A Lattice Framework for Option Pricing with Two State Variables

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Journal of Financial and Quantitative Analysis, Volume 23, Issue 1 (Mar., 1988), 1-12.

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Abstract

A procedure is developed for the valuation of options when there are two underlying state variables. The approach involves an extension of the lattice binomial approach developed by Cox, Ross, and Rubinstein to value options on a single asset. Details are given on how the jump probabilities and jump amplitudes may be obtained when there are two state variables. This procedure can be used to price any contingent claim whose payoff is a piece-wise linear function of two underlying state variables, provided these two variables have a bivariate lognormal distribution. The accuracy of the method is illustrated by valuing options on the maximum and minimum of two assets and comparing the results for cases in which an exact solution has been obtained for European options. One advantage of the lattice approach is that it handles the early exercise feature of American options. In addition, it should be possible to use this approach to value a number of financial instruments that have been created in recent years.

I. Introduction

The option pricing framework has proved extremely useful for the analysis of many aspects of corporate finance and investment practice. The volume and variety of options traded on world exchanges has expanded enormously since Black and Scholes developed their seminal approach. Corporations in many countries now offer increasingly sophisticated instruments to the investing public. Budd (1983) describes a number of these new types of securities. The valuation of these securities poses challenging problems since they may involve a complex package of imbedded options whose payoffs depend on several state variables.

Options involving two state variables have been discussed by Stulz (1982), Johnson (1981), Schwartz (1982), and Boyle and Kirzner (1985). These papers concentrated on the valuation of European options and were able to develop ana-

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lytic or quasi-analytic solutions involving the bivariate normal density function. American options with payoffs depending on two state variables satisfy the same basic generalization of the Black-Scholes equation as European options. However, the boundary conditions are more complex in the case of American options to accommodate the possibility of early exercise. It is possible to solve the equation for American options in the case of more than one state variable using a finite difference approach. With two or more state variables, the computations quickly become quite expensive in the case of the finite difference method. In addition, there are significant start-up tests to develop an efficient computing algorithm.

In the case of the valuation of American options when there is only one state variable, the lattice binomial approach developed by Cox, Ross, and Rubinstein (1979) has a strong intuitive appeal. It is extremely simple to implement. The basic idea is to replace the continuous distribution of stock prices by a two-point discrete distribution over successively smaller time intervals. Convergence to the true option value is obtained by increasing the number of steps. The terminal distribution of the stock price is developed using a binomial lattice. This method has won widespread acceptance in the case of a single state variable. Geske and Shastri (1985) outline its advantages and drawbacks as well as those of a number of other approximation techniques in the case of options with a single state variable.

The basic purpose of the present paper is to develop an extension of the Cox, Ross, Rubinstein (CRR) lattice binomial algorithm to handle the situation in which the payoff from the option depends on more than one state variable. Specifically, we consider the case in which the option is a function of two underlying state variables although it is possible that the procedure can be extended to situations involving a higher number of state variables. As a prelude to the development of a lattice approach in the case of two state variables, it proved convenient to modify the CRR approach in the case of one state variable. This modification consisted of replacing the two-point jump process used by CRR with a three-jump process. When this was done, we obtained a more efficient algorithm for the valuation of options when there is just one state variable; a result that may be of some interest in its own right.

Section II of the paper describes the modification of the CRR approach in the case of a single state variable. Section III develops a lattice type approach for the valuation of options whose payoff depends on two state variables. It turns out that a five-point jump process enables us to develop an appropriate lattice in three dimensions (two asset dimensions and one time dimension). We discuss the determination of the jump probabilities and jump amplitudes and are able to draw on some of the results of Section II in the development.

Numerical results are given in Section IV. These include cases in which exact solutions (available from Stulz (1982)), are used to illustrate the accuracy of the procedure. We also derive values for American put options on the minimum of two assets and illustrate the value of the early exercise provision.

The final section contains some concluding comments.

II. Modification of the Two State Approach for a Single State Variable

In this section, we develop a modification of the CRR lattice binomial approach for option valuation in the case of a single state variable. The modification introduced will be helpful in generalizing the approach to two state variables. One of the features of the CRR approach is that by assuming that the stock price can move only either upward or downward, the relationship between the hedging argument and the mathematical development is especially clear. Their approach provides a powerful demonstration that options can be priced by discounting their expected value in a risk-neutral world. However, if we know the distributional assumptions and are assured that a risk-neutral valuation procedure is appropriate, then other types of discrete approximation can be used. We can regard the option valuation problem as a problem in numerical analysis and replace the continuous distribution of stock prices by a suitable discrete process as long as the discrete distribution tends to the appropriate limit.[1] In this section, we use a three-jump process instead of the two-jump process used in the CRR method.

Three-jump models have been used before in the literature to analyze option valuation problems. Stapleton and Subrahmanyan (1984) discussed a three-jump model, but they did not examine its numerical efficiency or solve for the jump probabilities. Parkinson (1977) employed a three-jump model to value the American put, but his approach seems difficult to generalize to situations involving more than one state variable. Brennan and Schwartz (1978) related the coefficients of a transformed Black-Scholes equation to the probabilities of a three-jump process.

To proceed with our development, let us consider an asset (S) with a lognormal distribution of returns. Over a small time interval this distribution is approximated by a three-point jump process in such a way that the expected return on the asset is the riskless rate, and the variance of the discrete distribution is equal to the variance of the corresponding lognormal distribution. It is convenient to employ the following notation:

T = time to option maturity (in years),
X = exercise price of option,
r = the continuously compounded yearly interest rate,
σ^2 = the variance of the rate of return on the underlying asset (yearly),
n = the number of time steps into which the interval of length T is divided,
h = T/n: length of one time step,
Su = asset value after an up jump,
S = asset value after a horizontal jump,
Sd = asset value after a down jump.

[1] Proofs that the approximating discrete distribution tends to the appropriate continuous distribution for both the one-state-variable case and the two-state-variable case are available from the author. The author thanks the referee for affirming the importance of this convergence.

For the three-jump process considered in this section, the continuous-return distribution is approximated by the following discrete distribution:

Nature of Jump	Probability	Asset Price
Up	p_1	Su
Horizontal	p_2	S
Down	p_3	Sd

It is convenient to impose the condition that $ud = 1$. Our task is to find suitable values for the probabilities p_1, p_2, and p_3 and the stretch parameter u in terms of the known variables. This can be achieved by first imposing three conditions to solve for the probabilities in terms of u and the other variables. Then we show how a suitable value of u can be obtained.

The three conditions are:

(i) the probabilities sum to one,

(ii) the mean of the discrete distribution : SM, is equal to the mean of the continuous lognormal distribution

$$S\exp(rh) = SM, \text{ and}$$

(iii) the variance of the discrete distribution : S^2V, is equal to the variance of the continuous distribution

$$S^2M^2\left[\exp\left(\sigma^2 h\right) - 1\right] = S^2V.$$

Later on, when we specify the value of the stretch parameter u, we insist that each of the probabilities must be positive. The above three conditions are,

(1) $$p_1 + p_2 + p_3 = 1 ,$$

(2) $$p_1 Su + p_2 S + p_3 \frac{S}{u} = SM ,$$

(3) $$p_1\left(S^2u^2 - S^2M^2\right) + p_2\left(S^2 - S^2M^2\right) + p_3\left(\frac{S^2}{u^2} - S^2M^2\right) = S^2V.$$

It is convenient to divide the second equation by S and the third by S^2. Then the first equation can be used to remove p_2 from the last two equations. This yields

(4) $$p_1(u-1) + p_3\left(\frac{1}{u} - 1\right) = M - 1 ,$$

(5) $$p_1\left(u^2 - 1\right) + p_3\left(\frac{1}{u^2} - 1\right) = V + M^2 - 1 .$$

These equations can be solved to give explicit expressions for p_1 and p_2 as follows,

$$p_1 = \frac{\left(V + M^2 - M\right)u - (M-1)}{(u-1)\left(u^2-1\right)}, \tag{6}$$

$$p_3 = \frac{u^2\left(V + M^2 - M\right) - u^3(M-1)}{(u-1)\left(u^2-1\right)}, \tag{7}$$

$$p_2 = 1 - p_1 - p_3. \tag{8}$$

It is convenient to use the following notation,

$$p_1 = f(u,M,V), \tag{9}$$

$$p_3 = g(u,M,V). \tag{10}$$

Note that these equations imply that u is not equal to 1. This is reasonable since u corresponds to an up jump, and it will exceed 1.

So far, u has not been specified. In general, there will be a range of values of u that produce reasonable values for the probabilities. Recall that the CRR paper used

$$u = \exp\left[\sigma\sqrt{h}\right]. \tag{11}$$

If we use this value of u, then for many realistic parameter values p_2 will turn out to be negative. For example, suppose $\sigma = 0.20$, $r = 0.1$, $T = 1.0$, and $n = 20$. With these parameter values, Equation (11) implies that $u = 1.045736$, and the probabilities given by (6), (7), and (8) become $p_1 = 0.553859$, $p_2 = -0.018440$, and $p_3 = 0.464581$. The value of u given by Equation (11) is too low to produce a positive value for p_2 given the parameters of this example.[2] To rectify this, define u as

$$u = \exp\left[\lambda\sigma\sqrt{h}\right], \tag{12}$$

where λ is greater than 1. By using different values for λ, a range of values of u is obtained, and there is an interval within this range that produces acceptable values for all the probabilities and p_2 in particular. Table 1 displays the values of u and the corresponding probabilities for a range of values of λ.

In order to determine the most effective value of λ to use, a number of options were evaluated over a range of parameter values. Best results were obtained when the probabilities were roughly equal. Although the primary aim in

[2] Recall that for the CRR two-jump approximation, the variance of the discrete distribution is biased, whereas this is not the case for the three-jump process considered here. The stretch factor "u" will not be the same for both approximations to give meaningful results.

TABLE 1

Jump Amplitudes and Jump Probabilities
$\sigma = 0.2, r = 0.1, T = 1.0, n = 20$

λ	u	p_1	p_3	p_2
1.00	1.0457	0.5539	0.4646	−0.0184
1.10	1.0504	0.4610	0.3798	0.1592
1.20	1.0551	0.3900	0.3156	0.2943
1.30	1.0599	0.3346	0.2659	0.3995
1.40	1.0646	0.2904	0.2266	0.4829
1.50	1.0694	0.2547	0.1951	0.5502
1.60	1.0742	0.2253	0.1694	0.6053
1.70	1.0790	0.2008	0.1482	0.6510
1.80	1.0838	0.1802	0.1305	0.6892
1.90	1.0887	0.1627	0.1156	0.7216
2.00	1.0936	0.1477	0.1030	0.7493

developing the three-jump lattice approach to option valuation in the case of one state variable was to pave the way for extensions to two or more underlying assets, it may be of interest to summarize the results obtained by this algorithm with those of the two-jump approach used by CRR. For a range of parameter values, we found that the accuracy of the three-jump method with 5 time intervals was comparable to that of the CRR method with 20 time intervals.[3]

III. A Lattice Model for Valuation of Options on Two Underlying Assets

In this section, we develop a valuation algorithm for the pricing of options when there are two underlying assets. The method used to compute the jump probabilities within the lattice is outlined, and a procedure for selecting the jump amplitudes is explained. The essence of the idea is to derive a generalization of the method developed in the previous section.

It is assumed that the joint density of the two underlying assets is a bivariate lognormal distribution. When we implement the risk-neutral valuation procedure, it is implied that both assets earn the riskless rate. To specify the distribution, we need the variance-covariance matrix as well. The approximating discrete distribution has the same expected values and variance-covariance values as the discrete distribution. This means that one needs at least five degrees of freedom in constructing the discrete approximating distribution. In addition, we need to ensure that the jump probabilities add up to one, which gives another condition to be satisfied. A five-point jump process can be constructed to satisfy the various requirements and can be used to generate a two × one-dimensional lattice suitable for valuing the options we are considering.

It is convenient to rely on the basic notation used in the previous section. We use the subscript 1 to denote the first asset and the subscript 2 to denote the second asset:

[3] For space reasons, summary comparison tables are not included in this paper, but are available from the author on request.

	Asset 1	Asset 2
Current value	S_1	S_2
Variance rate	σ_1^2	σ_2^2

We assume that the option matures after time T and that its exercise price is X. We also asume that the time interval $[0, T]$ is divided into n equal steps each of length h. As before, the continuously compounded riskless rate of interest is r per annum. Suppose we consider a time interval of length h. The means and variances of the two assets at the end of this interval are obtained from the properties of the lognormal distribution and are:

	Asset 1	Asset 2
Mean	S_1M_1	S_2M_2
Variance	$S_1^2V_1$	$S_2^2V_2$

where

$$M_i = \exp[rh] , \tag{13}$$

$$V_i = M_i^2\left[\exp\left(\sigma_i^2 h\right) - 1\right] , \quad i = 1,2 . \tag{14}$$

The expected value of the product of the two variables S_1 and S_2 can be written down using the properties of the joint lognormal distribution

$$\begin{aligned} E(S_1S_2) &= S_1S_2M_1M_2\left[\exp\left(\rho\sigma_1\sigma_2 h\right)\right] \\ &= S_1S_2R . \end{aligned} \tag{15}$$

From this last equation, we can derive the covariance of the two assets. Thus, we have the means and variance-covariance matrix of the bivariate lognormal distribution of asset values at the end of the time internal h. To replace this distribution with a discrete distribution, we ensure that the first and second moments (including the covariance term) of the approximating discrete distribution are equal to those of the continuous distribution.

In order to obtain an efficient algorithm, we found that a five-point jump process was the most suitable. Given that the pair of assets has current value (S_1, S_2), there are five distinct outcomes for the proposed jump process. We can summarize the outcomes of the jump process as follows:

Event	Probability of Event	Value of Asset given that Event has Occurred	
		Asset 1	Asset 2
E_1	p_1	S_1u_1	S_2u_2
E_2	p_2	S_1u_1	S_2d_2
E_3	p_3	S_1d_1	S_2d_2
E_4	p_4	S_1d_1	S_2u_2
E_5	p_5	S_1	S_2

Thus, in analogy to the earlier case, u_1 represents the stretch factor for asset one, and u_2 represents the stretch factor for asset two. We impose the conditions

$$u_1d_1 = 1 , \qquad u_2d_2 = 1 .$$

Assume the process starts when the asset values are S_1 and S_2. We envisage a three-dimensional representation with time along the vertical axis. The asset price pair can take one of five values after time h depending on which of E_1, E_2, E_3, E_4, or E_5 has occurred. These five points correspond to points in S_1–S_2 space lying in a plane directly above the point of origin. The first four points lie at the corners of the rectangle and the final point lies in the center of the rectangle vertically above the point of origin. As this process is repeated, we generate a three-dimensional lattice structure resembling an inverted pyramid. The terminal distribution of the pair of assets is represented by the topmost of a layered series of rectangles in which the center of each rectangle lies vertically above the point of origin.

To evaluate the probabilities and the stretch parameters, it is convenient to proceed as we did in the previous section: first, find the probabilities in terms of u_1, u_2, and the other variables. By equating the mean of the discrete distribution to the mean of the continuous distribution, we obtain two equations

$$(16) \qquad (p_1+p_2)S_1u_1 + p_5S_1 + (p_3+p_4)S_1d_1 = S_1M_1 ,$$

$$(17) \qquad (p_1+p_4)S_2u_2 + p_5S_2 + (p_2+p_3)S_2d_2 = S_2M_2 .$$

Two more equations result from equating the variances

$$(18) \qquad \begin{aligned}&(p_1+p_2)(S_1^2u_1^2 - S_1^2M_1^2) + p_5(S_1^2 - S_1^2M_1^2)\\ &+(p_3+p_4)(S_1^2d_1^2 - S_1^2M_1^2) = S_1^2V_1 ,\end{aligned}$$

$$(19) \qquad \begin{aligned}&(p_1+p_4)(S_2^2u_2^2 - S_2^2M_2^2) + p_5(S_2^2 - S_2^2M_2^2)\\ &+(p_2+p_3)(S_2^2d_2^2 - S_2^2M_2^2) = S_2^2V_2 .\end{aligned}$$

Since the probabilities sum to 1 we also have

$$(20) \qquad p_1 + p_2 + p_3 + p_4 + p_5 = 1 .$$

Now we can use the results of the previous section to simplify the analysis. Note that there is a strong correspondence between the three equations (1), (2), and (3) and the triplet (20), (16), and (18) and also with the triplet (20), (17), and (19). In fact, if we regard (p_1+p_2), (p_3+p_4), and p_5 as new probabilities, we can solve for these new probabilities. The explicit expressions are given by Equations (6), (7), and (8) with appropriate modifications. In the same way, we can solve Equations (20), (17), and (19) for the "new" probabilities (p_1+p_4), (p_2+p_3), and p_5. Notice that this solution procedure corresponds to the projec-

tion of the two-variate distribution onto the unconditional distribution of a single variate. Using this procedure, we obtain the following results

$$p_1 + p_2 = f\left(u_1, M_1, V_1\right) = f_1 , \tag{21}$$

$$p_3 + p_4 = g\left(u_1, M_1, V_1\right) = g_1 , \tag{22}$$

$$p_1 + p_4 = f\left(u_2, M_2, V_2\right) = f_2 , \tag{23}$$

$$p_2 + p_3 = g\left(u_2, M_2, V_2\right) = g_2 . \tag{24}$$

In view of Equation (20), we have two distinct expressions for p_5 so that the following consistency equation needs to be satisified

$$f_1 + g_1 = f_2 + g_2 . \tag{25}$$

This gives a relationship between u_1 and u_2 that must be satisfied.

The final equation is obtained from relating the expected value of the product of the two assets under the continuous distribution and the approximating discrete distribution. Using Equation (15), this gives

$$\left(p_1 u_1 u_2 + p_2 u_1 d_2 + p_3 d_1 d_2 + p_4 d_1 u_2 + p_5\right) S_1 S_2 = R S_1 S_2 . \tag{26}$$

Dividing across by $S_1 S_2$ and eliminating p_5 yields

$$p_1\left(u_1 u_2 - 1\right) + p_2\left(u_1 d_2 - 1\right) + p_3\left(d_1 d_2 - 1\right) + p_4\left(d_1 u_2 - 1\right) = R - 1 .$$

If we substitute for d_1 and d_2 and use Equations (21), (22), (23), and (24), we can use this last equation to obtain the following expression for p_1

$$p_1 = \frac{u_1 u_2 (R-1) - f_1\left(u_1^2 - 1\right) - f_2\left(u_2^2 - 1\right) + \left(f_2 + g_2\right)\left(u_1 u_2 - 1\right)}{\left(u_1^2 - 1\right)\left(u_2^2 - 1\right)} . \tag{27}$$

Armed with this expression for p_1, we can find p_2, p_3, and p_4 using Equations (21), (22), (23), and (24). The results are

$$p_2 = \frac{f_1\left(u_1^2 - 1\right)u_2^2 + f_2\left(u_2^2 - 1\right) - \left(f_2 + g_2\right)\left(u_1 u_2 - 1\right) - u_1 u_2 (R-1)}{\left(u_1^2 - 1\right)\left(u_2^2 - 1\right)} , \tag{28}$$

$$p_3 = \frac{u_1 u_2 (R-1) - f_1\left(u_1^2 - 1\right)u_2^2 + g_2\left(u_2^2 - 1\right)u_1^2 + \left(f_2 + g_2\right)\left(u_1 u_2 - u_2^2\right)}{\left(u_1^2 - 1\right)\left(u_2^2 - 1\right)} , \tag{29}$$

$$p_4 = \frac{f_1\left(u_1^2 - 1\right) + f_2\left(u_2^2 - 1\right)u_1^2 - \left(f_2 + g_2\right)\left(u_1 u_2 - 1\right) - u_1 u_2 (R-1)}{\left(u_1^2 - 1\right)\left(u_2^2 - 1\right)} . \tag{30}$$

To illustrate the computation of these probabilities in a specific case, let us assume the following parameters: $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $r = 0.1$, $n = 20$, $T = 1.0$, and $\rho = 0.5$. Let

$$u_1 = \exp\left[(1.1)\sigma_1\sqrt{h}\right]$$

so that for these parameter values

$$u_1 = 1.05042358 .$$

In order to ensure consistency, Equation (25) must be satisfied. Equation (25) gives a relationship between u_1 and u_2. In order to solve for u_2, we use Newton's method and take the initial value of u_2 to be:

$$\begin{aligned} u_2 &= \exp\left[(1.1)\sigma_2\sqrt{h}\right] \\ &= 1.06342185. \end{aligned}$$

It turns out that the value of u_2 that satisfies Equation (25) is close to this starting value. When $u_2 = 1.0632918$, Equation (25) is satisfied. Armed with the values of u_1 and u_2, we can proceed to compute the probabilities. These are given for a range of values of λ in Table 2. Probabilities for the particular (u_1, u_2) combination just discussed are displayed in the second line of Table 2. Note from Table 2 that the sum of the probabilities p_1 and p_2 corresponds to the probability of an upward jump in the case of the single asset given by p_1 in Table 1. There is a similar relationship between the sum $(p_3 + p_4)$ from Table 2 and the probability of a down jump in Table 1. This illustrates the relationship between the bivariate distribution for S_1 and S_2 and the univariate unconditional distribution for S_1 and the univariate unconditional distribution for S_2.

TABLE 2

Jump Amplitudes and Jump Probabilities for 5-Jump Process
$\sigma_1 = 0.2, \sigma_2 = 0.25, r = 0.1, n = 20, T = 1, \rho = 0.5$

λ	u_1	u_2	p_1	p_2	p_3	p_4	p_5
1.00	1.0457	1.0574	0.4201	0.1337	0.3448	0.1198	−0.0184
1.10	1.0504	1.0633	0.3499	0.1111	0.2814	0.0984	0.1592
1.20	1.0551	1.0692	0.2962	0.0938	0.2334	0.0822	0.2943
1.30	1.0599	1.0752	0.2543	0.0803	0.1963	0.0696	0.3995
1.40	1.0646	1.0812	0.2208	0.0696	0.1670	0.0597	0.4829
1.50	1.0694	1.0873	0.1937	0.0609	0.1434	0.0517	0.5502
1.60	1.0742	1.0934	0.1715	0.0538	0.1243	0.0451	0.6053
1.70	1.0790	1.0995	0.1529	0.0479	0.1085	0.0397	0.6510
1.80	1.0838	1.1056	0.1373	0.0429	0.0953	0.0352	0.6892
1.90	1.0887	1.1118	0.1240	0.0387	0.0842	0.0314	0.7216
2.00	1.0936	1.1180	0.1126	0.0351	0.0748	0.0282	0.7493

IV. Numerical Examples

In this section, we present numerical examples to illustrate the operation of the method developed in the previous section. Both Johnson (1981) and Stulz (1982) have derived a number of exact results for European options when there

are two underlying assets. These results can be used to check the accuracy of the approach. In particular, expressions exist for a European call option on the maximum of two assets and for a European put option on the minimum of two assets. Johnson (1985) has obtained a generalization of these results for the case involving a European call option on the maximum or minimum of several assets. His results could be used as benchmarks to check extensions of our method to situations involving three or more assets.

The advantage of the present approach is that it permits early exercise and thus can be used to value American options and, in particular, American put options. Our approach can also be modified to handle dividends and other payouts.

For our numerical solutions, we assume $S_1 = 40$, $S_2 = 40$, $\sigma_1 = 0.20$, $\sigma_2 = 0.30$, $\rho = 0.5$, $r = 5$-percent per annum effective $= 0.048790$ continuously, $T = 7$ months $= 0.5833333$ years, and exercise prices $= 35, 40, 45$. We display in Table 3 the results for three types of options: European call options on the maximum of the two assets; European put options on the minimum of the two assets; and American put options on the minimum of two assets. Since we are assuming no dividend payments, the value of the European call on the maximum of the two assets will be (identical to) the value of an American call option with the same specifications. The agreement between the numbers obtained using the lattice approximation and the accurate values is quite good, especially when 50 time steps are used. In this case, the maximum difference is 0.005 and the accuracy would be adequate for most applications. For this set of parameter values, the American put is not much more valuable than its European counterpart.

TABLE 3

Different Types of Options Involving Two Assets:
Comparison of Lattice Approach of this Paper with Accurate Results

	Number of Steps			
Exercise Price	10	20	50	Accurate Value
	European Call on the Maximum of the Two Assets			
35	9.404	9.414	9.419	9.420
40	5.466	5.477	5.483	5.488
45	2.817	2.790	2.792	2.795
	European Put on the Minimum of the Two Assets			
35	1.425	1.394	1.392	1.387
40	3.778	3.790	3.795	3.798
45	7.475	7.493	7.499	7.500
	American Put on the Minimum of the Two Assets			
35	1.450	1.423	1.423	
40	3.870	3.885	3.892	
45	7.645	7.674	7.689	

V. Concluding Comments

We have seen how the lattice approach to option valuation that has been developed for options on a single state variable can be extended to handle situa-

tions in which there are two state variables. This means that American options in which there is a possibility of early exercise can be handled using this technique. It is suggested that approximation procedures of this type may be useful in the valuation of complex securities involving options that depend on two state variables. Thus, for example, the procedure developed here could be used in the pricing of secured debt when the underlying firm consists of two risky assets with a joint lognormal distribution. Recently, Stulz and Johnson (1985) have examined this problem and used a numerical approach to solve the resulting partial differential equation. Another area in which this new valuation technique might be useful is in the valuation of options with stochastic volatilities because, in this case, there are two stochastic state variables. Recently, a number of papers have appeared on the valuation of such options (c.f., Wiggins (1985), Hull and White (1985), Johnson and Shanno (1985)). To date, the techniques used to derive option values have involved the direct solution of the partial differential equation (Wiggins (1985)) or Monte-Carlo approaches (Johnson and Shanno (1985) and Hull and White (1985)). The method suggested in this paper may provide another approach.

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