# Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation

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## Abstract

This paper studies the binomial approximation to the continuous trading term structure model of Heath, Jarrow, and Morton (1987). The discrete time approximation makes the original methodology accessible to a wider audience, and provides a computational procedure necessary for calculating the contingent claim values derived in the continuous time paper. This paper also extends and generalizes Ho and Lee's (1986) model to include multiple random shocks to the forward rate process and to include an analysis of continuous time limits. The generalization provides insights into the limitations of the existing empirical implementation of Ho and Lee's model.

## I. Introduction

Recently, a new methodology has emerged to price default-free bonds and the term structure of interest rates (see Ho and Lee (1986) and Heath, Jarrow, and Morton (1987)). This methodology, using the martingale measure approach, provides arbitrage-free prices that do not explicitly depend on the "market price for risk," but rather depend on an exogenously specified initial forward rate curve. This subtle, but important, distinction differentiates this class of models from those previously employed (i.e., Langetieg (1980), Brennan and Schwartz (1979), and Cox, Ingersoll, and Ross (1985)) since it generates a "preference-free" pricing paradigm.

The initial paper by Ho and Lee (1986) studied a discrete trading economy where bond prices fluctuate stochastically through time according to a *single* binomial process. Due to its simplistic structure, this family of stochastic processes is most interesting when it is viewed as an approximation to a continuous trading economy. The continuous trading analogue or limit economy, however, was not analyzed by Ho and Lee (1986). Heath, Jarrow, and Morton (1987) analyzed this continuous trading analogue, significantly generalizing the earlier model to include *multiple* random shocks to the forward rate process (in the form of independent Brownian motions) and nonnegative interest rate processes.

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This paper provides a discrete time approximation to Heath, Jarrow, and Morton (1987) and, as such, makes three contributions to the literature beyond that contained in Heath, Jarrow, and Morton (1987). First, as a pedagogical piece, it makes the Heath, Jarrow, and Morton (1987) methodology accessible to a wider audience. The original paper by Heath, Jarrow, and Morton is very abstract and difficult to read. This paper illustrates how to obtain the continuous time Heath, Jarrow, and Morton model as the limit of the discrete case, thereby clarifying various concepts and proofs in the original paper.

Second, as a discrete time approximation to the continuous time models in Heath, Jarrow, and Morton, it provides a computational tool useful for empirical investigations of the continuous time model. This is especially true when computing values for interest rate dependent contingent claims with early exercise provisions, like callable Treasury bonds, Treasury futures, or options on Treasury futures. For multiple factor, nonnegative forward rate processes, there are no simple closed form solutions available for these financial securities. The approximation procedure employed below is designed to emphasize the economic insights, and may not be the most efficient from a computational prospective. More efficient computational procedures are needed (see Nelson and Ramaswamy (1989) in this regard), and their investigation is left for future research.

Third, this paper extends and generalizes the original Ho and Lee model (1986) in numerous ways. This generalization generates insights into the contributions and limitations of the existing implementations of the Ho and Lee model. First, on the pedagogical side, an alternative perspective and notation from that used by Ho and Lee is employed. Instead of focusing upon bond prices as in Ho and Lee (1986), we concentrate on forward rates. This "modification" makes the model easier to understand and to perform mathematical analysis. Second, we generalize the original model to include multiple random shocks to the forward rate process. This allows bond returns to be imperfectly correlated, in contrast to Ho and Lee (1986). Last, and perhaps most important, we study the continuous limit of their discrete trading economy. To so do, we reparameterize the discrete time process in terms of the continuous limit's volatilities. This reparameterization identifies Ho and Lee's path independence condition to be equivalent to a restriction that the volatility of the forward rate process is a constant. It also points out a limitation in the procedure suggested by Ho and Lee ((1986), p. 1025) for estimating the discrete processes' parameters. Ho and Lee recommend estimating the discrete processes' parameters, including the pseudo probability, implicitly by matching the model's value to the market value of various traded contingent claims. We show, however, in the limit that contingent claim values only depend on the volatility parameters, and not the pseudo probability. This is analogous to the situation that occurs with the binomial approximation to the Black-Scholes model. Consequently, implicit estimation of the pseudo probability in this manner will yield unstable estimates. Our reparameterization avoids this instability and allows one to estimate the Ho and Lee model's parameters, a single volatility, using only historic data.

An outline of this paper is as follows. Section II presents the terminology and notation, Section III presents the family of term structure movements, while Section IV introduces the arbitrage-free restrictions. Section V provides examples of single random shock models, one of which is the Ho and Lee (1986) model. Section VI provides an example of a two random shock model. Section VII studies continuous time limits, Section VIII analyzes contingent claim valuation, and Section IX concludes the paper.

## II. Terminology and Notation

This section presents the model's terminology and notation. We consider a discrete trading economy of length $[0,\tau]$ for a fixed $\tau > 0$. The interval between trades is of length $\Delta > 0$, where $N$ intervals of size $\Delta$ compose a unit in time (i.e., $\Delta = 1/N$). Given an arbitrary trading time $t \in [0,\tau]$, we write $\bar{t} = tN = t/\Delta$ to represent the number of trading periods of length $\Delta$ prior to and including time $t$.

A family of default-free discount bonds trade, one for each trading date $T \in [0,\tau]$. The $T$-maturity bond pays a certain dollar at date $T$. $P(t,T)$ will denote the time $t$ price of the $T$-maturity bond for all $T \in [0,\tau]$ and $t \in [0,T]$. We require that $P(T,T) = 1$ for all $T \in [0,\tau]$, and that $P(t,T) > 0$ for all $T \in [0,\tau]$ and $t \in [0,T]$.

Given bond prices, the forward rate structure is determined (and conversely). The *forward rate* at time $t$ for the time interval $[T,T+\Delta]$, $f(t,T)$, is defined by

$$f(t,T) = -[\log(P(t,T+\Delta)/P(t,T))]/\Delta, \quad \text{for all } T \in [\Delta,\ldots,\tau] \text{ and } t \in [0,\Delta,\ldots,T-\Delta]\ . \tag{1}$$

This implies

$$P(t,T) = \exp\left(-\sum_{j=\bar{t}}^{\bar{T}-1} f(t,j\Delta)\Delta\right), \quad \text{for all } T \in [\Delta,\ldots,\Delta\bar{\tau}] \text{ and } t \in \left[0,\Delta,\ldots,\Delta(\bar{T}-1)\right]\ . \tag{2}$$

Three aspects of Expression (2) need emphasis. First, the arguments of the forward rate are in units of time. The summation index, however, is over the number of trading intervals between $t$ and $T-\Delta$, and this accounts for the bars over the times $t$ and $(T-\Delta)$. Second, since the forward rate $f(t,T)$ corresponds to the future period $[T,T+\Delta]$, the upper limit of the summation index stops at the trading point $(\bar{T}-1)$. By definition, $P(T,T)$ is always one, and this parameter choice is excluded from the expression. This explains why the current date ranges from $[0,\ldots,\Delta(\bar{T}-1)]$.[1]

The *spot rate* at time $t$ (over $[t,t+\Delta]$), $r(t)$, is defined to be the forward rate at time $t$,[2] i.e.,

$$r(t) = f(t,t) . \tag{3}$$

Finally, we define an accumulation factor, $B(t)$, corresponding to the price of a money market account (rolling over at $r(t)$) initialized at time 0 with a dollar investment, i.e.,

$$\begin{aligned} B(0) &= 1 \text{ and} \\ B(t) &= \exp\left(\sum_{j=0}^{\bar{t}-1} r(j\Delta)\Delta\right) \text{ for all } t \in \left[\Delta,\ldots,\Delta\left(\bar{\tau}-1\right)\right] . \end{aligned} \tag{4}$$

The remainder of this paper studies the arbitrage-free pricing of the traded discount bonds. Intuitively, an *arbitrage opportunity* is any "trading strategy" with a zero initial investment, nonnegative cash flows with probability one, and strictly positive cash flows with positive probability. In the context of a discrete trading economy, Harrison and Pliska (1981) prove that there are no such arbitrage opportunities if and only if there exists a (equivalent) probability (measure) making relative asset prices (relative to $B(t)$) a martingale. In essence, the ability to invoke the Cox and Ross (1976) risk neutrality pricing argument is both necessary and sufficient for the absence of arbitrage opportunities. Consequently, the study of arbitrage-free bond pricing reduces to the study of conditions under which "martingale measures" exist. This is the perspective followed in this paper.

## III. Term Structure Movements

This section presents the family of stochastic processes representing forward rate movements. Once specified, this family uniquely determines the spot rate process and the bond price process.

[1] The limit of expression (1) as $\Delta \to 0$ gives the appropriate "instantaneous" expressions. Indeed, as $\Delta \to 0$, given $\partial P(t,T)/\partial T$ exists for all $T \in [0,\tau]$ and $t \in [0,T]$,

$$P(t,T) = \lim_{\Delta\to 0} \exp\left(-\sum_{j=\bar{t}}^{\bar{T}-1} f(t,j\Delta)\Delta\right) = \exp\left(-\int_t^T \tilde{f}(t,s)\,ds\right),$$

$$\begin{aligned} \text{where } \tilde{f}(t,s) &= \lim_{\Delta\to 0} f(t,s+\Delta) \\ &= \lim_{\Delta\to 0} -\left[\log P(t,s+\Delta) - \log P(t,s)\right]/\Delta \\ &= -\frac{\partial \log P(t,s)}{\partial s} . \end{aligned}$$

[2] The spot rate, $r(t)\Delta = \log(1/P(t,t+\Delta)) = f(t,t)\Delta$, corresponds to the continuously compounded equivalent *riskless* return earned on a $(t+\Delta)$-maturity bond over $[t,t+\Delta]$.

*(C.1) (Family of Forward Rate Processes).* For fixed, but arbitrary $T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]$, the forward rate $f(t,T)$ satisfies the following stochastic process,

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \left[ u_1(j\Delta,T) - v_1(j\Delta,T) \right] + \sum_{j=1}^{\bar{t}} v_1(j\Delta,T) \\
&\quad + \sum_{j=1}^{\bar{t}} b_j \left[ u_2(j\Delta,T) - v_2(j\Delta,T) \right] + \sum_{j=1}^{\bar{t}} v_2(j\Delta,T) ,
\end{aligned}
\tag{5}
$$

for all $t \in [\Delta, \ldots, T]$, where $\{f(0,t)\colon T \in [0, \ldots, \Delta(\bar{\tau}-1)]\}$ is a fixed nonrandom initial forward rate curve; $a_j$, $b_j$ for $j \in \{1, \ldots, \bar{T}-1\}$ are random variables taking on the values $\{0,1\}$ with joint probabilities, summing to one, given by

$$
\begin{aligned}
&q_{00}(j) \text{ if } a_j = 0, \quad b_j = 0 \\
&q_{01}(j) \text{ if } a_j = 0, \quad b_j = 1 \\
&q_{10}(j) \text{ if } a_j = 1, \quad b_j = 0 \\
&q_{11}(j) \text{ if } a_j = 1, \quad b_j = 1 .
\end{aligned}
$$

These probabilities are indexed by $j$ since, in general, they can depend on any information available prior to time $j\Delta$.

$u_1\colon [0,\tau] \times [0,\tau] \to R$, $v_1\colon [0,\tau] \times [0,\tau] \to R$, $u_2\colon [0,\tau] \times [0,\tau] \to R$, $v_2\colon [0,\tau] \times [0,\tau] \to R$ are *random* functions that, at time $t$, can depend on the information available prior to time $t$.

This forward rate process has two random shocks, represented by the random variables $\{a_j, b_j\}$, which are correlated or not, depending upon the specification of the joint probabilities. The magnitude of the "upward" movement of a jump at time $t$ is denoted $u_1(t,T)$ for $a_{\bar{t}}$ and $u_2(t,T)$ for $b_{\bar{t}}$. The magnitude of a "downward" movement of a jump at time $t$ is denoted $v_1(t,T)$ for $a_{\bar{t}}$ and $v_2(t,T)$ for $b_{\bar{t}}$. These magnitudes can depend on all the information available prior to time $t$. Also specified is a fixed initial forward rate curve $\{f(0,T)\colon T \in [0, \ldots, \tau - \Delta]\}$ from which the process moves. The key issue in the subsequent analysis is the restrictions required upon these jump magnitudes such that the resulting forward rate process is consistent with no arbitrage opportunities.

Condition (C.1) determines the spot rate process as

$$
\begin{aligned}
r(t) &= f(0,t) + \sum_{j=1}^{\bar{t}} a_j \left[ u_1(j\Delta,t) - v_1(j\Delta,t) \right] + \sum_{j=1}^{\bar{t}} v_1(j\Delta,t) \\
&\quad + \sum_{j=1}^{\bar{t}} b_j \left[ u_2(j\Delta,t) - v_2(j\Delta,t) \right] + \sum_{j=1}^{\bar{t}} v_2(j\Delta,t) .
\end{aligned}
\tag{6}
$$

Both arguments in the process shift simultaneously across time.

Define the relative price for a $T$-maturity bond as $Z(t,T) = P(t,T)/B(t)$ for $T \in [0,\tau]$ and $t \in [0,T]$. The relative bond price is the discount bond's price in terms of a new numeraire, the accumulation account. Given the expressions for $P(t,T)$

and $B(t)$ in terms of the forward rate and spot rate, respectively, we can rewrite the relative price $Z(t,T)$ as

$$Z(t,T) = \exp\left\{ -\sum_{j=\bar{t}}^{\bar{T}-1} f(t,j\Delta)\Delta - \sum_{j=0}^{\bar{t}-1} f(j\Delta,j\Delta)\Delta \right\}. \tag{7}$$

Substitution of the stochastic process for forward rates into Expression (7) yields

$$\begin{aligned} Z(t,T) = \exp\Bigg\{ & -\sum_{j=0}^{\bar{T}-1} f(0,j\Delta)\Delta \\ & -\sum_{j=\bar{t}}^{\bar{T}-1}\sum_{i=1}^{\bar{t}} \left[a_i\left(u_1(i\Delta,j\Delta) - v_1(i\Delta,j\Delta)\right) + v_1(i\Delta,j\Delta)\right]\Delta \\ & -\sum_{j=\bar{t}}^{\bar{T}-1}\sum_{i=1}^{\bar{t}} \left[b_i\left(u_2(i\Delta,j\Delta) - v_2(i\Delta,j\Delta)\right) + v_2(i\Delta,j\Delta)\right]\Delta \\ & -\sum_{j=1}^{\bar{t}-1}\sum_{i=1}^{j} \left[a_i\left(u_1(i\Delta,j\Delta) - v_1(i\Delta,j\Delta)\right) + v_1(i\Delta,j\Delta)\right]\Delta \\ & -\sum_{j=1}^{\bar{t}-1}\sum_{i=1}^{j} \left[b_i\left(u_2(i\Delta,j\Delta) - v_2(i\Delta,j\Delta)\right) + v_2(i\Delta,j\Delta)\right]\Delta \Bigg\}. \end{aligned} \tag{8}$$

Changing the order of summation[3] and combining terms gives

$$\begin{aligned} Z(t,T) = \exp\Bigg\{ & -\sum_{j=0}^{\bar{T}-1} f(0,j\Delta)\Delta \\ & -\sum_{i=1}^{\bar{t}}\sum_{j=i}^{\bar{T}-1} \left[a_i\left(u_1(i\Delta,j\Delta) + v_1(i\Delta,j\Delta)\right) + v_1(i\Delta,j\Delta)\right]\Delta \\ & -\sum_{i=1}^{\bar{t}}\sum_{j=i}^{\bar{T}-1} \left[b_i\left(u_2(i\Delta,j\Delta) - v_2(i\Delta,j\Delta)\right) + v_2(i\Delta,j\Delta)\right]\Delta \Bigg\}, \end{aligned} \tag{9}$$

for $T \in [\Delta, \ldots, \Delta\bar{\tau}]$, and $t \in [\Delta, \ldots, \Delta(\bar{T}-1)]$.

[3] The identity used in this change of summation is

$$\sum_{j=1}^{\bar{t}-1}\sum_{i=1}^{j} W(i,j) = \sum_{i=1}^{\bar{t}-1}\sum_{j=i}^{\bar{t}-1} W(i,j).$$

Expression (9) implies the following stochastic difference equation for $Z(t,T)$,

$$
\begin{aligned}
Z(t,T) \;=\; Z(t-\Delta,T)\exp\Bigg\{&-a_{\bar{t}}\left[\sum_{j=\bar{t}}^{\bar{T}-1}\Big(u_1(t,j\Delta) - v_1(t,j\Delta)\Big)\Delta\right] \\
&-\sum_{j=\bar{t}}^{\bar{T}-1} v_1(t,j\Delta)\Delta - b_{\bar{t}}\left[\sum_{j=\bar{t}}^{\bar{T}-1}\Big(u_2(t,j\Delta) - v_2(t,j\Delta)\Big)\Delta\right] \\
&-\sum_{j=\bar{t}}^{\bar{T}-1} v_2(t,j\Delta)\Delta\Bigg\}\,,
\end{aligned}
\tag{10}
$$

for $T \in [\Delta, \ldots, \Delta\bar{\tau}]$ and $t \in [\Delta, \ldots, \Delta(\bar{T}-1)]$.

## IV. Arbitrage-Free Pricing and Term Structure Movements

This section analyzes the restrictions (necessary and sufficient) on the jump magnitudes in the forward rate process for there to be no arbitrage opportunities in the economy. These restrictions, by implication, generate the bond pricing formulae satisfied by the discount bonds. The correspondences among these restrictions and those contained in Ho and Lee (1986) and Heath, Jarrow, and Morton (1987) are identified.

The following proposition provides the major result of the paper.

*Proposition 1.* (Characterization of Bond Prices and Forward Rate Processes). Given a family of forward rate jump magnitudes, $\{u_1(\cdot,T),\ v_1(\cdot,T),\ u_2(\cdot,T),\ v_2(\cdot,T)\colon T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]\}$, satisfying Condition (C.1), the following expressions are equivalent:

(11.a) The forward rate process given by (C.1) is an arbitrage-free pricing process.

(11.b) There exist probabilities, summing to one and denoted by $\{\pi_{00}(j),\ \pi_{01}(j),\ \pi_{10}(j),\ \pi_{11}(j)\}$ with respect to $\{a_j,\ b_j\}$ for each $j \in \{1, \ldots, \bar{\tau}-1\}$, such that $Z(t,T)$ is a martingale with respect to these probabilities for all $T \in [\Delta, \ldots, \Delta\bar{\tau}]$ and $t \in [0, \ldots, T-\Delta]$.

(11.c) There exist probabilities, summing to one and denoted by $\{\pi_{00}(j),\ \pi_{01}(j),\ \pi_{10}(j),\ \pi_{11}(j)\}$ with respect to $\{a_j, b_j\}$ for each $j \in \{1, \ldots, \bar{\tau}-1\}$ such that

$$
\begin{aligned}
&\pi_{00}(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}\big[v_1(t,j\Delta) + v_2(t,j\Delta)\big]\Delta\right\} + \\
&\pi_{01}(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}\big[v_1(t,j\Delta) + u_2(t,j\Delta)\big]\Delta\right\} + \\
&\pi_{10}(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}\big[u_1(t,j\Delta) + v_2(t,j\Delta)\big]\Delta\right\} + \\
&\pi_{11}(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}\big[u_1(t,j\Delta) + u_2(t,j\Delta)\big]\Delta\right\} \;=\; 1\,,
\end{aligned}
$$

for all $\bar{T} \in [2, \ldots, \bar{\tau}]$ and $\bar{t} \in [1, \ldots, \bar{T}-1]$.

*Proof.* Let expectation with respect to $\{\pi_{00}(\bar{t}), \pi_{01}(\bar{t}), \pi_{10}(\bar{t}), \pi_{11}(\bar{t})\}$, given the information available at time $t-\Delta$, be denoted by $E^*_{t-\Delta}(\cdot)$.

Condition (11.a) is true if and only if (11.b) is true (stated earlier in the text).

But, $Z(t,T)$ is a martingale with respect to these probabilities for all $t,T$ if and only if

$$E^*_{t-\Delta}(Z(t,T)) = Z(t-\Delta,T) \text{ for all } t,T.$$

Substitution of Expression (10) yields

$$E^*_{t-\Delta}\left\{\left(\exp\left(-a_{\bar{t}}\left(\sum_{j=\bar{t}}^{\bar{T}-1}\left(u_1(t,j\Delta) - v_1(t,j\Delta)\right)\Delta\right) - \sum_{j=\bar{t}}^{\bar{T}-1} v_1(t,j\Delta)\Delta\right)\cdot\right.\right.$$
$$\left.\left.\exp\left(-b_{\bar{t}}\left(\sum_{j=\bar{t}}^{\bar{T}-1}\left(u_2(t,j\Delta) - v_2(t,j\Delta)\Delta\right)\right) - \sum_{j=\bar{t}}^{\bar{T}-1} v_2(t,j\Delta)\Delta\right)\right)\right\} = 1 .$$

This last expression, in terms of probabilities, is Expression (11.c). □

Expression (11.b) is called the *pricing condition* because it asserts that $Z(t,T)$ is a martingale under the probabilities $\{\pi_{00}(\bar{t}), \pi_{01}(\bar{t}), \pi_{10}(\bar{t}), \pi_{11}(\bar{t})\}$ for $\bar{t} \in [0, \ldots, \bar{\tau}-1]$. Denoting expectation with respect to these probabilities, given the information available at time $t$ as $E_t^*(\cdot)$, yields

(12.a) $$Z(t,T) = E_t^*(Z(T,T))$$

(12.b) $$\text{or } P(t,T) = E_t^*(1/B(T))B(t) .$$

Expression (12.b) gives a pricing formula for the discount bonds. To calculate these values, one starts at time $T-\Delta$ and calculates the expectations backward in time (to time $t$) using the law of iterated expectations and Expression (9). Expression (12) corresponds exactly to the pricing formula of Heath, Jarrow, and Morton (1987).

It is important to point out that the pricing conditon (11.b) combines two restrictions on the probabilities $\{\pi_{00}(\bar{t}), \pi_{01}(\bar{t}), \pi_{10}(\bar{t}), \pi_{11}(\bar{t})\}$ for $t \in [0, \ldots, \bar{\tau}-1]$. The first is that, for a fixed $T$, these probabilities make $Z(t,T)$ a martingale. In this case, the probabilities could depend on $T$ and, hence, this restriction only implies that there are no arbitrage opportunities between the $T$-maturity bond and the accumulation factor $B(t)$. The second restriction is that the probabilities are independent of $T$, making *all* bonds martingales, implying that there are no arbitrage opportunities present simultaneously across all bonds. These two restrictions are separately imposed in the continuous time model of Heath, Jarrow, and Morton (1987).

Expression (11.c) is the *forward rate process restriction*. Given the probabilities $\{\pi_{00}(\bar{t}), \pi_{01}(\bar{t}), \pi_{10}(\bar{t}), \pi_{11}(\bar{t})\}$ for all $\bar{t} \in \{0, \ldots, \bar{\tau}-1\}$, it specifies the relationships among $\{u_1, v_1, u_2, v_2\}$ that must be satisfied for there to be no arbitrage opportunities present in the economy. The simultaneous existence of such probabilities and jump magnitudes will be demonstrated in subsequent sections through explicit examples.

For the one random shock case (obtained by setting $u_2(j\Delta,T) = v_2(j\Delta,T) = 0$ for all $j \in \{0,\ldots,\bar{\tau}-1\}$ and all $T \in [0,\ldots,\tau-\Delta]$), Condition (11.c) simplifies considerably. Let $q_{00}(j) \equiv 1-q(j)$, $q_{10}(j) \equiv q(j)$ denote the probabilities that $a_j$ takes on the values $\{0,1\}$, respectively. Similarly, let $\pi_{00}(j) \equiv 1-\pi(j)$, $\pi_{10}(j) \equiv \pi(j)$ denote the martingale probabilities. Dropping the subscripts on $\{u_1,v_1\}$ yields,

$$(13) \qquad \left(1-\pi(\bar{t})\right)\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v(t,j\Delta)\,\Delta\right\} + \pi(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u(t,j\Delta)\,\Delta\right\} = 1 \,,$$

for $\bar{T} \in [2,\ldots,\bar{\tau}-1]$ and $\bar{t} \in [1,\ldots,\bar{T}-1]$. This is identical to the no arbitrage condition contained in Ho and Lee ((1986), p. 1017, Expression (10)).[4] Hence, Expression (10.c) provides an appropriate generalization of Ho and Lee for the multiple binomial case.

In the continuous time model of Heath, Jarrow, and Morton (1987), Condition (11.c) corresponds to their forward rate drift restriction; however, there is a subtle difference. In Heath, Jarrow, and Morton (1987), also given was a volatility function, which implied the existence of a unique "martingale measure." The variance restriction has not yet been imposed in the discrete model.

In general, the martingale probabilities (if they exist) need not be unique. This nonuniqueness will be evidenced below in the examples. Ho and Lee (1986) get uniqueness in their model by imposing a second condition, called the path-independence condition. We show below that this is equivalent to a particular specification for the variance of the forward rate process. In fact, for the single random shock model, specification of the conditional variance of $[f(t+\Delta,T)-f(t,T)]$ uniquely identifies the martingale probabilities in terms of the probabilities $q(j)$ and the forward rate parameters $\{u,v\}$.

## V. Example (One Random Shock Processes)

This section presents the one random shock forward rate process to illustrate the application of Proposition 1. Ho and Lee's (1986) model is shown to be a special case of this example.

The forward rate process, under a single random shock, is written as,

$$(14) \qquad f(t,T) = f(0,T) + \sum_{j=1}^{\bar{t}} a_j\left[u(j\Delta,T)-v(j\Delta,T)\right] + \sum_{j=1}^{\bar{t}} v(j\Delta,T) \,,$$

[4] To make the identification, let $\pi(\bar{t})$ be independent of $\bar{t}$. In the notation of Ho and Lee (1986), our $\pi$ is Ho and Lee's $(1-\pi)$. Starting from $\bar{t} = 1$,

$$h^*(T) = \exp\left(-\sum_{j=1}^{\bar{T}-1} u(\Delta,j\Delta)\,\Delta\right) \text{ and}$$

$$h(T) = \exp\left(-\sum_{j=1}^{\bar{T}-1} v(\Delta,j\Delta)\,\Delta\right).$$

for $T \in [\Delta, \dots, \Delta(\bar{\tau}-1)]$ and $t \in [\Delta, \dots, T]$, where $q(j)$ equals the probability that $a_j$ equals 1. The subscripts are omitted for simplicity.

As explained in Expression (13), the necessary and sufficient conditions on the jump magnitudes $\{u(j\Delta,T),v(j\Delta,T)\}$ for the absence of arbitrage opportunities is the existence of probabilities $\pi(\bar{t})$ for $\bar{t} \in [0, \dots, \bar{\tau}-1]$ such that

$$(15) \qquad (1-\pi(\bar{t}))\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v(t,j\Delta)\Delta\right\} + \pi(\bar{t})\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u(t,j\Delta)\Delta\right\} = 1 ,$$

for $\bar{T} \in \{1, \dots, \bar{\tau}-1\}$ and $\bar{t} \in [0, \dots, \bar{T}-1]$. These probabilities $\{\pi(\bar{t})\}$ are, in general, nonunique.

To verify this assertion, we characterize the forward rate processes consistent with arbitrage-free pricing *and* the imposition of the following variance restriction,

$$(16) \qquad \operatorname{Var}_{t-\Delta}(f(t,T)-f(t-\Delta,T)) = \sigma^2(t,T)\Delta ,$$

where $\sigma(t,T)$ is a random factor that depends on the information available prior to time $t$. A simple calculation shows that Condition (16) is equivalent to

$$(17) \qquad \begin{aligned} &q(\bar{t})(1-q(\bar{t}))[u(t,T)-v(t,T)]^2 = \sigma^2(t,T)\Delta \\ &\text{or } u(t,T) - v(t,T) + \sigma(t,T)\left[\Delta/q(\bar{t})(1-q(\bar{t}))\right]^{1/2}. \end{aligned}$$

This yields

$$(18) \qquad \begin{aligned} &-\sum_{j=\bar{t}}^{\bar{T}-1} u(t,j\Delta)\Delta = \\ &\quad -\sum_{j=\bar{t}}^{\bar{T}-1}\left[v(t,j\Delta)\Delta + \sigma(t,j\Delta)\Delta^{3/2}/\left(q(\bar{t})(1-q(\bar{t}))\right)^{1/2}\right] . \end{aligned}$$

Substitution of this expression into (15) and algebra gives

$$(19) \qquad \begin{aligned} &\sum_{j=\bar{t}}^{\bar{T}-1} v(t,j\Delta)\Delta = \\ &\quad \log\left(1+\pi(\bar{t})\left(\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}\sigma(t,j\Delta)\Delta^{3/2}\Big/\left(q(\bar{t})(1-q(\bar{t}))\right)^{1/2}\right\}-1\right)\right) . \end{aligned}$$

The solution to Expression (19) is

$$\begin{aligned} &v(j\Delta,\bar{T}\Delta) = \\ &\left[\log\left(1+\pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\}-1\right)\right)\right. \\ &\left.-\log\left(1+\pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\}-1\right)\right)\right]\Big/\Delta . \end{aligned}$$

Summing across $j = 1, \ldots, \bar{t}$ generates

$$\sum_{j=1}^{\bar{t}} v(j\Delta, \bar{T}\Delta) =$$

$$\sum_{j=1}^{\bar{t}} \left[ \log\left(1 + \pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right) \right. \tag{20}$$
$$\left. - \log\left(1 + \pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right)\right] \Big/ \Delta .$$

The forward rate process consistent with no arbitrage is, therefore,

$$f(t,T) =$$

$$f(0,T) + \sum_{j=1}^{\bar{t}} a_j \sigma(j\Delta, T)(\Delta/[q(j)(1-q(j))])^{1/2}$$

$$+ \sum_{j=1}^{\bar{t}} \left[ \log\left(1 + \pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right) \right. \tag{21}$$
$$\left. - \log\left(1 + \pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right)\right] \Big/ \Delta ,$$

for all $T \in [\Delta, \ldots, \Delta(\bar{\tau} - 1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$.

Approximations to (21) for small $\Delta$ are analyzed in Section VII below. Two special cases of Expression (21) are worth studying further.

## A. Ho and Lee's (1986) Model

This special case provides an alternate derivation of Ho and Lee's model. This alternative derivation clarifies the use of their path independence condition in obtaining uniqueness. To specify Ho and Lee's model as a special case, set the variance of the forward rate process, $\sigma(t,T) \equiv \sigma$, where $\sigma > 0$ is a positive constant. Furthermore, let $q(\bar{t}) \equiv q > 0$, a strictly positive constant, so that the probabilities of jumps are constant across time. Last, we search for constant martingale probabilities, i.e., $\pi(\bar{t}) \equiv \pi > 0$, a strictly positive constant.

Under these constraints, (20) can be written in the simplified form,

$$\sum_{j=1}^{\bar{t}} v(j\Delta, T\Delta) = \left[\log\left(1 + \pi\left(e^{-\bar{T}\psi\Delta^{3/2}} - 1\right)\right)\right. \tag{22}$$
$$\left. - \log\left(1 + \pi\left(e^{-(\bar{T}-\bar{t})\psi\Delta^{3/2}} - 1\right)\right)\right] \Big/ \Delta ,$$

where $\psi = \sigma(q(1-q))^{-1/2}$.

This form of the expression (with notational changes) is identical to the unique forward rate solution, given the path independence condition, provided by Ho and Lee (1986).[5] This identification proves the assertion made previously that the path independence condition is equivalent to a particular restriction upon the variance of the forward rate process. The forward rate process is, therefore,

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \psi \sqrt{\Delta} \\
&\quad + \left[\log\left(1+\pi\left(e^{-\bar{T}\psi\Delta^{3/2}}-1\right)\right) - \log\left(1+\pi\left(e^{-(\bar{T}-\bar{t})\psi\Delta^{3/2}}-1\right)\right)\right]\Big/\Delta\,,
\end{aligned} \tag{23}
$$

for all $T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$, where $\psi = \sigma/(q(1-q))^{1/2}$.

This solution depends on two parameters, the martingale probability $\pi$ and the "adjusted" variance parameter, $\psi = \sigma/(q(1-q))^{1/2}$. The volatility parameter, $\sigma^2$, can be estimated from historical observations of the forward rates. The probabilities $\pi$ and $q$ can be specified arbitrarily to obtain the best fit.

Of course, given the simplistic structure of the forward rate process (23), this process will only provide reasonable approximations (if at all) for small $\Delta$. The limiting form of (23) is examined in Section VII. In the limit, the need to specify $\pi$ and $q$ vanishes, giving a parsimonious model of only one "observable" parameter $\sigma^2$.

## B. The Exponentially Decaying Variance Model

An alternative forward rate process consistent with no arbitrage opportunities can be obtained by making the following specifications,

$$\sigma^2(t,T) = \sigma^2 e^{-\lambda(T-t)}\Delta\,,$$

for positive constants $\sigma > 0$, $\lambda > 0$, and $q(\bar{t}) \equiv q > 0$.

Following the procedure employed in the previous example generates the forward rate process given by,

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \psi \exp\left\{-(\lambda/2)\left(\bar{T}-j\right)\Delta\right\}\sqrt{\Delta} \\
&\quad + \sum_{j=1}^{\bar{t}} \left[\log\left(1+\pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}} \psi \exp\{-(\lambda/2)(i-j)\Delta\}\Delta^{3/2}\right\}-1\right)\right)\right. \\
&\quad \left. - \log\left(1+\pi(j)\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1} \psi \exp\{-(\lambda/2)(i-j)\Delta\}\Delta^{3/2}\right\}-1\right)\right)\right]\Big/\Delta,
\end{aligned} \tag{24}
$$

where $\psi = \sigma/(q(1-q))^{1/2}$ for all $T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$.

[5] Using the identification in footnote 4, Expression (19) is equivalent to

$$h(T) = \frac{1}{(1-\pi) + \pi\delta^{(\bar{T}-1)}}\,,$$

where $1 > \delta \equiv e^{-\sigma\Delta^{3/2}/(q(1-q))^{1/2}} > 0$, which corresponds to the *unique* solution of Ho and Lee ((1986), p. 1019, Expression (19)).

## VI. An Example: Constant and Exponentially Decaying Model

This section studies an example of a forward rate process subject to two random shocks, a "short-term" and a "long-term" factor. The "long-term" factor uniformly affects all forward rates equally, while the "short-term" factor dampens exponentially with time to maturity. To obtain this model, we further restrict the process given in Condition (C.1).

First, we restrict the joint probabilities of the random variables $\{a_j,b_j\}$ to be constant across time, and to satisfy

$$\begin{aligned} q_{00}(j) &= (1-q_1)(1-q_2) \\ q_{01}(j) &= (1-q_1)q_2 \\ q_{10}(j) &= q_1(1-q_2) \\ q_{11}(j) &= q_1 q_2 , \end{aligned} \tag{25}$$

where $q_1$ is the probability that $a_j$ takes the value 1, and $q_2$ is the probability that $b_j$ takes the value 1.

This specification has two implications: (1) $\{a_j,b_j\}$ are independent across time; and (2) $a_j$ and $b_j$ are independent within a particular time period. We also restrict variances of the change in the forward rate process as follows,

$$\mathrm{Var}_{t-\Delta}\big(f(t,T) - f(t-\Delta,T) \mid b_{\bar{t}}\big) = \sigma_1^2\Delta \tag{26.a}$$

$$\mathrm{Var}_{t-\Delta}\big(f(t,T) - f(t-\Delta,T) \mid a_{\bar{t}}\big) = \sigma_2^2 e^{-\lambda(T-t)}\Delta , \tag{26.b}$$

for strictly positive constants $\sigma_1,\sigma_2,\lambda > 0$. Expression (26.a) captures the "long-term" variance, while Expression (25.b) captures the "short-term" variance. These conditions alone will not generate unique martingale probabilities. Consequently, to identify a unique element from the class of martingale probabilities, we add the following conditions.

First, we want the martingale probabilities to be constants that satisfy

$$\begin{aligned} \pi_{00}(\bar{t}) &= (1-\pi_1)(1-\pi_2) \\ \pi_{01}(\bar{t}) &= (1-\pi_1)\pi_2 \\ \pi_{10}(\bar{t}) &= \pi_1(1-\pi_2) \\ \pi_{11}(\bar{t}) &= \pi_1\pi_2 . \end{aligned} \tag{27}$$

This implies that the martingale probabilities preserve the statistical independence of $\{a_j,b_j\}$ across time and from each other. Substitution of Condition (27) into the no arbitrage condition, Expression (11.c) yields

$$\begin{aligned} &\left(\pi_1 \exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u_1(t,j\Delta)\Delta\right\} + (1-\pi_1)\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v_1(t,j\Delta)\Delta\right\}\right)\cdot \\ &\left(\pi_2 \exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u_2(t,j\Delta)\Delta\right\} + (1-\pi_2)\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v_2(t,j\Delta)\Delta\right\}\right) = 1 . \end{aligned} \tag{28}$$

We add an additional restriction that induces additivity in the forward rate process components. Although, by Conditions (25) and (26), the random variables generating the shocks $\{a_j, b_j\}$ are independent of each other, this does not necessarily imply that the two separate contributions each of these shocks have on the forward rate process (Expression (5)) are independent. A dependence could occur through a relationship between the jump magnitudes $\{u_1, v_1\}$ and $\{u_2, v_2\}$ themselves. This is evidenced in Expression (28), where the product could be 1 as long as both $\{u_1, v_1\}$ and $\{u_2, v_2\}$ depend on each other in offsetting ways. To remove this dependence, and to make the two separate shocks in the change in forward rates statistically independent, we add the following restrictions

$$\text{(29.a)} \qquad \pi_1 \exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u_1(t,j\Delta)\Delta\right\} + \left(1-\pi_1\right)\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v_1(t,j\Delta)\Delta\right\} = 1$$

$$\text{(29.b)} \qquad \pi_2 \exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} u_2(t,j\Delta)\Delta\right\} + \left(1-\pi_2\right)\exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1} v_2(t,j\Delta)\Delta\right\} = 1\,.$$

Under these restrictions, the previous two examples (Sections V.A and V.B) apply in an additive form, and the two separate shocks to the forward rate process are statistically independent.

$$\begin{aligned} f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \psi_1 \surd\Delta + \sum_{j=1}^{\bar{t}} b_j \psi_2 \exp\left\{-(\lambda/2)\left(\bar{T}-j\right)\Delta\right\}\surd\Delta \\ &\quad + \left[\log\left(1+\pi_1\left(\exp\left\{-\bar{T}\psi_1\Delta^{3/2}\right\}-1\right)\right)\right. \\ &\quad \left. -\log\left(1+\pi_1\left(\exp\left\{-(\bar{T}-\bar{t})\psi_1\Delta^{3/2}\right\}-1\right)\right)\right]\Big/\Delta \\ \text{(30)} \qquad &\quad + \sum_{j=1}^{\bar{t}}\left[\log\left(1+\pi_2\left(\exp\left\{-\sum_{i=j}^{\bar{T}}\psi_2\exp\{-(\lambda/2)(i-j)\Delta\}\Delta^{3/2}\right\}-1\right)\right)\right. \\ &\quad \left. -\log\left(1+\pi_2\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1}\psi_2\exp\{-(\lambda/2)(i-j)\Delta\}\Delta^{3/2}\right\}-1\right)\right)\right]\Big/\Delta\,, \end{aligned}$$

for all $T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$, where $\psi_1 = \sigma_1/(q_1(1-q_1))^{1/2}$ and $\psi_2 = \sigma_2/(q_2(1-q_2))^{1/2}$.

## VII. Limit Economies

This section discusses the limiting form of the discrete trading models in Sections V and VI for a special family of the forward rate processes. The family of processes considered has: (1) the variances of the changes in the forward rate process are deterministic functions of time; and (2) the original probabilities and the martingale probabilities of the random variables are constant and independent (as in Section VI). These conditions are imposed for expositional purposes in order to utilize the simple form of the Central Limit Theorem in the limiting processes. These restrictions can be readily generalized, but the mathematics becomes very sophisticated.

Using the additivity conditions of Section VI for multiple random variables, it suffices to consider the single random variable case. The specification of the variance of forward rate process is given by,

$$\text{(31)} \qquad \operatorname{Var}_{t-\Delta}(f(t,T) - f(t-\Delta,T)) = \sigma^2(t,T)\Delta ,$$

where $\sigma$: $[0,\tau] \times [0,\tau] \to R$ is a nonnegative, continuous, *deterministic* function.

Under Expression (31), the forward rate process is given by Expression (21), rewritten here for convenience,

$$\begin{aligned} f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta} \\ \text{(32)} \qquad &+ \sum_{j=1}^{\bar{t}} \left[ \log\left(1 + \pi\left(\exp\left\{-\sum_{i=j}^{\bar{T}} \sigma(j\Delta,i\Delta)(q(1-q))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right) \right. \\ &\left. - \log\left(1 + \pi\left(\exp\left\{-\sum_{i=j}^{\bar{T}-1} \sigma(j\Delta,i\Delta)(q(1-q))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right) \right] \Big/ \Delta , \end{aligned}$$

for all $T \in [\Delta, \ldots, \Delta(\bar{\tau} - 1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$.

To approximate this solution for small $\Delta$, we use the Taylor series expansion,

$$\begin{aligned} &\log\left(1 + \pi\left(\exp\left\{-\sum_{i=j}^{\bar{T}} \sigma(j\Delta,i\Delta)(q(1-q))^{-1/2}\Delta^{3/2}\right\} - 1\right)\right) = \\ \text{(33)} \qquad &- \pi \sum_{i=j}^{\bar{T}} \sigma(j\Delta,i\Delta)(q(1-q))^{-1/2}\Delta^{3/2} \\ &+ \frac{\pi(1-\pi)}{2}\left(\sum_{i=j}^{\bar{T}} \sigma(j\Delta,i\Delta)\right)^2 (q(1-q))^{-1}\Delta^3 + 0\left(\Delta^3\right) , \end{aligned}$$

where $\lim_{\Delta \to 0} \frac{0\left(\Delta^3\right)}{\Delta^3} = 0$ .

Substitution of (33) into (32) gives,

$$f(t,T) = f(0,T) + \sum_{j=1}^{\bar{t}} a_j \sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}$$

$$- \sum_{j=1}^{\bar{t}} \pi\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}$$

(34)

$$+ \pi(1-\pi)\sum_{j=1}^{\bar{t}}\left[\left(\sum_{i=j}^{\bar{T}}\sigma(j\Delta,i\Delta)\Delta\right)^2 - \left(\sum_{i=j}^{\bar{T}-1}\sigma(j\Delta,i\Delta)\Delta\right)^2\right] \Big/ 2q(1-q) + 0\left(\Delta^2\right),$$

for all $T \in [\Delta, \ldots, \Delta(\bar{\tau}-1)]$ and $t \in [\Delta, \ldots, \Delta\bar{T}]$. This is an approximate solution to the forward rate process for small $\Delta$.

We consider the limit of this expression as $\Delta \to 0$ by subdividing into three terms. First,

$$\text{(35)} \quad \lim_{\Delta\to 0} \frac{\sum_{j=1}^{\bar{t}}\left(\left(\sum_{i=j}^{\bar{T}}\sigma(j\Delta,i\Delta)\Delta\right)^2 - \left(\sum_{i=j}^{\bar{T}-1}\sigma(j\Delta,i\Delta)\Delta\right)^2\right)\Delta}{\Delta} =$$

$$\int_0^t\left(\frac{d}{dT}\left[\int_s^T \sigma(s,y)\,dy\right]^2\right)ds = 2\int_0^t \sigma(s,T)\left(\int_s^T \sigma(s,y)\,dy\right)ds .$$

The integrals are well-defined since $\sigma(t,T)$ is continuous and bounded. Second, by a generalized form of the central limit theorem (see the Appendix for a proof),

$$\text{(36)} \quad \lim_{\Delta\to 0} \frac{\sum_{j=1}^{\bar{t}} a_j\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta} - q\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}}{\left[\sum_{j=1}^{\bar{t}}\sigma^2(j\Delta,T)\right]^{1/2}\sqrt{\Delta}}$$

converges in distribution to a standard normal random variable. Third,

$$\text{(37)} \quad \lim_{\Delta\to 0}\sum_{j=1}^{\bar{t}}(q-\pi)\sigma(j\Delta,T)(q(1-q))^{1/2}\sqrt{\Delta} = \begin{array}{l} 0 \text{ if } \pi = q \\ +\infty \text{ if } q > \pi \\ -\infty \text{ if } q < \pi . \end{array}$$

This expression only converges if $\pi = q$. Hence, the approximate forward rate process $f(t,T)$ converges in distribution to a random variable only if $\pi = q$. In this case,

$$\text{(38)} \quad f(t,T) \xrightarrow{d} f(0,T) + \int_0^t \sigma(s,T)\,dW(s) + \int_0^t \sigma(s,T)\left(\int_s^T \sigma(s,y)\,dy\right)ds ,$$

as $\Delta \to 0$, where $\{W(t)\colon t \in [0,\tau]\}$ is a standard Brownian motion, since $\int_0^t \sigma(s,T)dW(s)$ is normally distributed with mean zero and variance $\int_0^t \sigma^2(s,T)ds$.

The form of the drift function in Expression (38) is identical to that obtained in Heath, Jarrow, and Morton (1987). The limiting form of the distribution in Expression (38) depends only on the volatility function, $\sigma(t,T)$, specified in Expression (31), and it is independent of the choice of $q$, as long as $q = \pi$. This independence is significant. It happens because (when $\pi = q$) the "drift" term is subtracted from the forward rate process in Expression (34). This is the third term on the right-hand side of Expression (34). The effect that different values of $\pi$ (or $q$) have on the forward process is, thus, neutralized. The fact that the process converges only if $\pi = q$ is due to the fact that, in the model studied here, the martingale probability $\pi$ is chosen to be a constant over time. This restriction was imposed for expositional purposes, and the analysis can be generalized in the appropriate fashion. The mathematics, however, rapidly becomes quite sophisticated.

We next specialize Expression (38) for the three cases previously studied.

## A. Ho and Lee's Model

If we set $\sigma^2(t,T) = \sigma^2$, then Expression (38) simplifies to,

$$f(t,T) \xrightarrow{d} f(0,T) + \sigma W(t) + \sigma^2\left(Tt - t^2/2\right), \tag{39}$$

where $\{W(t)\colon t \in [0,\tau]\}$ is a standard Brownian motion. Here, Expression (34), with $\tau = q = ½$ and $\sigma(t,T) = \sigma^2 > 0$, provides an approximation to the continuous time model given on the right-hand side of Expression (39). These substitutions generate

$$f(t,T) = f(0,T) + \sum_{j=1}^{t}\left(a_j - 1/2\right)2\sigma\sqrt{\Delta} + \left[Tt - t^2/2\right]\sigma^2. \tag{40}$$

For practical applications, the importance of this approximation is that the parameters of this model can be estimated directly, using historical forward rates. This is in contrast to the original form of Ho and Lee's model, where the parameters need to be estimated implicitly.

## B. The Exponentially Decaying Variance Model

If we set $\sigma^2(t,T) = \sigma^2 e^{-\lambda(T-t)}$, then Expression (38) simplifies to

$$\begin{aligned} f(t,T) &\xrightarrow{d} f(0,T) + \int_0^t \sigma e^{-(\lambda/2)(T-s)} dW(s) \\ &\quad + \sigma^2 \int_0^t e^{-(\lambda/2)(T-s)} \int_s^T e^{-(\lambda/2)(y-s)} dy ds, \end{aligned} \tag{41}$$

where $\{W(t)\colon t \in [0,\tau]\}$ is a standard Brownian motion. Unfortunately, the continuous time process on the right-hand side of Expression (41) depends on the path of the Brownian motion process. The binomial process given in Expression (34), with $\pi = q = ½$ and $\sigma^2(t,T) = \sigma^2 e^{-\lambda(T-t)}$, provides a useful numerical approximation of Expression (41) for pricing contingent claims, i.e.,

$$f(t,T) = f(0,T) + \sum_{j=1}^{\bar{t}} \left(b_j - 1/2\right) 2\sigma e^{-(\lambda/2)(\bar{T}-j)\Delta} \sqrt{\Delta} \tag{42}$$
$$+ \frac{1}{2} \sum_{j=1}^{\bar{t}} \left[ \left( \sum_{i=j}^{\bar{T}} \sigma e^{-(\lambda/2)(i-j)\Delta} \Delta \right)^2 - \left( \sum_{i=j}^{\bar{T}-1} \sigma e^{-(\lambda/2)(i-j)\Delta} \Delta \right)^2 \right] + 0\left(\Delta^2\right).$$

## C. Combined Constant and Exponentially Decaying Model

If we set

$$\begin{aligned} \mathrm{Var}_{t-\Delta}\left(f(t,T) - f(t-\Delta,T) \mid b_{\bar{t}}\right) &= \sigma_1^2 \Delta \\ \text{and} \quad \mathrm{Var}_{t-\Delta}\left(f(t,T) - f(t-\Delta,T) \mid a_{\bar{t}}\right) &= \sigma_2^2 e^{-\lambda(T-t)} \Delta , \end{aligned}$$

and impose the additivity condition (as in Section VI), then setting $\pi = q = ½$ yields the following forward rate process consistent with no arbitrage opportunities,

$$f(t,T) \xrightarrow{d} f(0,T) + \sigma_1 W_1(t) + \int_0^t \sigma_2 e^{-(\lambda/2)(T-s)} dW_2(s) \tag{43}$$
$$+ \sigma_1^2\left(Tt - t^2/2\right) + \sigma_2^2 \int_0^t e^{-(\lambda/2)(T-s)} \int_s^T e^{-(\lambda/2)(y-s)} dy ds ,$$

where $\{W_1(t): t \in [0,\tau]\}$ and $\{W_2(t): t \in [0,\tau]\}$ are independent Brownian motions. A numerical approximation for this model is given by

$$f(t,T) = f(0,T) + \sum_{j=1}^{\bar{t}} \left(a_j - 1/2\right) 2\sigma_1 \sqrt{\Delta}$$
$$+ \sum_{j=1}^{\bar{t}} \left(b_j - 1/2\right) 2\sigma_2 e^{-(\lambda/2)(\bar{T}-j)\Delta} \sqrt{\Delta} + \left[Tt - t^2/2\right] \sigma_1^2 \tag{44}$$
$$+ (1/2) \sum_{j=1}^{\bar{t}} \left[ \left( \sum_{i=j}^{\bar{T}} \sigma_2 e^{-(\lambda/2)(i-j)\Delta} \Delta \right)^2 - \left( \sum_{i=j}^{\bar{T}-1} \sigma_2 e^{-(\lambda/2)(i-j)\Delta} \Delta \right)^2 \right]$$
$$+ 0\left(\Delta^2\right) .$$

# VIII. Contingent Claim Valuation

The major application of this methodology is to price interest rate dependent contingent claims. The pricing condition in Proposition 1 (11.b) can be used to obtain pricing formulas for all interest rate dependent contingent claims, e.g., call options on bonds. The detailed analysis is contained in Heath, Jarrow, and Morton (1987). To illustrate this procedure, let $C(T)$ represent the only cash flow to the contingent claim, and let it be received at time $T$. Given that this cash flow can be duplicated by a self-financing trading strategy involving the bonds and the accumulation factor (this is our definition of an interest rate dependent contingent claim), the value of this contingent claim at time $t$, $C(t)$, will be

$$C(t) = B(t)E_t^*(C(T)/B(T)) . \tag{45}$$

This value can be calculated recursively.

As Expression (45) makes explicit, only the martingale probabilities $\{\pi_{00}(j), \pi_{01}(j), \pi_{10}(j), \pi_{11}(j): j \in \{1, \ldots, \vec{\tau}\}\}$ are needed to price contingent claims. It is, therefore, the dynamics of the forward rate process under these probabilities that are relevant in the recursive computation.

The continuous time approximations studied in Section VII are for a special case of the general model. This special case has the original probabilities $(q)$ and the martingale probabilities $(\pi)$ being constant across time, and the variances of the changes in the forward rate process $(\sigma(t,T))$ being deterministic functions of time. Under these restrictions, the discrete time process converges to a continuous time limit if and only if $\pi = q$, i.e., the original economy is risk neutral. This appears to make the approximation results of little use, but it is not the case. The approximating discrete time processes (Expressions (34), (40), (42), and (44) are exactly the proper processes to use when approximating contingent claim values in continuous time economies where the volatility functions are deterministic.

To prove this assertion, consider the continuous time economy, corresponding to the case where the volatility parameters are deterministic functions of time, i.e.,

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{2}\sigma_i(t,T)dW_i(t) , \tag{46}$$

where $\{W_1(t),W_2(t): t \in [0,\tau]\}$ are independent, standard Brownian motions initialized at zero,

$\sigma_i$: $\{(t,s): 0 \leq t \leq s \leq T\} \rightarrow R$ is jointly measurable for $i = 1, 2$, satisfies

$$\int_0^T \sigma_i(t,T)^2 dt < +\infty \text{ for } i = 1, 2,$$

$$\alpha(t,T) = -\sum_{i=1}^{2}\sigma_i(t,T)\phi_i(t) + \sum_{i=1}^{2}\sigma_i(t,T)\int_t^T \sigma_i(t,v)dv,$$

and $\phi_i(t)$ for $i = 1, 2$ are bounded, adapted processes representing arbitrary "market prices for risk."

The drift term in this expression is specified such that the process admits no arbitrage opportunities (see Heath, Jarrow, and Morton (1987)).

In this economy, Heath, Jarrow, and Morton (1987) show that there exists a martingale measure $\widetilde{Q}$, (corresponding to $\pi$) and independent Brownian motions $\{\widetilde{W}_1(t), \widetilde{W}_2(t): t \in [0,\tau]\}$ with respect to $\widetilde{Q}$ such that,

$$df(t,T) = \sum_{i=1}^{2} \sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv + \sum_{i=1}^{2} \sigma_i(t,T)\,d\widetilde{W}_i(t) . \tag{47}$$

(Compare this expression to Expression (38).)

Contingent claims are valued under this process with the $\widetilde{Q}$ probability measure. Hence, for the contingent claim described prior to Expression (45), the following valuation formula applies

$$C(t) = \widetilde{E}_t(C(T)/B(T))B(t) . \tag{48}$$

To (numerically) approximate this value, we need to approximate the stochastic process in (47), not (46). But Expression (47) is the process that exists in a *"risk-neutral"* economy. Hence, to approximate Expression (47), we need a discrete time economy where $\pi = q$. This, however, is the economy we derived our limiting approximations for in Section VII. Consequently, the analysis on Section VII is still relevant to practical applications.

The continuous time approximations of Section VII, of which the Ho and Lee (1986) model is a special case, also share the characteristic that the limiting process for forward rates is independent of the parameter $\pi$. Hence, the limits of the contingent claim values will be independent of $\pi$ as well. This insight implies that a procedure that involves estimating $\pi$ by inverting contingent claim values will provide unstable estimates for $\pi$. Indeed, for small step sizes, the discrete process will provide an approximation to the continuous time limit, and the continuous time limit is insensitive to $\pi$. This is a limitation of the estimation procedure suggested by Ho and Lee ((1986), p. 1025). An estimation procedure based on the parameters of the limiting process (Expressions (38), (39), (41), and (43)), however, will avoid this difficulty.

## IX. Summary

This paper analyzes the binomial approximation to the continuous trading term structure model of Heath, Jarrow, and Morton (1987). As such, it makes three additional contributions to the literature beyond that contained in the original Heath, Jarrow, and Morton (1987) piece. First, it makes the abstract methodology of the continuous time model accessible to a wider audience by obtaining it as the limit to a discrete time model. Second, it illustrates a numerical approximation procedure useful for calculating the continuous time models in the original paper. Unfortunately, many of these models do not have simple trees, and additional research is needed along these lines to develop faster and more efficient estimating procedures. Third, it also extends and generalizes the model of Ho and Lee (1986). First, it extends it by offering an alternative notational scheme that is more conducive to formal analysis. Second, it generalizes the Ho and Lee forward rate process to include multiple random shocks. Third, it provides an alternative reparameterization of their discrete trading model, which facilitates historical estimation of the parameters. This is in sharp contrast to the implicit estimation procedure suggested by Ho and Lee that can generate unstable estimates.

## Appendix: Proof of Expression (36).

Let $0 < \ell \leqslant \sigma(j\Delta,T) \leqslant L$ for all $j \in [1,\dots,\bar{T}]$ and $T \in [\Delta,\dots,\Delta(\bar{\tau}-1)]$, since $\sigma$ is a continuous function on a compact set. Consider $\Sigma_{j=1}^{\bar{t}} a_j\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}$. This is the sum of $\bar{t}$ independent random variables. First,

$$
\begin{aligned}
&E\left(|a_j\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta} - q\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}|^3\right) = \\
&\quad q(1-q)^3\sigma^3(j\Delta,T)\left(q(1-q)^{-1/2}\right)\Delta^{3/2} \\
&\quad + (1-q)q^3\sigma^3(j\Delta,T)(q(1-q))^{-1/2}\Delta^{3/2} = \\
&\quad \left[(1-q)^2+q^2\right]\left[(q(1-q))^{1/2}\sigma^3(j\Delta,T)\Delta^{3/2}\right]
\end{aligned}
$$

and $s_{\bar{t}}^2 = \sum_{j=1}^{\bar{t}} \mathrm{Var}\left(a_j\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta}\right) = \sum_{j=1}^{\bar{t}}\sigma^2(j\Delta,T)\Delta$.

$$
\text{Hence,}\quad \lim_{\bar{t}\to\infty} \frac{\sum_{j=1}^{\bar{t}}\left[(1-q)^2+q^2\right]\left[(q(1-q))^{1/2}\sigma^3(j\Delta,T)\Delta^{3/2}\right]}{\left[\sum_{j=1}^{\bar{t}}\sigma^2(j\Delta,T)\right]^{3/2}\Delta^{3/2}}
$$

$$
\leqslant \frac{2\sum_{j=1}^{\bar{t}}\sigma^3(j\Delta,T)}{\left[\sum_{j=1}^{\bar{t}}\sigma^2(j\Delta,T)\right]^{3/2}} \leqslant \frac{2\bar{t}L^3}{(\bar{t})^{3/2}\ell^3} \to 0 \text{ as } \bar{t}\to\infty .
$$

By Breiman ((1968), Theorem 9.2, p. 186), as $\bar{t}\to\infty$,

$$
\frac{\sum_{j=1}^{\bar{t}}\left[a_j\sigma(j\Delta,T)(q(1-q))^{-1/2}\sqrt{\Delta} - q\sigma(j\Delta,T)q(1-q)^{-1/2}\sqrt{\Delta}\right.}{\sqrt{\sum_{j=1}^{\bar{t}}\sigma^2(j\Delta,T)\Delta}}
$$

converges in distribution to a standard normal random variable.

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