$, evaluate Condition C1 at $T=t$,

$$\theta(r,t) \quad = \quad -\left(\left.\frac{\partial \xi(t,T)}{\partial t}\right|_{T=t} \right) r - h(t,t).$$

Substituting back into Condition C1 yields the following equation, which must hold for allowable volatility structures,

$$\gamma(r,t,t)^2 \xi(t,T) \int_t^T \xi(t,v)dv \quad = \quad \xi(t,T) \left[\left(- \left.\frac{\partial \xi(t,T)}{\partial t}\right|_{T=t} \right) r - h(t,t) \right] + \frac{\partial \xi(t,T)}{\partial t} r + h(t,T).$$

This equation can only have solutions if the spot interest rate volatility is of the form,

$$\gamma(r,t,t)^2 \quad = \quad a(t)r + b(t). \tag{14}$$

Consequently, the following two equations must hold.

First,
$$b(t)\xi(t,T) \int_t^T \xi(t,v)dv + \xi(t,T)h(t,t) \quad = \quad h(t,T).$$

Noting $\xi(t,T) = \gamma(r,t,T)/\gamma(r,t,t)$ implies $\xi(t,t) = 1$, there always exists a $h(t,T)$ for any arbitrary choice of $h(t,t)$, which ensures that this equation holds.

Second,

$$a(t)\xi(t,T) \int_t^T \xi(t,v)dv + \xi(t,T) \left(\left.\frac{\partial \xi(t,T)}{\partial t}\right|_{T=t} \right) \quad = \quad \frac{\partial \xi(t,T)}{\partial t}. \tag{15}$$

[22]See Appendix 5 for a reconciliation of these volatility structures with this example.

[23]The class of volatility structures considered by Ritchken and Sankarasubramanian (1995) are of the form $\sigma(\omega,t)\exp(\int_t^T \kappa(x)dx)$. If the spot interest rate process is Markov with respect to a single state variable, then $\sigma(\omega,t) = \sigma(r,t)$. Consequently, their class conforms to the structure considered in the example.

The solution to this equation needs to be considered under two cases, namely when $a(t) = 0$ and $a(t) \neq 0$.

Case 1. When $a(t) = 0$.

Under this scenario, the forward rate volatility structure is deterministic and of the form $\xi(t,T)\sqrt{b(t)}$. This structure also implies that $b(t)$ is nonnegative. For convenience, let the volatility structure $\gamma(r,t,T)$ be denoted by $\gamma(t,T)$ and $\gamma(t,t) = \sigma(t)$.

Solving equation (15) when $a(t) = 0$ shows that the forward rate volatility structure must be of the following functional form,[24]

$$(16) \quad \gamma(t,T) = \sigma(t)\exp\{\zeta(T) - \zeta(t)\} \quad \text{for arbitrary functions } \sigma(t) \text{ and } \zeta(t).$$

From equations (9), (16), Condition C2, and Theorem 1, the dynamic term structure model is

$$\begin{aligned} f(r,t,T) &= -e^{\zeta(T)}\int_0^t e^{-\zeta(s)}\theta(r,s)ds + \int_0^t \sigma(s)^2 e^{\zeta(T)-\zeta(s)}\int_s^T e^{\zeta(v)-\zeta(s)}dv\,ds \\ &\quad + re^{\zeta(T)-\zeta(0)} + k(T), \end{aligned}$$

where $k(0) = 0$ and $k(T)$ can be chosen to ensure an exact correspondence to the observed initial term structure.

Further, from equation (10) and Condition C2,

$$\begin{aligned} \int_0^t e^{-\zeta(s)}\theta(r,s)ds &= e^{-\zeta(t)}\Bigg[\int_0^t \sigma(s)^2 e^{\zeta(t)-\zeta(s)}\int_s^t e^{\zeta(v)-\zeta(s)}dv\,ds \\ &\quad - r + re^{\zeta(t)-\zeta(0)} + k(t)\Bigg]. \end{aligned}$$

On simplification, the term structure model is

$$(17) \quad f(r,t,T) = e^{\zeta(T)}\int_t^T e^{\zeta(v)}dv\int_0^t \sigma(s)^2 e^{-2\zeta(s)}ds + (r - k(t))\,e^{\zeta(T)-\zeta(t)} + k(T),$$

where $k(0) = 0$.

In summary, I show that any forward rate volatility structure that is exclusively a function of time t and maturity T can only occur in a Markovian spot interest rate based paradigm if it is of the form given in equation (16). This form is consistent with the Carverhill (1994) necessary and sufficient condition derived for deterministic forward rate volatility structures to imply a Markovian spot interest rate process. Further, this volatility structure is of the Ritchken and Sankarasubramanian (1995) form indicating that their deterministic volatility structures are allowable in a Markovian spot interest rate based paradigm. Finally, for volatility

[24] The solution was not written in the form $\gamma(t,T) = \sigma(t)A(T)/A(t)$ to avoid undefined $\gamma(t,T)$.

structures of the form given in equation (16), the corresponding term structure model is given in equation (17) where $k(T)$ is chosen to correspond to the actual observed initial term structure.[25]

Case 2. When $a(t) \neq 0$.

Under this scenario, the forward rate volatility structure is dependent on the stochastic spot interest rate. For convenience, denote $\gamma(r,t,t)$ by $\sigma(r,t)$ and from equation (14), $\sigma(r,t)^2 = a(t)r + b(t)$; consequently, it is necessary that $a(t)r+b(t) \geq 0$. This restriction indicates the following regions for which the spot interest rate can exist:

i) if $a(t) < 0$, then the spot interest rate must be in the region $r \leq (b(t)/a(t))$;
ii) if $a(t) > 0$, then the spot interest rate must be in the region $r \geq (b(t)/a(t))$.

An individual modeling the term structure dynamics may consider the most reasonable region for spot interest rates to lie within is the latter since a lower bound for the spot interest rate is defined[26] rather than an upper bound as is the case when $a(t) < 0$.

Solving equation (15) shows that the forward rate volatility structure must be of the following functional form,[27]

$$\gamma(r,t,T) = \frac{2A(T)\left(C'(t) - A(t)\right)\sqrt{a(t)r + b(t)}}{a(t)\left(\int_t^T A(s)ds + C(t)\right)^2} \tag{18}$$

for an arbitrary function $C(t)$ where

$$A(t) = \frac{C'(t) \pm \sqrt{C'(t)^2 - 2a(t)C(t)^2}}{2};$$

$$C'(t)^2 \geq 2a(t)C(t)^2; \quad \text{and } C'(t) \text{ represents } \frac{dC(t)}{dt}.$$

Now that the class of allowable volatility structures has been determined, Condition C2 and Theorem 1 state that the initial forward rate curve must be of the form,

$$f(r,0,T) = \frac{2A(T)\left(C'(0) - A(0)\right)}{a(0)\left(\int_0^T A(s)ds + C(0)\right)^2} \cdot r + k(T),$$

where $k(0) = 0$, and $k(T)$ can be chosen to ensure an exact correspondence to the observed initial term structure.

[25] Hull and White (1990) supply a closed form solution for their "Extended Vasicek" model. This model has a volatility structure of the form $D(t,T)\sigma(t)$ and, therefore, the closed form expression of the entire term structure given in equation (17) is the same as the Extended Vasicek model expressed in terms of forward rates.

[26] Here, the spot interest rate volatility is of the form $\sqrt{a(t)r + b(t)}$ so if the modeler wishes to set a lower bound to the spot interest rate process, he simply needs to ensure that $a(t) > 0$.

[27] See Appendix 6.

In summary, I show that any forward rate volatility structure that is of the form $\xi(t,T)\gamma(r,t,t)$ can only occur in a Markovian spot interest rate based paradigm if it is of the form given in equation (18). The class of Ritchken and Sankarasubramanian (1995) volatility structures allowable in such a paradigm is also captured by the form $\xi(t,T)\gamma(r,t,t)$. However, the only case where equation (18) is of the Ritchken and Sankarasubramanian class is when $a(t) = 0$.[28] Therefore, the only volatility structures of the Ritchken and Sankarasubramanian class that are allowable in a Markovian spot interest rate based paradigm are those that are deterministic. Finally, since a closed form expression for the term structure dynamics could not be found, knowledge of allowable volatility structures and the form of the initial forward rate curve enables the use of numerical methods to model the term structure dynamics via either i) equations (9) and (10) from Section III, or ii) PDE (13) from Section IV.

VI. Summary and Conclusions

In this paper, I study and discuss a representation for those HJM dynamic term structure models that are allowable in a Markovian spot interest rate based paradigm. Two necessary and sufficient conditions emerged that define the combinations of functional forms for the initial forward rate curve and the forward rate volatility structure to be allowable. Further, I show that only allowable volatility structures of the form $\xi(t,T)\gamma(r,t,t)$ have enough flexibility in their corresponding initial term structure such that the resulting term structure model can fit any observed initial forward rate curve. To unify existing preference-free frameworks in a Markovian spot interest rate based paradigm, a PDE representation of the term structure dynamics has also been derived. This PDE representation is consistent with other preference-free frameworks in the term structure literature since it does not involve the market price of risk and the drift component of the spot interest rate process.

To demonstrate the usefulness of the dynamic term structure representations presented in Sections III and IV, the class of allowable forward rate volatility structures of the form $\xi(t,T)\gamma(r,t,t)$ has been derived. For the case where the volatility structure is deterministic, a closed form expression for the resulting class of dynamic term structure models has been provided. For the general case, the set of allowable volatility structures and corresponding initial forward rate curves has been derived. This is useful as numerical methods may be utilized to solve for the term structure dynamics via either equations (9) and (10) from Section III, or the preference-free PDE representations from Section IV. Since the class of models considered in this paper has the property that the term structure at time t is a function of the spot interest rate at time t, time and maturity only, they can easily be incorporated into lattice methods to price interest rate contingent claims such as the methods of Nelson and Ramaswamy (1990) and Li, Ritchken, and Sankarasubramanian (1994).

[28]See Appendix 7.

Appendix 1. Spot Rate Process Implied by the HJM Framework

Using the fact that $r(t) = f(t,t)$, the spot interest rate dynamics implied by equation (3) is

$$r(t) = f(0,t) + \int_0^t \alpha(\omega, s, t)ds + \int_0^t \gamma(\omega, s, t)dz(s).$$

Differentiating with respect to t results in the following specification for the dynamics of the spot rate,

$$\begin{aligned} dr(t) &= \frac{\partial f(0,t)}{\partial t}dt + \left[\int_0^t \frac{\partial \alpha(\omega,s,t)}{\partial t}ds\right]dt + \left[\int_0^t \frac{\partial \gamma(\omega,s,t)}{\partial t}dz(s)\right]dt \\ &\quad + \alpha(\omega,t,t)dt + \gamma(\omega,t,t)dz(t) \\ &= \left\{\frac{\partial}{\partial T}\left[f(0,T) + \int_0^t \alpha(\omega,s,T)ds + \int_0^t \gamma(\omega,s,T)dz(s)\right]\Bigg|_{T=t}\right\}dt \\ &\quad + \alpha(\omega,t,t)dt + \gamma(\omega,t,t)dz(t) \\ &= \left[\alpha(\omega,t,t) + \frac{\partial f(t,T)}{\partial T}\Bigg|_{T=t}\right]dt + \gamma(\omega,t,t)dz(t). \end{aligned}$$

From equation (2), $\alpha(\omega,t,t) = \gamma(\omega,t,t)\lambda(\omega,t)$. Therefore, the spot rate process implied by the HJM framework is

$$dr(t) = \left[\gamma(\omega,t,t)\lambda(\omega,t) + \frac{\partial f(t,T)}{\partial T}\Bigg|_{T=t}\right]dt + \gamma(\omega,t,t)dz(t). \qquad \square$$

Appendix 2. Analysis of the HJM Framework Subject to a Markovian Spot Interest Rate Based Paradigm

The term structure dynamics is defined by the system of equations (7) and (8). Using equation (8), rewrite equation (7) as

$$\text{(A-1)} \qquad \gamma(r,t,T)\int_t^T \gamma(r,t,v)dv = \frac{\gamma(r,t,T)}{\sigma(r,t)}\theta(r,t) + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2$$

where $\theta(r,t) = \mu(r,t) - \lambda(r,t)\sigma(r,t)$. Further, from equation (8),

$$\text{(A-2)} \qquad \frac{\partial^2 f(r,t,T)}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\gamma(r,t,T)}{\sigma(r,t)}\right].$$

Putting (A-2) into (A-1) and noting that $\sigma(r,t) = \gamma(r,t,t)$ yields

$$\text{(A-3)} \quad \gamma(r,t,T)\int_t^T \gamma(r,t,v)dv = \frac{\gamma(r,t,T)}{\gamma(r,t,t)}\theta(r,t) + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\gamma(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,t,T)}{\gamma(r,t,t)}\right].$$

The term structure dynamics can be represented as follows by integrating equation (A-3) with respect to t,

$$\text{(A-4)} \quad f(r,t,T) = -\int_0^t \frac{\gamma(r,s,T)}{\gamma(r,s,s)}\theta(r,s)ds - \frac{1}{2}\int_0^t \gamma(r,s,s)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,s,T)}{\gamma(r,s,s)}\right]ds + \int_0^t \gamma(r,s,T)\left[\int_s^T \gamma(r,s,v)dv\right]ds + f(r,0,T).$$

The fact that $r = f(r,T,T)$ implies

$$\text{(A-5)} \quad r = -\int_0^T \frac{\gamma(r,s,T)}{\gamma(r,s,s)}\theta(r,s)ds - \frac{1}{2}\int_0^T \gamma(r,s,s)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,s,T)}{\gamma(r,s,s)}\right]ds + \int_0^T \gamma(r,s,T)\left[\int_s^T \gamma(r,s,v)dv\right]ds + f(r,0,T).$$

Subtracting (A-5) from (A-4) provides the following representation for the term structure dynamics,

$$\text{(A-6)} \quad f(r,t,T) = \int_t^T \frac{\gamma(r,s,T)}{\gamma(r,s,s)}\theta(r,s)ds + \frac{1}{2}\int_t^T \gamma(r,s,s)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,s,T)}{\gamma(r,s,s)}\right]ds - \int_t^T \gamma(r,s,T)\left[\int_s^T \gamma(r,s,v)dv\right]ds + r.$$

Equation (A-5) demonstrates that $\theta(r,t)$ can be defined in terms of the initial forward rate curve and the forward rate volatility structure. Consequently, equations (A-5) and (A-6) together demonstrate that the specification of the initial forward rate curve and the forward rate volatility structure is sufficient to solve the system of equations (7) and (8) to provide a unique $f(r,t,T)$.

Now, equation (8) implies that if a solution $f(r,t,T)$ exists to the system of equations (7) and (8), then it must be of the form,

$$f(r,t,T) = \int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm + g(t,T) \quad \text{for some appropriate } g(t,T).$$

The boundary condition $f(r,t,t) = r$ for all t implies that $g(t,T)$ is of the form $g(t,T) = -\int_t^T h(s,T)ds$. Therefore, $f(r,t,T)$ must be of the form,

$$f(r,t,T) = \int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm - \int_t^T h(s,T)ds \tag{A-7}$$

for some appropriate $h(t,T)$.

Note that equations (A-6) and (A-7) represent the same term structure dynamics, therefore, the following equation must always hold,

$$\int_t^T \frac{\gamma(r,s,T)}{\gamma(r,s,s)}\theta(r,s)ds + \frac{1}{2}\int_t^T \gamma(r,s,s)^2 \frac{\partial}{\partial r}\left[\frac{\gamma(r,s,T)}{\gamma(r,s,s)}\right] ds + r \tag{A-8}$$
$$= \int_t^T \gamma(r,s,T)\left[\int_s^T \gamma(r,s,v)dv\right] ds + \int_0^r \frac{\gamma(r,t,T)}{\gamma(m,t,t)}dm - \int_t^T h(s,T)ds.$$

Differentiating (A-8) with respect to t provides the following necessary and sufficient condition for $\gamma(r,t,T)$ to be an allowable volatility structure in a Markovian spot interest rate based paradigm.

Condition C1. A volatility structure $\gamma(r,t,T)$ is allowable in a Markovian spot interest rate based paradigm if there exists a pair of functions $\theta(r,t)$ and $h(t,T)$ that satisfy the following equation,

$$\gamma(r,t,T)\int_t^T \gamma(r,t,v)dv = \frac{\gamma(r,t,T)}{\gamma(r,t,t)}\theta(r,t) + \frac{\partial}{\partial t}\left[\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm\right]$$
$$+ h(t,T) + \frac{1}{2}\gamma(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,t,T)}{\gamma(r,t,t)}\right].$$

The following necessary and sufficient condition, which follows directly from equation (A-7) evaluated at $t = 0$, indicates which initial term structures are allowable for a given volatility structure satisfying Condition C1.

Condition C2. Given a volatility structure that satisfies Condition C1, the corresponding initial forward rate curve must be of the following form,

$$f(r,0,T) = \int_0^r \frac{\gamma(m,0,T)}{\gamma(m,0,0)}dm + k(T)$$

where $k(T) = -\int_0^T h(s,T)ds$ for any valid $h(t,T)$ in Condition C1.

Evaluating Condition C1 at $T = t$ shows that the structural form for $\theta(r,t)$ must be

$$\theta(r,t) = -\frac{\partial}{\partial t}\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm\bigg|_{T=t} - h(t,t). \tag{A-9}$$

Consequently, equation (A-6) shows that given an allowable volatility structure $\gamma(r,t,T)$, $h(t,t)$ defines the term structure dynamics. Therefore, to be able to fit any initial term structure given an allowable volatility structure, it is necessary for $h(t,t)$ to be arbitrary. I now determine if $h(t,t)$ can be arbitrary for any allowable volatility structure $\gamma(r,t,T)$, and if it is, determine if I can fit any observed initial forward rate curve to the required form in Condition C2. This is the essence of Theorem 1 and the proof is as follows.

Consider an allowable volatility structure (that is, Condition C1 is satisfied) and the corresponding general $h(t,T)$ for Condition C1. Given the functional form for $\theta(r,t)$ in equation (A-9) and Condition C1, the following equation must hold,

$$\begin{aligned} -\frac{\gamma(r,t,T)}{\gamma(r,t,t)}h(t,t)+h(t,T) &= \gamma(r,t,T)\int_t^T \gamma(r,t,v)dv \\ &- \frac{1}{2}\gamma(r,t,t)^2\frac{\partial}{\partial r}\left[\frac{\gamma(r,t,T)}{\gamma(r,t,t)}\right] - \frac{\partial}{\partial t}\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm \\ &+ \frac{\gamma(r,t,T)}{\gamma(r,t,t)}\left[\frac{\partial}{\partial t}\int_0^r \frac{\gamma(m,t,T)}{\gamma(m,t,t)}dm\right]\Bigg|_{T=t}. \end{aligned}$$

If $h_p(t,T)$ is a particular $h(t,T)$ for Condition C1, then

$$h(t,T) = \frac{\gamma(r,t,T)}{\gamma(r,t,t)}\left(h(t,t)-h_p(t,t)\right)+h_p(t,T). \tag{A-10}$$

If $\gamma(r,t,T)/\gamma(r,t,t)$ depends on r, then equation (A-10) can only hold if $h(t,T) = h_p(t,T)$. Consequently, if an allowable volatility structure is chosen that is not of the form $\xi(t,T)\gamma(r,t,t)$, where $\xi(t,T)$ is some function of t and T, then there is only one $h(t,T)$ for Condition C1. Consequently, $k(T)$ in Condition C2 is uniquely determined by the choice of an allowable $\gamma(r,t,T)$.

Further, if $\gamma(r,t,T)/\gamma(r,t,t)$ does not depend on r, then, in equation (A-10), there always exists a $h(t,T)$ for any choice of $h(t,t)$. Consequently, if an allowable $\gamma(r,t,T)$ is chosen that is of the form $\xi(t,T)\gamma(r,t,t)$, then $h(t,t)$ is arbitrary. For convenience, denote $h(t,t)$ as $c(t)$. To show for Condition C2 that any $k(T)$, where $k(0)=0$ can be obtained by an appropriate choice of $c(t)$, proceed as follows. Fix any particular $h(t,T)$ for Condition C1, for example $h_p(t,T)$. From Condition C2 and equation (A-10), the relationship between $k(T)$ and $h(t,T)$ is

$$k(T) = \int_0^T h_p(s,T)-\xi(s,T)h_p(s,s)ds+\int_0^T \xi(s,T)c(s)ds.$$

Given $h_p(t,T)$ and $\xi(t,T)$, the above equation is a Volterra integral equation of the first kind for which there exists a unique $c(s)$ for any choice of $k(T)$ subject to $k(0)=0$. □

Appendix 3. The Bond Price Partial Differential Equation

From the specification of the spot interest rate process in equation (5), the bond price dynamics can be obtained by applying Itô's lemma to $P(r,t,T)$, which yields

$$\begin{aligned} dP(r,t,T) &= \left[\frac{\partial P(r,t,T)}{\partial r}\mu(r,t) + \frac{\partial P(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 P(r,t,T)}{\partial r^2}\sigma(r,t)^2\right] dt \\ &+ \left[\frac{\partial P(r,t,T)}{\partial r}\sigma(r,t)\right] dz(t). \end{aligned}$$

Using the standard no-arbitrage condition for bond prices as in Vasicek (1977), results in the following partial differential equation describing the term structure dynamics,

$$\text{(A-11)} \qquad \begin{aligned} &\frac{\partial P(r,t,T)}{\partial r}\left[\mu(r,t) - \lambda(r,t)\sigma(r,t)\right] + \frac{\partial P(r,t,T)}{\partial t} \\ &+ \frac{1}{2}\frac{\partial^2 P(r,t,T)}{\partial r^2}\sigma(r,t)^2 - rP(r,t,T) = 0, \end{aligned}$$

where $P(r,T,T) = 1$. Differentiating (A-11) with respect to T, yields

$$\text{(A-12)} \qquad \begin{aligned} &\frac{\partial^2 P(r,t,T)}{\partial r \partial T}\left[\mu(r,t) - \lambda(r,t)\sigma(r,t)\right] + \frac{\partial^2 P(r,t,T)}{\partial t \partial T} \\ &+ \frac{1}{2}\frac{\partial^3 P(r,t,T)}{\partial r \partial r \partial T}\sigma(r,t)^2 - r\frac{\partial P(r,t,T)}{\partial T} = 0. \end{aligned}$$

Rearranging (A-11) and incorporating it into (A-12),

$$\begin{aligned} &\frac{\partial^2 P(r,t,T)}{\partial r \partial T}\left[-\frac{\partial P(r,t,T)}{\partial t} - \frac{1}{2}\frac{\partial^2 P(r,t,T)}{\partial r^2}\sigma(r,t)^2 + rP(r,t,T)\right] \\ &+ \frac{\partial P(r,t,T)}{\partial r}\left[\frac{\partial^2 P(r,t,T)}{\partial t \partial T} + \frac{1}{2}\frac{\partial^3 P(r,t,T)}{\partial r \partial r \partial T}\sigma(r,t)^2 - r\frac{\partial P(r,t,T)}{\partial T}\right] = 0, \end{aligned}$$

where $P(r,T,T) = 1$. This is the preference-free partial differential equation for bond prices, which represents the term structure dynamics. □

Appendix 4. The Forward Rate Partial Differential Equation

The term structure dynamics is characterized by the system of equations (7) and (8). Using equation (8), express equation (7) as

$$\text{(A-13)} \qquad \begin{aligned} &\frac{\partial f(r,t,T)}{\partial r}\mu(r,t) + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2 \\ &= \frac{\partial f(r,t,T)}{\partial r}\sigma(r,t)\left[\int_t^T \frac{\partial f(r,t,v)}{\partial r}\sigma(r,t)dv + \lambda(r,t)\right], \end{aligned}$$

where $f(r,t,t) = r$. For ease of manipulation, rearrange (A-13) as follows

$$\text{(A-14)}\quad \frac{\partial f(r,t,T)}{\partial r}\left[\mu(r,t) - \sigma(r,t)\lambda(r,t) - \sigma(r,t)\int_t^T \frac{\partial f(r,t,v)}{\partial r}\sigma(r,t)dv\right] + \frac{\partial f(r,t,T)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2 = 0.$$

Partially differentiating (A-14) with respect to T yields

$$\text{(A-15)}\quad \frac{\partial^2 f(r,t,T)}{\partial r\partial T}\left[\mu(r,t) - \sigma(r,t)\lambda(r,t) - \sigma(r,t)\int_t^T \frac{\partial f(r,t,v)}{\partial r}\sigma(r,t)dv\right] - \left[\sigma(r,t)\frac{\partial f(r,t,T)}{\partial r}\right]^2 + \frac{\partial^2 f(r,t,T)}{\partial t\partial T} + \frac{1}{2}\frac{\partial^3 f(r,t,T)}{\partial r\partial r\partial T}\sigma(r,t)^2 = 0.$$

Rearranging (A-14) and incorporating it into (A-15),

$$\begin{aligned}\frac{\partial^2 f(r,t,T)}{\partial r\partial T}\left[-\frac{\partial f(r,t,T)}{\partial t} - \frac{1}{2}\frac{\partial^2 f(r,t,T)}{\partial r^2}\sigma(r,t)^2\right] & \\ - \frac{\partial f(r,t,T)}{\partial r}\left[\frac{\partial f(r,t,T)}{\partial r}\sigma(r,t)\right]^2 & \\ + \frac{\partial f(r,t,T)}{\partial r}\left[\frac{\partial^2 f(r,t,T)}{\partial t\partial T} + \frac{1}{2}\frac{\partial^3 f(r,t,T)}{\partial r\partial r\partial T}\sigma(r,t)^2\right] &= 0,\end{aligned}$$

where $f(r,t,t) = r$. This is the preference-free partial differential equation for forward rates, which represents the term structure dynamics. □

Appendix 5. Models with Volatility Structure Form $\xi(t,T)\sigma(r,t)$

In a Markovian spot interest rate based paradigm, $\gamma(r,t,T) = [\partial f(r,t,T)/\partial r]\,\sigma(r,t)$, where $\sigma(r,t)$ is the spot interest rate volatility.

A. Vasicek (1977)

The model in terms of forward rates is

$$f(r,t,T) = (r-\phi)e^{-\kappa(T-t)} + \phi + \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)e^{-\kappa(T-t)}$$
for constants ϕ, κ, σ.

Here, $\sigma(r,t) = \sigma$, consequently, the forward rate volatility structure is

$$\gamma(r,t,T) = e^{-\kappa(T-t)}\sigma \quad \text{for constants } \kappa \text{ and } \sigma.$$

B. CIR (1985b)

The model, in terms of forward rates, is

$$f(r,t,T) = \phi_1 \left\{ \frac{2\delta e^{\delta(T-t)}}{\phi_2\left(e^{\delta(T-t)}-1\right)+2\delta} - 1 \right\} + \frac{4\delta^2 e^{\delta(T-t)} r}{\left[\phi_2\left(e^{\delta(T-t)}-1\right)+2\delta\right]^2},$$

for some constants δ, ϕ_1, ϕ_2. Here $\sigma(r,t) = \sigma\sqrt{r}$, consequently, the forward rate volatility structure is

$$\gamma(r,t,T) = \frac{4\delta^2 e^{\delta(T-t)}}{\left(\phi\left(e^{\delta(T-t)}-1\right)+2\delta\right)^2}\sigma\sqrt{r} \quad \text{for constants } \phi, \delta, \sigma.$$

C. Hull and White (1990) and Duffie (1992)

In terms of forward rates, Hull and White (1990) and Duffie (1992) only consider term structure dynamic models of the form $f(r,t,T) = C(t,T) + D(t,T)r$. Given that $\gamma(r,t,T) = [\partial f(r,t,T)/\partial r]\sigma(r,t)$, where $\sigma(r,t)$ captures all possible spot interest rate volatilities, the form of the forward rate volatility structure is $\gamma(r,t,T) = D(t,T)\sigma(r,t)$ for some appropriate function $D(t,T)$.

D. HJM (1992)

The volatility structures considered are the constant $\gamma(\omega,t,T) = \sigma$ and exponential decay $\gamma(\omega,t,T) = \exp(-\kappa(T-t))\,\sigma$ volatility structures. It is clear that these conform to the structure considered in the example. □

Appendix 6. Solution to Equation (15)

To solve this equation, transform it into the following partial differential equation,

$$\text{(A-16)} \qquad a(t)\xi(t,T) = -\xi(t,T)^{-2}\frac{\partial\xi(t,T)}{\partial T}\frac{\partial\xi(t,T)}{\partial t} + \xi(t,T)^{-1}\frac{\partial^2\xi(t,T)}{\partial t\partial T},$$

where $\xi(t,t) = 1$. To solve this partial differential equation, let $X(t,T) = \text{Ln}[a(t)\,\xi(t,T)]$, which transforms equation (A-16) into the reduction of Liouville's equation,

$$\frac{\partial^2 X(t,T)}{\partial t\partial T} = \exp\{X(t,T)\}.$$

This partial differential equation, without the condition $\xi(t,t) = 1$, can be solved by a Backlünd transformation, which yields,

$$X(t,T) = \text{Ln}\left[\frac{2\exp\{\alpha(T)-\beta(t)\}}{\left(\int_{T_0}^{T}\exp\{\alpha(s)\}ds + \int_{t_0}^{t}\exp\{-\beta(s)\}ds\right)^2}\right],$$

for arbitrary functions $\alpha(T)$ and $\beta(t)$, and arbitrary constants T_0 and t_0. Consequently, $\xi(t,T)$ is of the form,

$$\text{(A-17)} \quad \xi(t,T) = \left[\frac{2a(t)^{-1}A(T)B(t)}{\left(\int_0^T A(s)ds + \int_0^t B(s)ds + K \right)^2} \right] \quad \text{for some constant } K.$$

To determine the structure of $\xi(t,T)$ such that the condition $\xi(t,t) = 1$ holds, let

$$C(t) = \int_0^t A(s) + B(s)ds + K.$$

Consequently,

$$\text{(A-18)} \qquad B(t) = C'(t) - A(t), \quad \text{where } C'(t) = \frac{dC(t)}{dt}.$$

Therefore, the condition $\xi(t,t) = 1$ implies

$$A(t)^2 - C'(t)A(t) + \frac{a(t)C(t)^2}{2} = 0,$$

and, consequently,

$$\text{(A-19)} \qquad A(t) = \frac{C'(t) \pm \sqrt{C'(t)^2 - 2a(t)C(t)^2}}{2}.$$

To guarantee that the discriminant in equation (A-19) is nonnegative, ensuring only real solutions for $A(t)$, the following condition must hold

$$\text{(A-20)} \qquad C'(t)^2 \geq 2a(t)C(t)^2.$$

Incorporating equations (A-18), (A-19), and Condition (A-20) into equation (A-17) results in the following solution to equation (A-16),

$$\xi(t,T) = \frac{2A(T)\left(C'(t) - A(t)\right)}{a(t)\left(\int_t^T A(s)ds + C(t)\right)^2} \quad \text{for an arbitrary function } C(t),$$

$$\text{where} \quad A(t) = \frac{C'(t) \pm \sqrt{C'(t)^2 - 2a(t)C(t)^2}}{2},$$

$$C'(t)^2 \geq 2a(t)C(t)^2, \quad \text{and}$$

$$C'(t) \quad \text{represents} \quad \frac{dC(t)}{dt}. \qquad \square$$

Appendix 7. Ritchken and Sankarasubramanian Volatility Structures in a Markovian Spot Interest Rate Paradigm

This can be answered by determining when the following holds,

$$\frac{2A(T)\left(C'(t)-A(t)\right)}{a(t)\left(\int_t^T A(s)ds + C(t)\right)^2} = \exp\left(-\int_t^T \kappa(x)dx\right).$$

Differentiating both sides with respect to T and then dividing by $\exp(-\int_t^T \kappa(x)dx)$ implies

$$\frac{A'(T)}{A(T)} - \frac{2A(T)}{\int_t^T A(s)ds + C(t)} = \kappa(T).$$

For this equation to hold, then $\int_t^T A(s)ds = \eta(T) - C(t)$, for some appropriate $\eta(T)$. Differentiating both sides with respect to t implies $A(t) = C'(t)$. To satisfy this with respect to the class of allowable volatility structures (equation (18)), $A(t) = 0$ (implying zero volatility) or $a(t) = 0$. Consequently, any spot interest rate volatility that depends on the spot interest rate itself cannot have a forward rate volatility structure of the Ritchken and Sankarasubramanian form in a Markovian spot interest rate based paradigm. □

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