THE JOURNAL OF FINANCE • VOL. XXXIV, NO. 5 • DECEMBER 1979

Path Dependent Options: "Buy at the Low, Sell at the High"

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I. Introduction

IT IS WELL KNOWN that the valuation of European puts and calls with fixed exercise prices is solely dependent on the distribution of the terminal price of the underlying stock. This paper examines the properties of European options with exercise prices that are functions of the realized sample path of the stock. In particular, the commonplace shareholder desire to "buy at the low" and sell at the high" can be satisfied with a combination of a call on the stock with an exercise price equal to $\min_{0\leq\tau\leq T} S(\tau)$ and a put with exercise price equal to $\max_{0\leq\tau\leq T} S(\tau)$ where S is the stock price and T is the term of the option.[1]

Strictly speaking, the creation of these new options in a *frictionless* context would not expand the investor's opportunity set. However, in a *realistic* market setting such new options might very well acquire substantial popularity. The appeal would be threefold: (1) the options would guarantee the investor's fantasy of buying at the low and selling at the high, (2) the options would, in some loose intuitive sense, minimize regret, and (3) the options would allow investors with special information on the range (but possibly without special information on the terminal stock price) to directly take advantage of such information.

In this paper we analyze the hedging, pricing, and economic properties of these options. Wherever possible we compare and contrast these options with their traditional counterparts. In section two we establish that these options can be hedged and that closed-form valuation equations exist. Particular emphasis here centers on the hedgeability of these options when the stock is at an extremum (i.e., equal to its current maximum or minimum). In section three, by analysis and simulation, we establish the properties of these options and contrast them with those of their traditional counterparts. In particular we examine: (1) the functional dependence of these options with respect to two state variables—stock price and time to expiration, and (2) the pricing of these options relative to the stock and traditional options at the time of inception. Due to the ungainliness of the general pricing relations developed in section two, we found it convenient throughout section three to provide detailed analyses and explicit derivations of the properties of these options for the particularly intuitive case where the logarithm of the adjusted geometric mean return of the stock is zero. We then illustrate by simulation that, qualitatively, the results of our specific example carry over to the general case. We conclude in section four with a general discussion of path-dependent options.

[1] Another example of a path-dependent option that has been examined in the pricing of an American put. See Parkinson [7] for details.

Throughout the paper we will adhere to the following notation

	Timing conventions:
O:	All options are assumed to be written at time zero,
T:	The expiration date of all options,
τ:	The current date,
t:	$T - \tau$, the time to expiration.
	Remaining notation:
$S(\tau)$:	Stock price at time τ, (occasionally abbreviated to S),
X:	Exercise price,
r:	Risk free rate of interest,
σ^2:	Variance per unit time of log of stock price return,[2]
$C[S(\tau), X, t]$:	Value of an ordinary European call,
$P[S(\tau), X, t]$:	Value of an ordinary European put,
$M(\tau)$:	$\max_{0 \le \delta \le \tau} S(\delta)$, (occasionally abbreviated to M),
$Q(\tau)$:	$\min_{0 \le \delta \le \tau} S(\delta)$, (occasionally abbreviated to Q),
$C_{\min}[S(\tau), Q(\tau), t]$:	Value of a European option to buy the stock at its realized minimum, when the current stock price is $S(\tau)$, the realized minimum to date is $Q(\tau)$, and the time remaining on the option is t,
$P_{\max}[S(\tau), M(\tau), t]$:	Value of a European option to sell the stock at its realized maximum, when the current stock price is $S(\tau)$, the realized maximum to date is $M(\tau)$, and the time remaining on the option is t,
$N\{\cdot\}, N'\{\cdot\}$:	Standard normal cumulative distribution and density functions,
d_1:	$\dfrac{[\ln(S/X) + (r + \sigma^2/2)t]}{\sigma\sqrt{t}}$,
d_2:	$\dfrac{[\ln(S/X) + (r - \sigma^2/2)t]}{\sigma\sqrt{t}}$,
E:	Expectation operator,
dy:	Limit of a Wiener process, and
α:	Drift term of the rate of the return on the stock.

II. Hedging and Valuation

In order to hedge a position, the writer of any option must find a way to invest the proceeds from the sale of the option in an initial portfolio and to then alter the composition of this portfolio as is required to guarantee that in all states of nature (i.e., with probability one) the terminal value of the portfolio is adequate to meet his terminal obligation. This portfolio strategy is termed a *perfect hedge* and has the following properties: (1) the value of the terminal portfolio is *exactly* equal to the terminal obligation, and (2) the hedging policy is *self-financing*-each portfolio revision undertaken is exactly financed by the proceeds from the sale of the previous position.

[2] By "return" we denote $(S_{t+\Delta t}/S_t)$; The proportional gain or loss, $(S_{t+\Delta t} - S_t)/S_t$, we call "rate of return".

Recent work by Black and Scholes [1] and Merton [6] has established that in a perfect (i.e., frictionless) market, when the natural logarithm of the underlying stock price follows a Wiener process with drift (i.e., $dS/S = \alpha \cdot dt + \sigma \cdot dy$) so that $d(\ln S) = (\alpha - \sigma^2/2)\, dt + \sigma \cdot dy$, the payoff of a European put or call with a fixed exercise price can be identically duplicated by a portfolio consisting of shares of stock and units of riskless bond. Thus, this portfolio meets the criteria established for a perfect hedge. Cox and Ross [3] have illustrated and Harrison and Kreps [5] have proved that if a contingent claim can be perfectly hedge it can be priced as if it existed in a risk-neutral world. This result implies that the instantaneous rate of return (α) of the underlying stock is of no consequence to the pricing of the option and in fact it may be assumed equal to the riskless rate (r). The drift of the logarithm of the stock price becomes effectively $(r - \sigma^2/2)$ per unit time, and after making this substitution, the option will be priced at its discounted expected terminal value.

Together these authors have illustrated that in equilibrium, to prevent riskless arbitrage, ordinary European puts and calls will have the following closed-form valuation equations:

$$C[S, X, t] = S \cdot N(d_1) - e^{-rt} X \cdot N(d_2) \tag{1}$$

$$P[S, X, t] = C[S, X, t] - S + X \cdot e^{-rt} \tag{2}$$

In this section we illustrate that puts on the maximum ($P_{\max}$) and calls on the minimum ($C_{\min}$) can be perfectly hedged and that closed-form valuation equations can be derived. The major insight in this section concerns the hedgeability of these options when the stock is at an extremum (i.e., at its current maximum or minimum). We begin this section by deriving the hedging portfolio when $r = \sigma^2/2$ (i.e., the logarithm of the underlying stock has zero effective drift). Here a hedging portfolio for writers of these options may be deduced as is illustrated in the following theorem.

THEOREM 1: *When* $r = \sigma^2/2$,

$$P_{\max}[S(\tau), M(\tau), t] = P[S(\tau), M(\tau), t] + C[S(\tau), M(\tau), t] \tag{3}$$

and

$$C_{\min}[S(\tau), Q(\tau), t] = P[S(\tau), Q(\tau), t] + C[S(\tau), Q(\tau), t]. \tag{4}$$

Proof: For the proof of this result we refer the reader to the derivation of the general pricing relations as summarized in relations (10) and (11) (i.e., letting $r = \sigma^2/2$ and using the definitions of puts and call affirms the equivalence of $P_{\max}$ and $C_{\min}$ to straddles).

Note that at $\tau = 0$, $S(0) = M(0) = Q(0)$. Thus, the theorem implies that an initial hedging portfolio for $P_{\max}$ (for $C_{\min}$) is an ordinary put plus an ordinary call (i.e., a straddle) both with an exercise price equal to the initial stock price and term to maturity equal to the term to maturity of $P_{\max}(C_{\min})$. By examination, theorem 1 also implies that as long as the stock never rises above (falls below) its initial value, the composition of the hedging portfolio is unaltered. Clearly, if, over the life of the option, the stock price never rises above (falls below) its initial value, the initial straddle would exactly satisfy the writer's terminal obligation. For $P_{\max}$, $M(T) = S(0)$ (for $C_{\min}$, $Q(T) = S(0)$), the call (put) in the straddle would be valueless but the put (call) in the straddle would be just what was required to meet the terminal obligation. When the stock price is equal to its old maximum (minimum) (i.e., the stock is at an extremum) and then achieves a new maximum (minimum), theorem 1 implies that the old portfolio—the straddle with exercise price equal to the old maximum (minimum)—should be sold and a new porfolio established—a straddle with exercise price equal to the new maximum (minimum). Theorem 2 establishes that this is a self-financing portfolio strategy.

THEOREM 2: *If $r = \sigma^2$ and $X = S$ then*

$$P[S + ds, S, t] + C[S + ds, S, t] = P[S + ds, S + ds, t] + C[S + ds, S + ds, t] \quad (5)$$

and hence this is a self-financing portfolio strategy.

Proof: Taking a Taylor Series Expansion of the LHS of relation (5) around $X = S$ (holding the stock price and the term to expiration constant) yields:

$$P[S + ds, X, t] + C[S + ds, X, t] = P[S + ds, X + ds, t] + C[S + ds, X + ds, t] - \left(\frac{\partial P[S + ds, X + ds, t]}{\partial X} + \frac{\partial C[S + ds, X + ds, t]}{\partial X}\right) ds + o(ds)$$

where $o(ds) \equiv$ terms of higher order than ds. These may be ignored since the derivations of $P[\cdot]$ and $C[\cdot]$ are bounded. Using the Black-Scholes formulae (relations (1) and (2)) to evaluate the terms in $(\cdot)$ yields

$$-(-2e^{-rt}N\{d_2\} + e^{-rt}) = 0$$

since, $N\{d_2\} = ½$ when $r = \sigma^2/2$ and $S = X$. Q.E.D.

Another way to state the result of theorem 2 is that when $r = \sigma^2/2$ and

$$S(\tau) = M(\tau)(\text{or } S(\tau) = Q(\tau)), \frac{\partial P_{max}}{\partial M} = 0 \left(\frac{\partial C_{min}}{\partial Q} = 0\right).$$

Thus, in contrast to ordinary puts (calls) where increases (decreases) in the exercise price always increase the value of the option, for these special options, when the stock is at its current maximum (minimum) infinitesimal changes in the maximum (minimum) are valueless.

Hedging: The General Case

Theorem 3 below illustrates that the results of theorem 2 concerning changes in the value of P_{max} (or C_{min})caused by changes in the maximum (minimum) when the stock price is equal to its current maximum (minimum) hold in general. However, before presenting the proof, we will first motivate the result. To this end it is useful to present a simplifying concept.

The Joint Distribution of M(T) and S(T): A Red Herring

If these options are hedgeable, then after making the Cox-Ross transformation (i.e., substituting the risk-free rate of interest for the logarithm of the stock's expected return and assuming that these securities are priced in a risk neutral world) we know that P_{max} will be priced equal to the probability weighted, conditional (over non-negative payouts) expected value of the realized maximum over the life of the option minus the terminal stock price, discounted back to the present; C_{min} will be priced equal to the probability weighted, conditional expected value of the terminal stock price minus the realized minimum, discounted back to the present:

$$P_{max}[S(\tau), M(\tau), t] = e^{-rt}E\,|_{m \geq S}[M(T) - S(T)]Prob(M \geq S) \tag{6}$$

$$C_{min}[S(\tau), Q(\tau), t] = e^{-rt}E\,|_{Q \leq S}[S(T) - Q(T)]Prob(Q \leq S) \tag{7}$$

where E is a conditional expectation operator.

A casual examination of (6) and (7) seems to indicate that knowledge of the conditional joint distribution of the maximum (minimum) and the terminal value of a Wiener process with drift (conditioned on the current price, the curent maximum (minimum) to date and the length of time remaining to expiration) is required. However, since these options are always exercised we know that $Prob(M \geq S) = 1$, and $Prob(Q \leq S) = 1$, hence we can use the distributive property of expectation. Since $e^{-rt}E[S(T)] = S(\tau)$, relations (6) and (7) may be rewritten as

$$P_{max}[S(\tau), M(\tau), t] = e^{-rt}E[M(T)] - S(\tau) \tag{6'}$$

$$C_{min}[S(\tau), Q(\tau), t] = S(\tau) - e^{-rt}E[Q(T)]. \tag{7'}$$

Thus, knowledge of the joint distribution is unnecessary; all that is required is knowledge of the conditional distribution of the maximum (minimum). The importance of this observation is that in order to value P_{max} it is sufficient to value a security that pays off the realized maximum (call it V_{max}) and to then subtract the current stock price; to value C_{min} it is sufficient to subtract from the current stock price the value of a security that pays off the realized minimum (call it V_{min}). Hence

$$P_{max}[S(\tau), M(\tau), t] = V_{max}[S(\tau), M(\tau), t] - S(\tau) \tag{8}$$

$$C_{min}[S(\tau), Q(\tau), t] = S(\tau) - V_{min}[S(\tau), Q(\tau), t]. \tag{9}$$

Further, in order to prove that P_{max} and C_{min} are hedgeable it suffices to prove that V_{max} and V_{min} are hedgeable since we know that the $S(T)$ can be hedged with $S(\tau)$.

It is intuitive that for $P_{max}(C_{min})$ when $S(\tau) < M(\tau)$, $(S(\tau) > Q(\tau))$ instantaneous changes in its value can only be effected by changes in the stock price and changes in time. That is, the assumed Wiener process for the stock insures that over the infinitesimal horizon the stock price will not rise above (fall below) its current maximum (minimum). Thus, when the stock price is not equal to its current maximum (minimum) potential changes in the maximum (minimum) can be ignored (i.e., $dM = 0$, or $dQ = 0$). However, when $S(\tau) = M(\tau)$, $(S(\tau) = Q(\tau))$ it might appear that infinitesimal changes in the value of the maximum (minimum) would cause changes in the value of $V_{max}(V_{min})$. This result, if it were true, has the disturbing implication that when the stock price is at an extremum there would appear to be two hedging ratios: one if the stock price rises, and one if the stock price falls. Two hedging ratios at a point in time would imply that a portfolio of stock and bonds would not span these options.

Theorem 3 establishes that when $S(\tau) = M(\tau)$, the distribution of $M(T)$, (i.e., the distribution of the maximum for the entire interval) is unaffected by marginal changes in the current maximum. Since V_{max} is only dependent upon the distribution of $M(T)$, this theorem establishes that $\left.\frac{\partial V_{max}}{\partial M(\tau)}\right|_{M(\tau)=S(\tau)} = 0$. Although not presented, it is also true that $\left.\frac{\partial V_{min}}{\partial Q(\tau)}\right|_{Q(\tau)=S(\tau)} = 0$.

THEOREM 3: *When $S(\tau) = M(\tau)$, holding S constant, all of the moments of $M(T)$ are unaffected by marginal changes in $M(\tau)$. Thus when $S(\tau) = M(\tau)$ the distribution of $M(T)$ and hence the value of P_{max} is unaffected by marginal changes in the maximum.*

Proof: See Appendix 2.

The intuition behind this result is that when $S(\tau) < M(\tau)$ there is always a positive probability that $M(\tau)$ will be the final maximum (i.e., $M(\tau) = M(T)$) and hence marginal increases in $M(\tau)$ will have value. However, in the limit as $S(\tau) \to M(\tau)$ if the stock follows a Wiener process then the probability that $M(\tau) = M(T)$ approaches zero. At $M(\tau) = S(\tau)$ infinitesimal increases in $M(\tau)$ will be of no value. Thus the value of marginal changes in $M(\tau)$ may be thought of as being proportional to the probability that $M(\tau)$ will be the terminal maximum.

The General Pricing Relations

Equations (10) and (11) present the pricing relations for V_{max} and V_{min} obtained by making the Cox-Ross transformation and then computing the discounted expected value of the terminal maximum and minimum. (Remember $P_{max} = V_{max} - S$, $C_{min} = S - V_{min}$).

$$V_{max}[S(\tau), M(\tau), t] = M(\tau)e^{-rt}\left[N\left\{\frac{a - \mu t}{\sigma\sqrt{t}}\right\} - \frac{\sigma^2}{2r}e^{2\mu a/\sigma^2}N\left\{\frac{-a - \mu t}{\sigma\sqrt{t}}\right\}\right]$$
$$+ S(\tau)\left[1 + \frac{\sigma^2}{2r}\right]\left[1 - N\left\{\frac{a - (\mu + \sigma^2)t}{\sigma\sqrt{t}}\right\}\right] \quad (10)$$

$$V_{min}[S(\tau), Q(\tau), t] = Q(\tau)e^{-rt}\left[N\left\{\frac{b + \mu t}{\sigma\sqrt{t}}\right\} - \frac{\sigma^2}{2r}e^{-2\mu b/\sigma^2}N\left\{\frac{-b + \mu t}{\sigma\sqrt{t}}\right\}\right]$$
$$+ S(\tau)\left[1 + \frac{\sigma^2}{2r}\right]N\left\{\frac{-b - (\mu + \sigma^2)t}{\sigma\sqrt{t}}\right\} \quad (11)$$

where:

$$a \equiv \ln[M(\tau)/S(\tau)] \geqslant 0,$$
$$b \equiv \ln[S(\tau)/Q(\tau)] \geqslant 0, \quad \text{and}$$
$$\mu \equiv r - \sigma^2/2.$$

Casual observation reveals that these relations meet their respective terminal conditions. That is,

$$V_{max}[S(T), M(T), 0] = M(T)$$

$$V_{max}[0, M(\tau), t] = M(\tau)e^{-rt}$$

and,

$$V_{min}[S(T), Q(T), 0] = Q(T)$$

$$V_{min}[0, 0, t] = 0.$$

To establish that relations (10) and (11) are self-financing it is necessary to examine the evolution of V_{max} and V_{min} through time. By hypothesis (i.e., since the logarithm of the underlying stock follows a Wiener process with drift) $V = V_{max}[S(\tau), M(\tau), t]$ is a smooth function of S and t. Dependence on $M(\tau)$ will be suppressed since theorem 3 established that $\left.\frac{\partial V_{max}}{\partial M}\right|_{M=S} = 0$ and since dM is zero otherwise[3]. Ito's Lemma implies that

$$dV = V_1 dS + V_3\, dt + ½\, V_{11}(dS)^2 \tag{12}$$

where:

$$(dS)^2 = -S^2\sigma^2\, dt,$$

$$dS = \alpha\cdot S\cdot dt + \sigma\cdot S\cdot dy, \quad \text{and}$$

$$\alpha \equiv \text{drift term.}$$

The only stochastic term (dS) may be hedged out by combining V with a short position in stock equal to V_1S dollars (where $V_1 = \partial V/\partial S$). The coefficient of dy will be zero hence the portfolio is riskless and must satisfy the following relation:

$$dV - V_1 dS = (V - V_1 S)r\cdot d\tau. \tag{13}$$

The LHS is the return on the option, short the stock. The term in parentheses on the RHS is the value of the portfolio remaining after shorting the stock. Noting that $d\tau = -dt$ and substituting (12) into (13) yields

$$V_3\, dt - ½\, V_{11}S^2\sigma^2\, dt = -(V - V_1 S)r\cdot dt.$$

Rearranging and canceling defines a differential equation for the value of V_{max}.

$$(V - V_1 S)r + V_3 - ½\, V_{11}S^2\sigma^2 = 0. \tag{14}$$

It is important to note that not only is relation (14) the differential equation for V_{max} but in addition it is the basic differential equation for *all* contingent claims written on a stock whose rates of return follow a Wiener process. In particular it is the differential equation for V_{min}, P_{max}, C_{min} and for ordinary puts and calls. Of course each option will have its own boundary conditions.

[3] This fact guarantees the arbitrage. However, (12) is exact only when $S \neq M$ and must be modified at the boundary $S = M$ since the second partials w.r.t. M do not disappear on the boundary. This is a mathematical curiosity and does not affect the solution technique. See Appendix 1 for details.

Careful differentiation of (10) and (11) reveals that they satisfy relation (14). Thus, relations (10) and (11) meet the criteria for a perfect hedge and will be the equilibrium valuation equations for V_{max} and V_{min}.

III. Properties of P_{max} and C_{min}

In this section we briefly explore some of the properties of these path-dependent options. In the first two subsections we describe these options as a function of the state variables S and t and compare the prices of these options with the prices of traditional options and the price of the underlying stock. As a benchmark, whenever illuminating, we briefly review the behavior of traditional puts and calls before proceeding to analyze the properties of P_{max} and C_{min}. In Table I, we contrast the inception values of these options for three alternative relations between r and T^2, and four alternative terms.

Table 1

Option Prices at Inception
($S=M=Q=X=100¢$, $r=0.06$)

$T = 0.2$ years,

	$r = 2\sigma^2$	$r = \sigma^2/2$	$r = \sigma^2/4$
Put	2.51¢	5.56	8.09
Call	3.70	6.75	9.28
P_{max}	5.72	12.31	18.02
C_{min}	6.62	12.31	16.82

$T = 0.5$ years

Put	3.48	8.20	12.12
Call	6.43	11.15	15.08
P_{max}	8.62	19.35	28.88
C_{min}	10.84	19.35	25.92

$T = 1.0$ years

Put	4.19	10.64	16.01
Call	10.01	16.46	21.84
P_{max}	11.51	27.10	41.33
C_{min}	15.88	27.10	35.50

$T = 5.0$ years

Put	4.14	15.11	24.47
Call	30.06	41.03	50.39
P_{max}	19.79	56.14	93.27
C_{min}	39.22	56.14	67.35

Options as Functions of the State Variables S and t: Properties of Traditional Puts and Calls

It is well known (see Merton [6]) that ordinary calls are convex-increasing functions of the stock price and increasing functions of the time to expiration. Calls have the following boundary conditions:

$$C[S, X, 0] = \max[S - X, 0], C[S, X, \infty] = S, \quad \text{and} \quad C[0, X, t] = 0.$$

Less often exhibited are the properties of ordinary European puts. Puts are convex-decreasing functions of the stock price. All "in-the-money" puts ($S < X$) are first decreasing functions of the time to expiration (i.e., at $t = 0$, $\partial P/\partial t\,|_{S>X} = -rX$). Puts sufficiently in-the-money are decreasing-convex functions of the time to expiration. Puts only slightly in-the-money first decrease, then increase and finally decrease as a function of the time of expiration. Out-of-the-money puts, ($S > X$) first increase (although at $t = 0$, $\left.\frac{\partial P}{\partial t}\right|_{S>X} = 0$) and then decrease as a function of t. The boundary conditions for a put are:

$$P[S, X, 0] = \max[0, X - S], P[0, X, t] = Xe^{-rt}, \quad \text{and} \quad P[S, X, \infty] = 0.$$

Graphs for ordinary puts and calls are presented in panel a of Figures 1–4. Since the case where $r = \sigma^2/2$ plays an important role in the analysis, and since the qualitative results are unaffected, we have chosen to plot ordinary puts and calls assuming this risk specification.

Properties of P_{max}

Panels *b–d* of Figure 1 present pictures of P_{max} as a function of S for three alternative relations between the riskless rate of interest and the variance of the stock. For the case where $r = \sigma^2/2$ (Panel *b*), theorem 1 has established that P_{max} is equivalent to a straddle with exercise price equal to the current maximum. The composition of this straddle is an in-the-money put plus an out-of-the-money call (i.e., the exercise price for both options is M and by construction $M > S$). The following theorem proves that for $r = \sigma^2/2$ and M normalized to unity, P_{max} as a function of S first behaves like an in-the-money put and then like an ordinary call. That is, P_{max} is convex in S-first decreasing and then increasing.

THEOREM 4: *When $r = \sigma^2/2$, P_{max} is convex in S for $t < \infty$. P_{max} as a function of S first decreases, reaches a unique minimum and then increases. Its unique minimum occurs at $-\ln(S/M) = \sigma^2 t$.*

Proof: see Appendix 2.

The intuition behind the behavior of P_{max} is that for $S(\tau)$ sufficiently below $M(\tau)$, not enough time (probabilistically) remains to establish a new maximum and to then establish a larger $M(T) - S(T)$. Thus increases in $S(\tau)$ lead to a smaller expected $M(T) - S(T)$. However, for $S(\tau)$ sufficiently close to $M(\tau)$ enough time remains to establish a new maximum and to then establish a larger expected $M(T) - S(T)$. In other words, at inception and at all other times when $M(\tau) = S(\tau)$ and throughout the life of the option whenever $M(\tau) - S(\tau)$ is

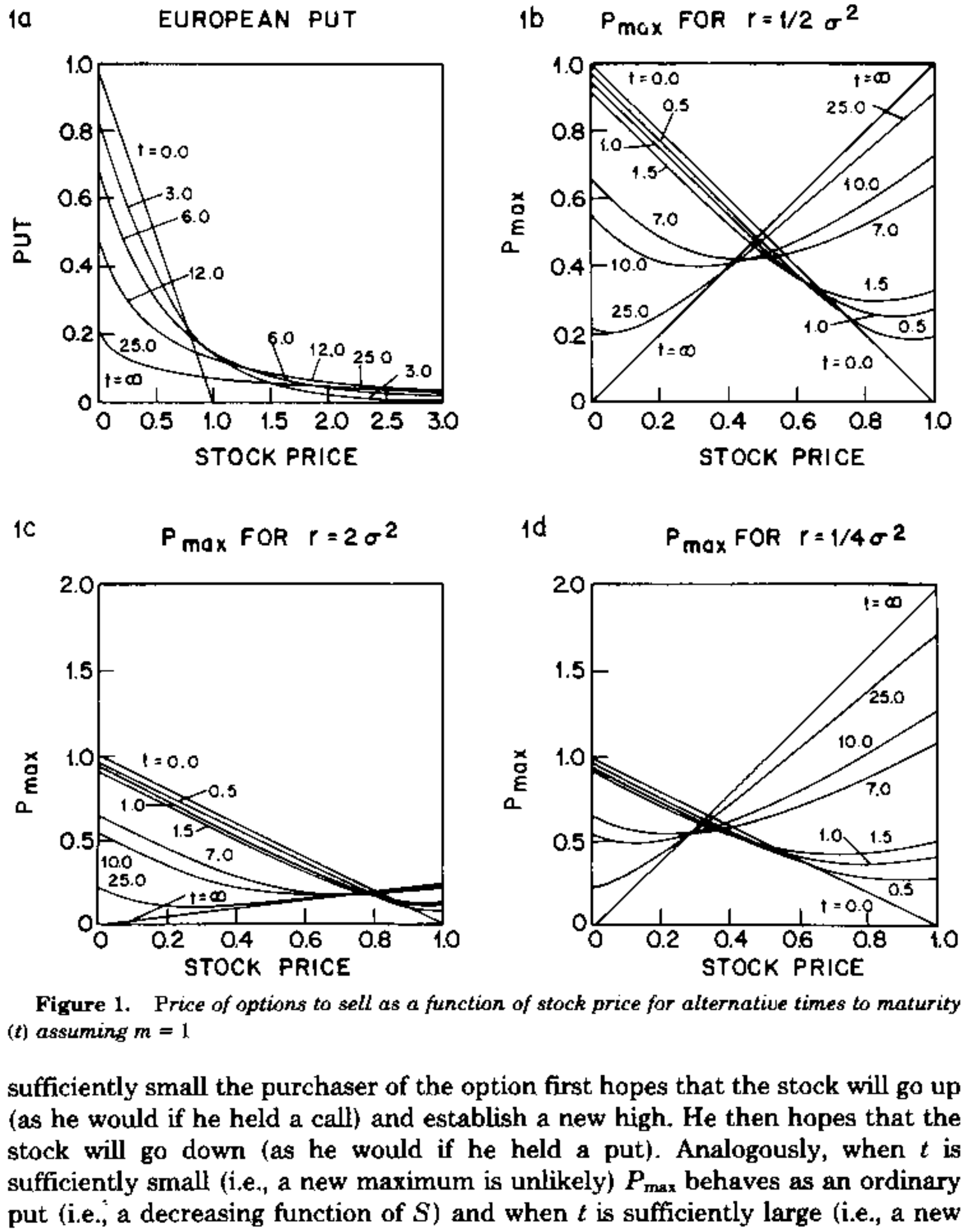


Figure 1. *Price of options to sell as a function of stock price for alternative times to maturity (t) assuming m = 1*

sufficiently small the purchaser of the option first hopes that the stock will go up (as he would if he held a call) and establish a new high. He then hopes that the stock will go down (as he would if he held a put). Analogously, when t is sufficiently small (i.e., a new maximum is unlikely) P_{max} behaves as an ordinary put (i.e., a decreasing function of S) and when t is sufficiently large (i.e., a new maximum is likely) P_{max} behaves as an ordinary call (i.e., an increasing function of S).

Qualitatively, the convexity of P_{max} for the case of $r \neq \sigma^2/2$ is exhibited in panels c and d of Figure 1. Analytical confirmation of convexity is straightforward. Using the notation of Theorem 3:

$$\frac{\partial^2 P_{max}}{\partial S^2} = e^{-rt}\left[\frac{\partial^2 L(1)}{\partial S^2}\right] - \frac{\partial^2 S}{\partial S^2} = e^{-rt}\left[\frac{M^2(\tau)}{S^3(\tau)}\right]\frac{dH(\hat{Z})}{d\hat{Z}}\bigg|_{\hat{Z}=M/S} > 0.$$

From before we know that as a function of t the value of a call is maximized with $t = \infty$ and that a sufficiently in-the-money put is maximized with $t = 0$. The next theorem establishes the somewhat surprising result that for $r = \sigma^2/2$ and $M \equiv 1$, as a function of t, for any S, the value of P_{max} is *uniquely* maximized with either $t = 0$ or $t = \infty$ except for $S = ½$ where $P_{max}[½, 0] = P_{max}[½, \infty]$.

THEOREM 5: *Let* $r = \sigma^2/2$, $\gamma = \sigma^2 t$ *and* $M \equiv 1$. *If* $S \leqslant ½$, $P_{max}[S, \gamma]$ *will be maximized at* $\gamma = 0$. *If* $S \geqslant ½$, $P_{max}[S, \gamma]$ *will be maximized at* $\gamma = \infty$. (*Note*: $P_{max}[½, 0] = P_{max}[½, \infty] = ½$).

Proof: See Appendix 2.

Theorems 4 and 5 established that for $r = \sigma^2/2$ and M normalized to unity, P_{max} first acts like a put and then a call and that the critical value of S below which zero time is preferred and above which infinite time is preferred occurs at the intersection of $P_{max}[S, 0] = 1 - S$ with $P_{max}[S, \infty] = S$ or at $S = ½$. Panels *c* and *d* illustrate that the qualitative properties of theorems 4 and 5 hold for arbitrary relations between r and σ^2. The critical point determining whether zero or infinite time is preferred will now occur at the intersection of $P_{max}[S, 0] = 1 - S$ and $P_{max}[S, \infty] = (\sigma^2/2r)\cdot S$ or at $S = \left[\frac{2r}{\sigma^2 + 2r}\right]$. Thus the riskier the stock the earlier the critical value of S.

Panels *b*–*d* of Figure 2 graph P_{max} as a function of t first for $r = \sigma^2/2$ and then for $r = 2\sigma^2$ and $r = \sigma^2/4$. These figures reinforce the results of theorems 4 and 5 for it is clear that for $S > \left[\frac{2r}{2r + \sigma^2}\right]$, P_{max} is maximized at $t = \infty$; for $S < \left[\frac{2r}{2r + \sigma^2}\right]$, P_{max} is maximized at $t = 0$ and for $S = \left[\frac{2r}{2r + \sigma^2}\right]$ the value of P_{max} at $t = 0$ and $t = \infty$ are identical.

Properties of C_{min}

When $r = \sigma^2/2$, C_{min} has been characterized as a straddle with exercise price equal to the current minimum. The straddle consists of an out-of-the-money put plus an in-the-money call. By an argument similar to that used for P_{max}, C_{min} is a convex function of S. However, in contrast to P_{max}, C_{min} is a strictly increasing function of S and hence acts only like a call. That is,

$$\frac{\partial C_{min}}{\partial S} = N\left\{\frac{\ln(S/Q) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}\right\} + \left[N\left\{\frac{\ln(S/Q) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}\right\} - 1\right] \geqslant 0 \quad (16)$$

since $S \geqslant Q$ by construction. Since $\frac{\partial C}{\partial S} > 0$ and $\frac{\partial P}{\partial S} < 0$, relation (16) illustrates that (regardless of the relation between r and σ^2) changes in the value of an in-the-money call (the first term on the RHS) dominates changes in the value of an

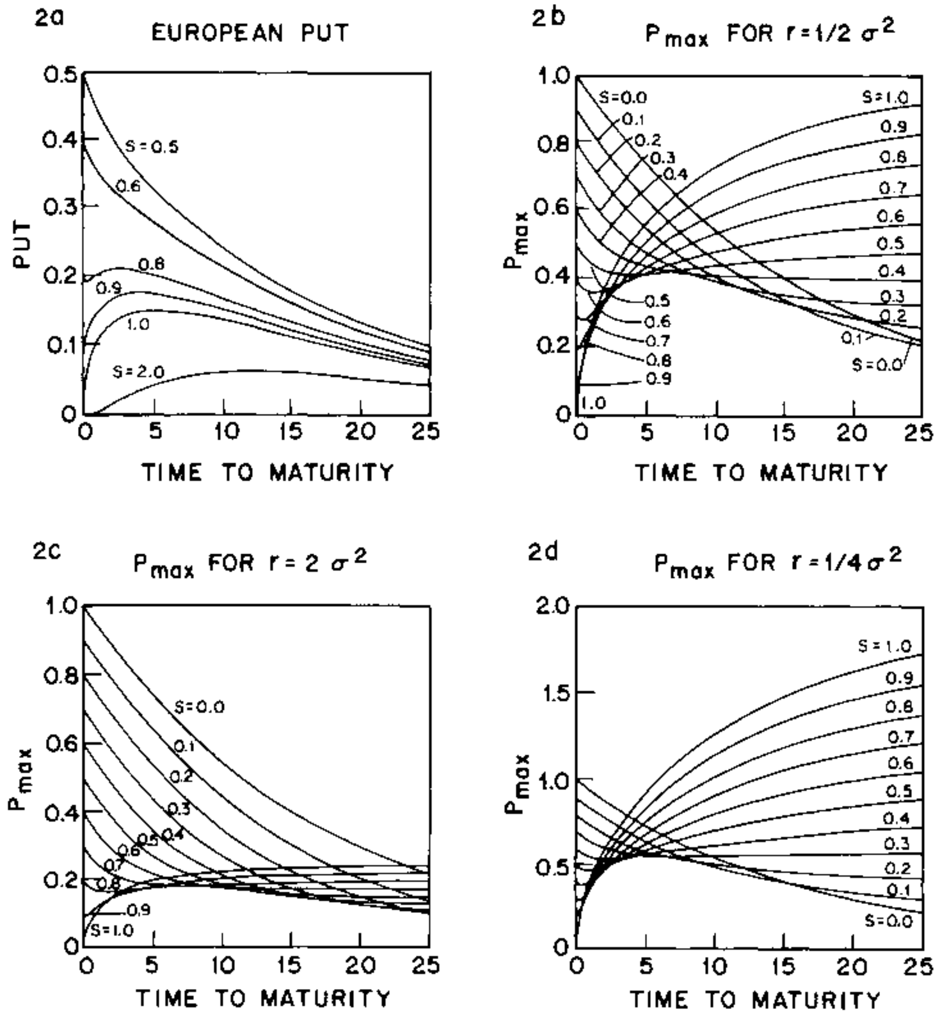


Figure 2. *Price of options to sell as a function of time to maturity for alternative stock prices (s) assuming m = 1*

out-of-the-money put (the second term on the RHS) if both have the same exercise price.

Like a call, C_{min} is a strictly increasing function of time to expiration.

IV. General Discussion

Hitherto, we have focused our attention upon the mathematical aspects of some path dependent options, to wit, the existence of pricing formulae and the qualitative properties of such formulae. Let us now speak more loosely about the possible application of this knowledge.

The financial community expends considerable resources in attempts to better predict the path of stock prices. Yet a few capital market instruments are designed to take direct advantage of such information. Obviously, path dependent options could be designed to exploit these real or perceived informational opportunities. For example an analyst with "special" knowledge of a stock's range over a particular period of time could use P_{max} and C_{min} to achieve a tidy profit [Range forecasts are regularly published. The value of such forecasts could be empirically investigated by using strategies of P_{max} and C_{min} that are functions of the discrepancy between published forecasts and naive forecasts]. Other path dependent options could be designed to profitably utilize special information about other aspects of the price distribution. Thus technicians could use path dependent options to automatically implement the classical technical strategies.

Recent work by Harrison and Kreps [5] is an attempt to specify the set of options that can be arbitraged. A modificiation of V_{max} which can be arbitraged and has certain advantages over V_{max} is forthwith described: A program that obtains the V_{max} payout holds the stock long until the high is attained and then sells the stock holding onto the proceeds in the form of non-interest bearing cash. However, a clever investor would prefer to switch from a stock position to a riskless bond position (that does pay interest). Further, the time of the switch would be a function of both the price path and the rate of interest. Thus, we could define new options that pay (ex-post) the maximum payouts obtainable by going long stock (bonds) and then at some intermediate time substituting a bond (stock) position for the stock (bond) position. A further discussion of such path dependent options may be found in Goldman-Sosin and Shepp [4].

Our casual empiricism makes compelling the demand for P_{max} and C_{min} and other path dependent options since they allow *direct* and *effective* speculation based on standard forecasts of share price distributions. Of course the valuation formulae of this paper are only guidelines to the true values of such options since the assumptions of our model are but a rough description of reality. In fact, it is these very market imperfections (the deviations from our model's perfect market context) that give meaning to the new assets. The information heterogeneity of investors and the costliness of creating perfect hedges of the new assets make the options particularly desirable.

If these options are desirable then why don't they already exist? We believe that markets inherently take advantage of scale economies and attempt to internalize various externalities. The creation of a market is frought with danger—sufficient scale may not be immediately achieved, the benefits of creation may in large part not be captureable by the creators, legal impediments may prove overly burdensome to the creators etc. Accordingly, a desirable and viable security may not currently exist in the market. The test of a security's viability is not its existence but rather its capacity to survive in a fully developed market. Notice that the new flourishing CBOE bears little resemblance to the OTC options market of the preceding era.

Appendix 1[4]

Technically, (12) should be:

$$dV = V_1\,dS + V_3\,dt + V_2\,dM + \tfrac{1}{2}\,V_{11}(dS)^2 + V_{12}\,dS\,dM + \tfrac{1}{2}\,V_{22}(dM)^2 + o(dl) \quad (12')$$

[4] We thank Mr. William Boyce for his persistence in the clarification of this issue.

When $S < M$, $dM = 0$ and all partial derivative terms taken with respect to M disappear—hence, yielding (12). However, when $M = S$ and $dS > 0$, V_{12} and V_{22} do not disappear ($V_2 = 0$ by Theorem 3). Accordingly, when $M = S$ (12′) becomes:

$$dV = V_1\, dS + V_3\, dt + \tfrac{1}{4}\, V_{11}(dS)^2 + o(dt) \qquad (12'')$$

To derive (12″) from (12′), set V_2 equal to zero (by Theorem 3). Theorem 3 implies that when $S = M$ and $dS > 0$:

(a) $0 = dV_2 \equiv V_{21}\, dS + V_{22}\, dM$

(b) $0 = dV_1 \equiv V_{11}\, dS + V_{12}\, dM$

Putting (a) and (b) together, when $S = M$ yields:

(c) $-V_{12} = V_{11} = V_{22}$

Notice that relation (*c*), although obtained assuming $dS > 0$, is in fact independent of dS since the values of the second-order partial derivatives are independent of dS.

For our dynamical system, by the properties of stochastic differential equations, when $S = M$: $dS > 0 => dM = dS$, $dS \leq 0 => dM = 0$, and

(d) $(dS)^2 \approx \tfrac{1}{2}\, (dM\, dS) \approx \tfrac{1}{2}\, (dM)^2$

(c) and (d) require that when $S = M$

$$\tfrac{1}{2}\, V_{11}(dS)^2 + V_{12}\, dS\, dM + \tfrac{1}{2}\, V_{22}(dM)^2 + o(dt) = \tfrac{1}{4}\, V_{11}(dS)^2 + o(dt)$$

and hence (12″).

The switch from (12′) to (12″) is a mathematical curiosity which has no bearing on our solution technique since in all contingencies we can short V_1 shares of stock to form the riskless hedge (note that $V_1 + V_2 = V_1$ *iff* $S = M$).

Appendix 2

Proof of theorem 3: We know that $M(T) = \max[M(\tau), M(\tau, T)]$ where $M(\tau, T)$ is the maximum realized over the interval from τ to T. We want to examine the moments of $(M(T)|\, \tau, S(\tau), M(\tau))$ in the limit as $S(\tau) \to M(\tau)$. Let $\tilde{Z} = \tilde{M}(\tau, T)/S(\tau)$ and

$$\tilde{\lambda} = \begin{cases} M(\tau)/S(\tau) & \text{for} \quad \tilde{M}(\tau, T) < M(T) \\ \tilde{M}(\tau, T)/S(\tau)(\equiv \tilde{Z}) & \text{otherwise.} \end{cases}$$

Note that $\tilde{Z}$ is like a return and since S follows a random walk, the distribution of $\tilde{Z}$ is independent of the level of $S(\tau)$. Let $\tilde{Z}$ have a distribution function $H(Z)$. Also note that $M(T) = S(\tau)\tilde{\lambda}$.

The n^{th} raw moment of $M(T)$ may be written as

$$L(n) = \int_{S(\tau)}^{\infty} [M(T)]^n\, dG[M(T)]$$

where $G(\cdot)$ is the CDF of $M(T)$. By substitution this becomes

$$L(n) = \int_1^{\infty} [S(\tau)\hat{\lambda}]^n \, dF(\hat{\lambda}).$$

Using the definition of λ this may be decomposed into

$$L(n) = \left[[M(\tau]^n \int_1^{M(\tau)/S(\tau)} dH(\bar{Z}) \right] + \left[[S(\tau)]^n \int_{M(\tau)/S(\tau)}^{\infty} \bar{Z}^n \, dH(\bar{Z}) \right].$$

Differentiation of $L(n)$ with respect to $M(\tau)$ yields

$$\frac{\partial L(n)}{\partial M(\tau)} = n[M(\tau)]^{n-1} \int_1^{M(\tau)/S(\tau)} dH(\bar{Z})$$

Finally, $\lim_{S(\tau)\to M(\tau)} \frac{\partial L(n)}{\partial M(\tau)} = 0$ since the assumed Wiener process guarantees that there is no probability mass associated with the point $M(\tau)/S(\tau)$.

Q.E.D.

Proof of theorem 4: P_{max} has been characterized as the sum of a put (which is convex-decreasing in S) and a call (which is convex-increasing in S). The sum of two convex functions is convex and hence P_{max} is convex. The uniqueness of the minimum is also a result of convexity.

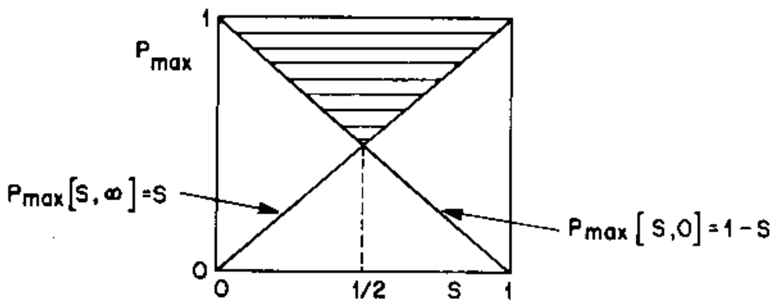
To establish the minimum, let $\gamma \equiv \sigma^2 t$ and normalized $M = 1$ (remember by construction $S \leq M = 1$). Then, $P_{max}[S, \gamma] = P[S, \gamma] + C[S, \gamma]$ for $S \leq 1$, $\frac{\partial P_{max}}{\partial S} = 2N\left\{\frac{\ln(S) + \gamma}{\sqrt{\gamma}}\right\} - 1$, $\left.\frac{\partial P_{max}}{\partial S}\right|_{\gamma=0} = -1$, hence P_{max} decreases initially. $\left.\frac{\partial P_{max}}{\partial S}\right|_{\gamma=\infty} = 1$, hence P_{max} eventually increases. $\frac{\partial P_{max}}{\partial S} = 0$ *iff* $N\{\cdot\} = ½$ or $-\ln(S) = \gamma$.

Q.E.D.

Proof of theorem 5:

From theorem 4, $[-1 \leq \partial P_{max}/\partial S \leq 1]$. Further, $\partial P_{max}/\partial S|_{\gamma=0} = -1$ and $P[S, 0] = 1 - S$; $\partial P_{max}/\partial S|_{\gamma=\infty} = 1$ and $P[S, \infty] = S$. Thus, if $P_{max}[½, \gamma] < ½ \; \forall \; \gamma$ other than $\gamma = 0$, $\gamma = \infty$ the proof is established. To see this, consider Figure A which plots

FIGURE A



$P_{\max}[S, 0]$ and $P_{\max}[S, \infty]$. Given the bounds on $\partial P_{\max}/\partial S$, if $P[\tfrac{1}{2}, \gamma] < \tfrac{1}{2}\ \forall\ \gamma$ other than $\gamma = 0$ and $\gamma = \infty$, then it will be impossible for any $P_{\max}$ as a function of S to "invade" the shaded region. Hence the value of either $P_{\max}[S, 0]$ or $P_{\max}[S, \infty]$ will dominate.

We will now prove that $\left.\dfrac{\partial P_{\max}}{\partial \gamma}\right|_{S=1/2}$ first decreases and then increases as a function of γ.

$$L = \partial P_{\max}/\partial\gamma\Big|_{S=1/2} = \frac{1}{2}e^{-\gamma/2}\left[\frac{2}{\sqrt{\gamma}}N'\left\{\frac{-\theta}{\sqrt{\gamma}}\right\} + 2N\left\{\frac{-\theta}{\sqrt{\gamma}}\right\} - 1\right]$$

$$= \frac{1}{2}e^{-\gamma/2}\left[\frac{2}{\sqrt{\gamma}}N'\left\{\frac{-\theta}{\sqrt{\gamma}}\right\} + N\left\{\frac{-\theta}{\sqrt{\gamma}}\right\} - N\left\{\frac{\theta}{\sqrt{\gamma}}\right\}\right] \tag{15}$$

where $\theta = \ln 2$. Notice that at $\gamma = 0$, $L = -1$ and at $\gamma = \infty$, $L = 0$. We now show that γ sufficiently large implies $L > 0$. Clearly $L > 0$ if $[\cdot]$ in relation (15) is positive. For the normal density, $\dfrac{2\theta}{\sqrt{\gamma}}N'\{0\} > N\left\{\dfrac{\theta}{\sqrt{\gamma}}\right\} - N\left\{\dfrac{-\theta}{\sqrt{\gamma}}\right\}$. Hence $[\cdot] > 0$ if $\dfrac{2}{\sqrt{\gamma}}N'\left\{\dfrac{-\theta}{\sqrt{\gamma}}\right\} \geqslant \dfrac{2\theta}{\sqrt{\gamma}}N'\{0\}$ or whenever $N'\{0\}/N'\left\{\dfrac{-\theta}{\sqrt{\gamma}}\right\} = e^{\theta^2/2\gamma} \leqslant \dfrac{1}{\theta} \cong 1.44$. Thus we know that L initially decreases with γ and that for γ sufficiently large L is an increasing function of γ.

Now consider $\gamma \geqslant \dfrac{\theta^2}{1-\theta}$. $e^{\theta^2/2\gamma}$ will be maximized at the smallest γ, or at $\gamma = \dfrac{\theta^2}{1-\theta}$. At $\gamma = \dfrac{\theta^2}{1-\theta}$, $e^{\theta^2/2\gamma} = e^{(1-\theta)/2} = e^{.154} \cong 1.18 < \dfrac{1}{\theta} \cong 1.44$. Thus $L > 0$ if $\gamma \geqslant \dfrac{\theta^2}{1-\theta}$. Consider $0 \leqslant \gamma < \dfrac{\theta^2}{1-\theta}$. At $\gamma = 0$, $L < 0$ and $[\cdot] < 0$, $\dfrac{\partial[\cdot]}{\partial\gamma} = N'\left\{\dfrac{-\theta}{\sqrt{\gamma}}\right\}\gamma^{-3/2}\left[-1 + \dfrac{\theta^2}{\gamma} + \theta\right]$, which implies, $\dfrac{\partial[\cdot]}{\partial\gamma} > 0$ if $\gamma < \dfrac{\theta^2}{1-\theta}$. Thus $= \left.\dfrac{\partial P_{\max}}{\partial S}\right|_{S=1/2}$ can change signs only once during interval $0 \leqslant \gamma \leqslant \dfrac{\theta^2}{1-\theta}$. Thus L starts out negative at $\gamma = 0$, and is positive at $\gamma = \dfrac{\theta^2}{1-\theta}$ and remains positive thereafter.

Q.E.D.

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