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Valuing Derivative Securities Using the Explicit Finite Difference Method

John Hull and Alan White*

Abstract

This paper suggests a modification to the explicit finite difference method for valuing derivative securities. The modification ensures that, as smaller time intervals are considered, the calculated values of the derivative security converge to the solution of the underlying differential equation. It can be used to value any derivative security dependent on a single state variable and can be extended to deal with many derivative security pricing problems where there are several state variables. The paper illustrates the approach by using it to value bonds and bond options under two different interest rate processes.

I. Introduction

Two of the most popular procedures for valuing derivative securities are the lattice (or tree) approach and the finite difference approach. The lattice approach was suggested by Cox, Ross, and Rubinstein (1979) and has been extended by Rendleman and Bartter (1979), Boyle (1986), (1988), and Hull and White (1988). The finite difference approach was suggested by Schwartz (1977) and Brennan and Schwartz (1978), and has been extended by Courtadon (1982b). Both approaches involve discrete approximations to the processes followed by the underlying variables.

There are two alternative ways of implementing the finite difference approach. The first, known as the explicit finite difference method, relates the value of the derivative security at time t to three alternative values at time $t + \Delta t$. The second, known as the implicit finite difference method, relates the value of the derivative security at time $t + \Delta t$ to three alternative values at time t. Brennan and Schwartz (1978) show that the explicit finite difference method is equivalent to a trinomial lattice approach. They also show that the implicit finite difference method corresponds to a multinomial lattice approach where, in the limit, the underlying variable can move from its value at time t to an infinity of possible values at time $t + \Delta t$.

Geski and Shastri (1985) provide an interesting comparison of different lattice and finite difference approaches. They conclude that the explicit finite differ-

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ence method, with logarithmic transformations, is the most efficient approach when large numbers of stock options are being evaluated. The explicit finite difference method is also attractive for a number of other reasons. It is computationally much simpler than the implicit method since it does not require the inversion of matrices. It is conceptually simpler than the implicit method since it is, in effect, nothing more than an application of the trinomial lattice approach. Furthermore, as will be explained in Section II, it can avoid the need to specify some boundary conditions. The method's only disadvantage is that the numerical solution does not necessarily converge to the solution of the differential equation as Δt tends to zero.

We modify the explicit finite difference method so that convergence of the calculated values to the correct solution is ensured. Brennan and Schwartz (1978) and Geske and Shastri (1985) show how a transformation of variables ensures convergence when stock options are being valued. The procedure in this paper involves both a transformation of variables and a new branching process. It can be used for the valuation of any derivative security dependent on a single state variable and for the valuation of many derivative securities that are dependent on several state variables. We illustrate the procedure by valuing bonds and bond options when interest rates are governed by the Cox, Ingersoll, and Ross (1985b) and the Brennan and Schwartz (1982) models.

The paper is organized as follows. Section II describes the explicit finite difference method and discusses its relation to lattice approaches. Section III discusses issues associated with convergence and describes a modification of the explicit finite difference method that ensures convergence in single state variable models. Section IV applies the explicit finite difference method to valuing bonds and bond options under the Cox, Ingersoll, and Ross (1985b) model. Section V discusses the application of explicit finite difference methods to problems with two state variables. It values bonds and bond options using the Brennan and Schwartz (1982) model. Conclusions are in Section VI.

II. The Explicit Finite Difference Method

Consider a derivative security, with price f, that depends on a single stochastic variable, θ. Suppose that the stochastic process followed by θ is

$$d\theta = \mu(\theta,t)\theta dt + \sigma(\theta,t)\theta dz ,$$

where dz is a Wiener process. The variables μ and σ, which may be functions of θ and t, are the instantaneous proportional drift rate and volatility of θ.

If λ is the market price of risk of θ, then, as shown by Garman (1976) and Cox, Ingersoll, and Ross (1985a), f must satisfy the following differential equation,

$$\text{(1)} \qquad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta}(\mu - \lambda\sigma)\theta + \frac{1}{2}\theta^2\sigma^2\frac{\partial^2 f}{\partial \theta^2} = rf ,$$

where r is the risk-free interest rate. Both r and λ may be functions of θ and t. When θ is the price of a nondividend-paying stock, $\mu - \lambda\sigma = r$ and (1) reduces to the well-known Black and Scholes (1973) differential equation.

To implement the explicit finite difference method, a small time interval, Δt, and a small change in θ, $\Delta\theta$, are chosen. A grid is then constructed for considering values of f when θ is equal to

$$\theta_0, \theta_0 + \Delta\theta, \theta_0 + 2\Delta\theta, \ldots, \theta_{\max},$$

and time is equal to

$$t_0, t_0 + \Delta t, t_0 + 2\Delta T, \ldots, T.$$

The parameters θ_0 and θ_{max} are the smallest and largest values of θ considered by the model, t_0 is the current time, and T is the end of the life of the derivative security.

We will denote $t_0 + i\Delta t$ by t_i, $\theta_0 + j\Delta\theta$ by θ_j, and the value of the derivative security at the (i,j) point on the grid by f_{ij}. The partial derivatives of f with respect to θ at node $(i-1,j)$ are approximated as follows,

(2) $$\frac{\partial f}{\partial \theta} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta\theta},$$

(3) $$\frac{\partial^2 f}{\partial \theta^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{ij}}{\Delta\theta^2},$$

and the time derivative is approximated as

(4) $$\frac{\partial f}{\partial t} = \frac{f_{ij} - f_{i-1,j}}{\Delta t}.$$

Substituting (2), (3), and (4) into (1) gives

(5) $$f_{i-1,j} = a_{j-1} f_{i,j-1} + a_j f_{ij} + a_{j+1} f_{i,j+1},$$

where $$a_{j-1} = \frac{1}{1+r\Delta t}\left[-\frac{(\mu-\lambda\sigma)\theta_j\Delta t}{2\Delta\theta} + \frac{1}{2}\frac{\theta_j^2\sigma^2\Delta t}{\Delta\theta^2}\right],$$

$$a_j = \frac{1}{1+r\Delta t}\left[1 - \frac{\theta_j^2\sigma^2\Delta t}{\Delta\theta^2}\right], \text{ and}$$

$$a_{j+1} = \frac{1}{1+r\Delta t}\left[\frac{(\mu-\lambda\sigma)\theta_j\Delta t}{2\Delta\theta} + \frac{1}{2}\frac{\theta_j^2\sigma^2\Delta t}{\Delta\theta^2}\right].$$

These equations form the basis of the explicit finite difference method.[1] They

[1] The equations for the implicit finite difference method are obtained in a similar way with the approximation

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{ij}}{\Delta t}$$

being used in place of (4) and the partial derivatives being assumed to apply to node (i,j). The implicit method relates $f_{i+1,j}$ to $f_{i,j-1}$, f_{ij}, and $f_{i,j+1}$.

relate the value $f_{i-1,j}$ of the derivative security at time t_{i-1} to three alternative values of the derivative security at time t_i. The value of f at time T is known. The value of f at time t can be obtained by using (5) repeatedly to work back from time T to time t in steps of Δt.[2]

Define

$$p_{j,j-1} = -\frac{\theta_j(\mu - \lambda\sigma)\Delta t}{2\Delta\theta} + \frac{1}{2}\theta_j^2\sigma^2\frac{\Delta t}{\Delta\theta^2},$$

$$p_{jj} = 1 - \theta_j^2\sigma^2\frac{\Delta t}{\Delta\theta^2}, \text{ and}$$

$$p_{j,j+1} = \theta_j(\mu - \lambda\sigma)\frac{\Delta t}{2\Delta\theta} + \frac{1}{2}\theta_j^2\sigma^2\frac{\Delta t}{\Delta\theta^2},$$

so that Equation (5) becomes

$$(6) \qquad f_{i-1,j} = \frac{1}{1+r\Delta t}\left[p_{j,j-1}f_{i,j-1} + p_{jj}f_{ij} + p_{j,j+1}f_{i,j+1}\right].$$

It is easy to show that $p_{j,j-1}$, p_{jj}, and $p_{j,j+1}$ can be interpreted as the probabilities of moving from θ_j to θ_{j-1}, θ_j, and θ_{j+1}, respectively, during time Δt, in a world where the proportional drift rate of θ is $\mu - \lambda\sigma$. ($p_{j,j-1}$, p_{jj}, and $p_{j,j+1}$ sum to unity and give a drift rate of $(\mu - \lambda\sigma)\theta$. Also, when terms of $0(\Delta t^2)$ are ignored, these values imply a variance rate of $\sigma^2\theta^2$.) When the p's are interpreted in this way, (6) gives the value of f at time t_i as its expected value at time t_{i+1}, in a world where the drift rate of θ is $\mu - \lambda\sigma$, discounted to time t_i at the risk-free rate of interest. This corresponds to the procedure suggested in Cox, Ingersoll, and Ross ((1985a), Lemma 4) for valuing derivative securities.

We can conclude from this that the explicit finite difference method is equivalent to a trinomial lattice approach. In Section III, this equivalence is used to explain the conditions required to ensure convergence.

The explicit finite difference method has the advantage that it can require the specification of fewer boundary conditions than the implicit method. Consider, for example, the valuation of a derivative security dependent on a stock price S. The implicit method requires the user to specify boundary conditions for the derivative security as $S \rightarrow 0$ and $S \rightarrow \infty$. The explicit method, when implemented as a trinomial lattice, does not require these boundary conditions.

Partial differential equations can be classified as either boundary value problems (where a full set of boundary conditions must be specified) or initial value problems (where only the value of the function at one particular time needs to be specified). Many derivative security pricing problems, including most option valuation problems, are initial value problems. Ames ((1977), p. 62) makes the point that the explicit finite difference method is the best approach for initial value problems. This is because errors are introduced by the extra boundary conditions used in the implicit finite difference method. Consider, for example, the valuation of a derivative security dependent on a stock price S. Errors are intro-

[2] The presentation here assumes that f is a European-style derivative security that pays no income. The arguments easily can be extended to other situations.

duced because the implicit method's boundary condition as $S \to \infty$ is applied to a finite value of S.

III. The Proposed Procedure

A. The Transformation of Variables

As pointed out by Brennan and Schwartz (1978), Geski and Shastri (1985), and others, when θ is a stock price, it is efficient to use $\mathrm{Ln}\theta$ rather than θ as the underlying variable when finite difference methods are applied. This is because, when σ is constant, the instantaneous standard deviation of $\mathrm{Ln}\theta$ is constant, i.e., the standard deviation of changes in $\mathrm{Ln}\theta$ in a time interval Δt is independent of θ and t.

Generalizing from this, it is always appropriate, when applying the explicit finite difference method, to define a new state variable $\phi(\theta,t)$ that has a constant instantaneous standard deviation. From Ito's lemma, the process followed by ϕ in a risk-neutral world is

(7) $$d\phi = q(\theta,t)dt + \frac{\partial \phi}{\partial \theta}\sigma\theta dz ,$$

where

(8) $$q(\theta,t) = \frac{\partial \phi}{\partial t} + (\mu - \lambda\sigma)\theta\frac{\partial \phi}{\partial \theta} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2 \phi}{\partial \theta^2} .$$

We, therefore, wish to choose the variable ϕ so that

(9) $$\sigma\theta\frac{\partial \phi}{\partial \theta} = v ,$$

for some constant v.

The state variable ϕ can be modeled in the same way as θ. A grid is constructed for values of ϕ equal to $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$, where $\phi_j = \phi_0 + j\Delta\phi$, and the probabilities in (6) become

(10) $$p_{j,j-1} = -q\frac{\Delta t}{2\Delta\phi} + \frac{1}{2}v^2\frac{\Delta t}{\Delta\phi^2} ,$$

(11) $$p_{jj} = 1 - v^2\frac{\Delta t}{\Delta\phi^2} , \text{ and}$$

(12) $$p_{j,j+1} = q\frac{\Delta t}{2\Delta\phi} + \frac{1}{2}v^2\frac{\Delta t}{\Delta\phi^2} .$$

If θ is a nondividend-paying stock with σ and r constant, $\mu - \lambda\sigma = r$ and q is constant. The lattice then corresponds to Boyle's (1986) trinomial extension of the Cox, Ross, and Rubinstein (1979) binomial lattice. Since q is constant, it has the simplifying property that the probabilities are the same at all nodes (i.e., $p_{j,j-1}$, p_{jj}, and $p_{j,j+1}$ are independent of j). If the grid is selected so that $\Delta t/\Delta\phi^2 = 1/\sigma^2$, the Cox, Ross, and Rubinstein binomial lattice results.

B. The Modification to the Branching Process

When using the explicit finite difference method, it is important to ensure that as Δt and $\Delta\phi \rightarrow 0$, the estimated value of the derivative security converges to its true value.[3] From a theorem in Ames ((1977), p. 45), a sufficient condition for convergence is that $p_{j,j-1}$, p_{jj}, and $p_{j,j+1}$ be positive as Δt and $\Delta\phi \rightarrow 0$. This can be seen intuitively from the equivalence of the explicit finite difference method and the trinomial lattice approach. From Equations (10), (11), and (12), this condition is satisfied if

$$v^2 \frac{\Delta t}{\Delta\phi^2} < 1 \tag{13}$$

and

$$q < \frac{v^2}{\Delta\phi} \tag{14}$$

as $\Delta t, \Delta\phi \rightarrow 0$.[4]

If q is bounded, (13) and (14) can be satisfied and convergence can be ensured. The simplest procedure is to let Δt and $\Delta\phi$ approach 0 in such a way that $\Delta t/\Delta\phi^2$ remains constant and less than $1/v^2$. The desirability of keeping the ratio of Δt to $\Delta\phi^2$ constant in order to ensure convergence also has been mentioned by Brennan and Schwartz (1978) and by Geski and Shastri (1985).

There are some situations where q is unbounded. As will be shown in Section IV, one such situation occurs when θ is an interest rate following a mean-reverting process. The explicit finite difference method, as it has been described so far, may not then converge. However, the method can be modified to overcome this problem. Instead of insisting that we move from ϕ_j to one of ϕ_{j-1}, ϕ_j, and ϕ_{j+1} in time Δt, we allow a movement from ϕ_j to one of ϕ_{k-1}, ϕ_k, and ϕ_{k+1}, where k is not necessarily equal to j. In Figure 1, (a)–(e) show the situations where $k = j$, $k = j+1$, $k = j-1$, $k < j-1$, and $k > j+1$, respectively.

In all cases, we choose k so that ϕ_k is the value of ϕ on the grid closest to $\phi_j + q\Delta t$.[5] The probabilities of ϕ_j moving to ϕ_{k-1}, ϕ_k and ϕ_{k+1} are chosen to make the first and second moments of the change in ϕ in the time interval Δt correct in the limit as $\Delta t \rightarrow 0$. The equations that must be satisfied are:

$$p_{j,k-1}(k-1)\Delta\phi + p_{jk}k\Delta\phi + p_{j,k+1}(k+1)\Delta\phi = E(\phi),$$

$$p_{j,k-1}(k-1)^2\Delta\phi^2 + p_{jk}k^2\Delta\phi^2 + p_{j,k+1}(k+1)^2\Delta\phi^2 = v^2\Delta t + E(\phi)^2, \text{ and}$$

$$p_{j,k-1} + p_{jk} + p_{j,k+1} = 1,$$

[3] Strictly speaking, we are interested in both stability and convergence. A stable procedure is one where the results are relatively insensitive to round-off and other small computational errors (See Ames (1977), p. 28). In practice, the conditions for stability and convergence are the same in most derivative security pricing problems.

[4] When $\phi = \text{Ln}\theta$ and $\mu - \lambda\sigma = r$, these correspond to the conditions in Brennan and Schwartz (1978).

[5] In most situations $k = j-1$, j, or $j+1$. This means that the branching process corresponds to Figure 1(a), 1(b), or 1(c).

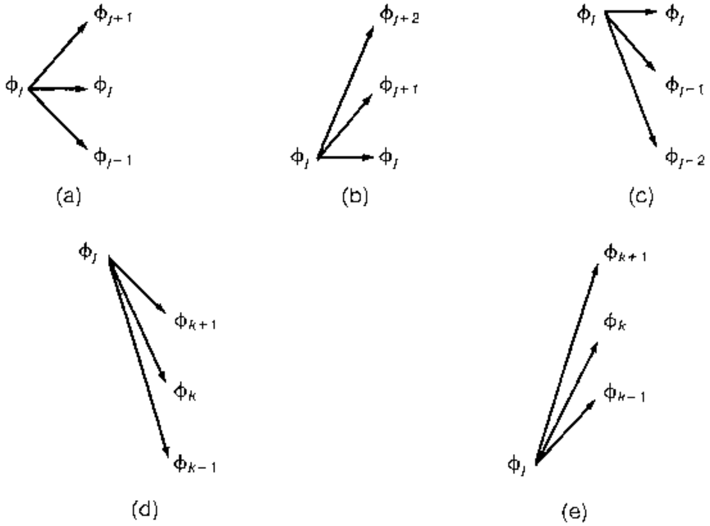

FIGURE 1

Alternative Branching Procedures in the Explicit Finite Difference Method Designed to Ensure that the Probabilities Associated with all Three Branches Remain Positive

The normal situation is illustrated in (a). If the expected change in ϕ_j is sufficiently large and positive, it may be necessary to use the branching processes illustrated in (b) or (e). The branching processes illustrated in (c) and (d) may be required for negative expected changes in ϕ_j.

where $E(\phi) = j\Delta\phi + E(\delta\phi)$ is the expected value of $\phi - \phi_0$ at the end of the time interval, Δt. The solution to these equations is

$$(15) \qquad p_{j,k-1} = \frac{1}{2}\left[k^2 + k - (1+2k)\frac{E(\phi)}{\Delta\phi} + \frac{E(\phi)^2}{\Delta\phi^2} + \frac{v^2\Delta t}{\Delta\phi^2}\right],$$

$$(16) \qquad p_{jk} = 1 - k^2 + \frac{2kE(\phi)}{\Delta\phi} - \frac{E(\phi)^2}{\Delta\phi^2} - \frac{v^2\Delta t}{\Delta\phi^2}, \text{ and}$$

$$(17) \qquad p_{j,k+1} = \frac{1}{2}\left[k^2 - k + (1-2k)\frac{E(\phi)}{\Delta\phi} + \frac{E(\phi)^2}{\Delta\phi^2} + \frac{v^2\Delta t}{\Delta\phi^2}\right].$$

The procedure suggested in this section can be used to deal with jumps in θ. Suppose for example that θ is the price of a security and a dividend at time τ is expected to cause θ to jump down by $\delta(\theta)$. For the time interval in which the dividend occurs, we can switch from Figure 1(a) to Figure 1(d). We define k in Figure 1(d) so that ϕ_k is the value of ϕ on the grid closest to $\phi(\theta_j - \delta(\theta_j), \tau)$.[6]

[6] Hull and White (1988) and Hull (1989) suggest that known dividends should be dealt with by defining θ as the security price less the present value of the known income. This has some theoretical appeal. If the dollar amount of future income is known, the price of the security logically should be divided into two components: a nonstochastic component that will be used to pay the known dividend, and a residual stochastic component. The approach suggested here may be more appropriate when long time periods are considered (e.g., in the valuation of warrants or convertible stock).

Finally, it is worth noting that the explicit finite difference method provides one degree of freedom: the choice of $v^2\Delta t/\Delta\phi^2$. We will denote this by w. One constraint on w is that it should always be possible to find a k such that $p_{j,k-1}$, p_{jk}, and $p_{j,k+1}$ are positive. It can be shown that this constraint implies that $0.25 < w < 0.75$. If q is small, an examination of the errors in the way in which the differential equation is approximated suggests that a sensible value for w is ⅓.[7] We find that this works well.

IV. Application to a One State Variable Interest Rate Model

A number of authors have suggested that an appropriate process for the short-term interest rate is

$$d\theta = a(b-\theta)dt + c\theta^{\alpha}dz ,$$

where a, b, c, and α are constants, and θ is the short-term interest rate. In Vasicek (1977), $\alpha = 0$; in Cox, Ingersoll, and Ross (1985b), $\alpha = ½$; in Courtadon (1982a), $\alpha = 1$. The modified version of the explicit finite difference method can be used for any value of α. We will illustrate its use for $\alpha = ½$.

From (9), the appropriate transformation of θ is

$$\phi = \sqrt{\theta} .$$

In a risk-neutral world,

$$d\theta = [a(b-\theta) - uc\theta]\,dt + c\sqrt{\theta}dz ,$$

where $u\sqrt{\theta}$ is the market price of risk. This means that

$$d\phi = qdt + vdz ,$$

where $v = c/2$, and

$$\begin{aligned} q &= [a(b-\theta) - uc\theta]\frac{\partial\phi}{\partial\theta} + \frac{c^2\theta}{2}\frac{\partial^2\phi}{\partial\theta^2} \\ &= \frac{4ab - c^2}{8\phi} - \frac{\phi}{2}(a+uc) . \end{aligned}$$

Since ϕ can take on any positive value, q is not bounded. It follows that the standard explicit finite difference method may not converge. However, the variation on the standard method described in Section III can be used.

Define

$$\alpha_1 = \frac{4ab - c^2}{8} ; \ \alpha_2 = \frac{1}{2}(a+uc) .$$

[7] If $q = 0$, the errors are $O(\Delta t^2)$ rather than $O(\Delta t)$ when $v^2\Delta t/\Delta\phi^2 = ⅙$. If q is constant (as is the case when θ is a stock price), it is efficient to define a new "zero q" variable ψ,

$$\psi = \phi - qt .$$

When this variable is modeled using $p_{j,j+1} = ⅙$, $p_{jj} = ⅔$, and $p_{j,j-1} = ⅙$, the errors are $O(\Delta t^2)$.

As suggested in Section III, we choose $v^2\Delta t/\Delta\phi^2 = 1/3$. It is easy to show $k = j$ when

$$-\frac{1}{2} \leqslant \left[\frac{\alpha_1}{\phi} - \alpha_2\phi\right]\frac{\Delta t}{\Delta\phi} \leqslant \frac{1}{2}.$$

Assuming α_1 and α_2 are positive, this condition reduces to[8]

$$\phi_{\min} \leqslant \phi \leqslant \phi_{\max},$$

where
$$\phi_{\min} = \frac{-\beta + \sqrt{\beta^2 + 4\alpha_1\alpha_2}}{2\alpha_2},$$

$$\phi_{\max} = \frac{\beta + \sqrt{\beta^2 + 4\alpha_1\alpha_2}}{2\alpha_2}, \text{ and}$$

$$\beta = \frac{\Delta\phi}{2\Delta t}.$$

Using the approach outlined in Section III, the values of ϕ considered on the grid for the explicit finite difference method are $\phi_0, \phi_1 \ldots \phi_n$, where ϕ_0 is the largest multiple of $\Delta\phi$ less than $\phi_{\min}$, $\phi_j = \phi_0 + j\Delta\phi$, and n is the smallest integer such that $\phi_n \geqslant \phi_{\max}$. (It is assumed that $\Delta\phi$ is also chosen so that some multiple of $\Delta\phi$ equals the current value of ϕ.) Note that, as Δt and $\Delta\phi$ tend to 0, $\phi_{\min}$ approaches 0 and $\phi_{\max}$ increases.

When $1 \leqslant j \leqslant n-1$, the explicit finite difference method (trinomial lattice) approach operates in the usual way. The probabilities of moving from ϕ_j to ϕ_{j-1}, ϕ_j and ϕ_{j+1} in time interval Δt are given by (15), (16), and (17). In this example, when the value ϕ_0 is reached, the three possible values that might be obtained after a time interval Δt are ϕ_0, ϕ_1, and ϕ_2. The probabilities p_{00}, p_{01}, and p_{02} are calculated from Equations (15), (16), and (17), with $j = 0$ and $k = 1$. Similarly, when the value ϕ_n is reached, the three possible values of ϕ after a time interval of Δt has elapsed are ϕ_{n-2}, ϕ_{n-1}, and ϕ_n. The probabilities $p_{n,n-2}$, $p_{n,n-1}$, and p_{nn} of moving to these values are calculated from Equations (15), (16), and (17) with $j = n$ and $k = n-1$. Clearly, it is unnecessary to consider values of ϕ less than ϕ_0 or greater than ϕ_n, since these can never be reached.

The modified explicit finite difference method can be used to value any interest rate contingent claim. Table 1 shows the results of using the procedure to value a discount bond with face value of \$100. Define B_{ij} as the value of the bond at the (i,j) node, and assume that the bond matures at time $t_0 + m\Delta t$. We know that $B_{mj} = 100$ for all j. Since the short-term interest rate is ϕ^2, the value of the bond prior to maturity can be calculated using

$$B_{ij} = \frac{1}{1 + \phi_j^2\Delta t}\left[p_{j,j-1}B_{i+1,j-1} + p_{jj}B_{i+1,j} + p_{j,j+1}B_{i+1,j+1}\right], \tag{18}$$

[8] Note that if $\alpha_2 < 0$, the risk-adjusted drift rate of the short-term interest rate is always positive. This would imply infinite discount rates in a risk-neutral world. If $\alpha_1 < 0$, the effective $\phi_{\min}$ is zero; if $\beta^2 + 4\alpha_1\alpha_2 < 0$, there is no effective $\phi_{\max}$.

for $j = 1, 2 \ldots n-1$,

$$(19) \quad B_{i0} = \frac{1}{1+\phi_0^2 \Delta t}\left[p_{00}B_{i+1,0} + p_{01}B_{i+1,1} + p_{02}B_{i+1,2}\right] \text{ , and}$$

$$(20) \quad B_{in} = \frac{1}{1+\phi_n^2 \Delta t}\left[p_{n,n-2}B_{i+1,n-2} + p_{n,n-1}B_{i+1,n-1} + p_{nn}B_{i+1,n}\right] .$$

Table 1 compares the calculated bond price with the analytic solution given by Cox, Ingersoll, and Ross (1985b) for the interest rate process parameters

$$a = 0.4 \text{ ,}$$
$$b = 0.1 \text{ ,}$$
$$c = 0.06 \text{ , and}$$
$$u = 0 \text{ .}$$

These parameters produce an interest rate model where the short-term interest rate reverts to 10 percent. The instantaneous volatility of the short rate is about 19 percent when the short-term rate is 10 percent. The parameter values chosen are, therefore, not unreasonable. Table 1 shows that the numerical solution is very close to the analytic solution.

TABLE 1

Bond Prices Given by the Explicit Finite Difference Method for the Cox, Ingersoll, and Ross Model in Which the Interest Rate Obeys the Process $dr = a(b-r)dt + c\sqrt{r}\,dz$

Bond Maturity (Years)	Current Short-Term Interest Rate				
	6%	8%	10%	12%	14%
5	66.31 (66.24)	63.53 (63.45)	60.86 (60.78)	58.30 (58.23)	55.85 (55.78)
10	40.92 (40.83)	38.98 (38.89)	37.14 (37.05)	35.38 (35.29)	33.71 (33.62)
15	25.02 (24.94)	23.82 (23.74)	22.68 (22.59)	21.59 (21.51)	20.55 (20.47)
20	15.28 (15.21)	14.54 (14.48)	13.85 (13.78)	13.18 (13.11)	12.55 (12.48)

Note: The market price of interest rate risk is $u\sqrt{r}$. True prices given by the analytic solution are shown in parentheses. Face value of bond = \$100, $\Delta t = 0.05$ years, $a = 0.4$, $b = 0.1$, $c = 0.06$, and $u = 0$.

Table 2 shows the results of using the method to value American call options on a bond paying a coupon at a continuous rate of $\gamma = \$10$ per unit time. First, the bond price at each node of the lattice was calculated using a similar approach to that described above.[9] The option price was then evaluated by working back through the lattice from the end of the option's life and applying the

[9] To adjust for coupons, the expressions in the square brackets in Equations (18), (19), and (20) were each increased by $\gamma\Delta t$.

boundary conditions for an American call option.[10] From Table 2, we see that the procedure converges fairly rapidly.

TABLE 2

Price of a 5-Year American Call Option on a 10-Year Bond ($100 Face Value) Using the Cox, Ingersoll, and Ross Interest Rate Model in Which the Interest Rate Obeys the Process $dr = a(b-r)dt + c\sqrt{r}\,dz$

Δt (Years)	Exercise Price 90	95	100	105	110
0.500	10.34	6.09	2.69	0.47	0.02
0.250	10.36	6.21	2.46	0.44	0.01
0.100	10.47	6.13	2.54	0.45	0.01
0.050	10.51	6.15	2.54	0.46	0.01
0.025	10.52	6.14	2.53	0.45	0.01
0.010	10.52	6.14	2.53	0.45	0.01

Note: The market price of interest rate risk is $u\sqrt{r}$. The current short-term interest rate is 10 percent per annum and the bond pays a coupon at the rate of $10 per annum.
The current price of the bond is 100.39, $a = 0.4$, $b = 0.1$, $c = 0.06$, and $u = 0$.

V. Dealing with More Than One State Variable

To illustrate how the ideas presented above can be extended to deal with several state variables, consider the two state variable case. Suppose the variables are θ_1 and θ_2. These must first be transformed to two new variables ϕ_1 and ϕ_2, so that the instantaneous standard deviation of each is constant. Assume that the volatility of θ_i depends only on θ_i and t ($i = 1, 2$). The correct transformations can then be determined in the way indicated in Section III. The processes for ϕ_1 and ϕ_2 have the form

$$d\phi_1 = q_1 dt + k_1 dz_1 \text{ , and}$$
$$d\phi_2 = q_2 dt + k_2 dz_2 \text{ ,}$$

where k_1 and k_2 are constants, and q_1 and q_2 are defined analogously to q in (8).

There is likely to be an instantaneous correlation ρ between the Wiener processes dz_1 and dz_2. Assume this is constant. The next stage is to transform variables again to eliminate the correlation. This is achieved by defining new variables ψ_1 and ψ_2 as follows,

$$\psi_1 = \left[k_2\phi_1 + k_1\phi_2\right] \text{ , and}$$
$$\psi_2 = \left[k_2\phi_1 - k_1\phi_2\right] \text{ .}$$

These follow the processes

$$d\psi_1 = \left(k_2 q_1 + k_1 q_2\right)dt + k_1 k_2 \sqrt{2(1+\rho)}\, dz_3 \text{ , and}$$
$$d\psi_2 = \left(k_2 q_1 - k_1 q_2\right)dt + k_1 k_2 \sqrt{2(1-\rho)}\, dz_4 \text{ ,}$$

where the Wiener processes dz_3 and dz_4 are uncorrelated.

[10] Suppose C, B, and X are the call price, the bond price, and the exercise price, respectively. At the end of the option's life, $C = \max(B - X, 0)$. At each node, the boundary condition $C \geqslant B - X$ is imposed.

The possible unconditional movements of ψ_1 in the time interval Δt, together with their associated probabilities, are chosen in the same way as they are for ϕ in Section III. The same is true for ψ_2. Unbounded drift rates are dealt with using the same approach as that described in Section III. The variables ψ_1 and ψ_2 are, therefore, modeled using a two-dimensional lattice with nine branches emanating from each node. The probability of any given point being reached is the product of the unconditional probabilities associated with the corresponding movements in ψ_1 and ψ_2.

Tables 3 and 4 show results from the application of the explicit finite difference method to the Brennan-Schwartz two state variable interest rate model. In this model,

$$dr = a(\ell - r)dt + \sigma_1 r\,dz_1 \text{ , and}$$
$$d\ell = \mu(\ell, r, t)dt + \sigma_2 \ell\, dz_2 \text{ ,}$$

where ℓ is the yield on a consol bond, r is the instantaneous risk-free interest rate, and a, σ_1, and σ_2 are constants. The market price of short-term interest rate risk, λ, and the instantaneous correlation, ρ, between r and ℓ also are assumed to be constant. Table 3 shows bond prices, while Table 4 shows European call prices. The rates of convergence are encouragingly fast.

TABLE 3

Prices for a 3-Year Face Value 8-Percent Continuous Coupon Bond using the Brennan-Schwartz Model

	Current Short-Term Rate, r		
Δt (Years)	6%	8%	10%
0.2	102.93	98.02	93.36
0.1	102.93	98.06	93.43
0.067	102.93	98.07	93.45
0.05	102.93	98.07	93.46
0.04	102.93	98.08	93.47

Note: The interest rates obey the processes $dr = a(\ell - r)dt + \sigma_1 r dz$ and $d\ell = \mu(r, \ell, t)dt + \sigma_2 \ell dz_2$, with $a = 0.1$ $\sigma_1 = 0.1$, $\sigma_2 = 0.03$. The market price of short-term interest risk is -0.4, the correlation between dz_1 and dz_2 is 0.2, and the initial long-term rate $\ell = 10$ percent.

TABLE 4

Price of a European Call Option on a 3-Year, 8-Percent Coupon Bond Using the Brennan-Schwartz Model

	Call Price Current Short-Term Rate, r		
Δt (Years)	6%	8%	10%
0.1	3.17	0.60	0.015
0.067	3.17	0.60	0.014
0.05	3.17	0.59	0.014
0.04	3.18	0.59	0.014

Note: The interest rates obey the processes $dr = a(\ell - r)dt + \sigma_1 r dz$ and $d\ell = \mu(r, \ell, t)dt + \sigma_2 \ell dz_2$, with $a = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.03$. The market price of short-term interest rate risk is -0.4, the correlation between dz_1 and dz_2 is 0.2, and the initial long-term rate $\ell = 10$ percent. The option has 1 year to maturity and an exercise price of 98. The corresponding bond prices are given in Table 3.

VI. Summary and Conclusions

The explicit finite difference method is both easier to implement and conceptually simpler than the implicit method. The explicit method's disadvantage is that it does not necessarily converge. This paper provides a systematic procedure for implementing a modified version of the method in such a way that convergence is ensured. This should make the method attractive to both practitioners and researchers.

Geske and Shastri ((1985), Table 2) found that the explicit finite difference method, when implemented in the most efficient way for valuing stock options, uses about 60 percent as much CPU time as the implicit method. We have applied both the explicit and implicit methods to a variety of different problems using an IBM AT Personal Computer. Our results are similar to those of Geske and Shastri. We find that the explicit method uses between 40 and 70 percent as much time as the implicit method to provide the same level of accuracy. One reason for the extra efficiency of the explicit method is that most derivative security pricing problems are initial value problems, *not* boundary value problems. Errors are introduced by the redundant boundary conditions in implicit methods.

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