Chapter 18

Volatility Smiles

Problem 18.1

When both tails of the stock price distribution are less heavy than those of the lognormal distribution, Black-Scholes will tend to produce relatively high prices for options that are either significantly out of the money or significantly in the money. This leads to an implied volatility pattern similar to that in Figure 18.7. When the right tail is heavier and the left tail is less heavy, Black-Scholes will tend to produce relatively high prices for out-of-the-money calls and in-the-money puts. It will tend to produce relatively low prices for out-of-the-money puts and in-the-money calls. This leads to the implied volatility being an increasing function of strike price.

Problem 18.2

When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing less heavy left tails and heavy right tails. Implied volatility is an increasing function of the strike price.

Problem 18.3

Jumps tend to make both tails of the stock price distribution heavier than those of the lognormal distribution. This creates a volatility smile similar to that Figure 18.1 of the text. The volatility smile is likely to be more pronounced for a a-years option than a 3-years option.

Problem 18.4

The put–call parity relationship

$$c - p = S_0 e^{-qT} - Ke^{-rT}$$

should hold for all option pricing models. Because the terms on the right hand side of this equation are independent of the option pricing model used, $c - p$ is independent of the option pricing model used.

Problem 18.5

Because the implied probability distribution in Figure 18.4 has a less heavy right tail than the lognormal distribution, it should lead to lower prices for out-of-the-money calls. Because it has a heavier left tail, it should lead to higher prices for out-of-the-money puts. This argument shows that,

if σ^* is the volatility corresponding to the lognormal distribution in Figure18.4, the implied

volatility for high strike price calls must be less than σ^*, and the implied volatility for low strike price puts must be greater than σ^*. It follows that Figure 18.3 is consistent with Figure 18.4.

Problem 18.6

With the notation in the text

$$c_{\text{bs}} + Ke^{-rT} = p_{bs} + S_0 e^{-qT}$$

$$c_{\text{mkt}} + Ke^{-rT} = p_{mkt} + S_0 e^{-qT}$$

It follows that

$$c_{\text{bs}} - c_{\text{mkt}} = p_{\text{bs}} - p_{\text{mkt}}$$

In this case $c_{\text{mkt}} = 3.00$; $c_{\text{bs}} = 3.50$; and $p_{\text{bs}} = 1.00$. It follows that p_{mkt} should be 0.50.

Problem 18.7

Literally, "crashophobia" means phobia against a terrible crash, just as October 1987. In practice, the term "crashophobia" is referred to strong negative skewness in the physical stock returns distribution, suggesting that the probability of a large decrease in stock prices exceeds the probability of a large increase. The economic rationale for crashophobia is that put options are used as hedging instruments to protect against large downward movements in stock prices. This demand by investors due to portfolio insurance strategies has increased the price of put options and therefore the left tail of the implied distribution has more weight.

Problem 18.8

The probability distribution of the stock price in one month is not lognormal. Possibly it consists of two lognormal distributions superimposed upon each other and is bimodal. Black–Scholes is inappropriate because it assumes that the stock price at any future time is lognormal.

Problem 18.9

When the volatility is positively correlated to the stock price, the volatility tends to increase as the stock price increases. Thus the probability has a less heavy left tail and a heavier right tail. This would lead to a volatility skew with a positive slop.

Problem 18.10

There are a number of problems in testing an option pricing model empirically. These include the problem of obtaining synchronous data on stock prices and option prices, the problem of estimating the dividends that will be paid on the stock during the option's life, the problem of distinguishing between situations where the market is inefficient and situations where the option pricing model is incorrect, and the problems of estimating stock price volatility.

Problem 18.11

In this case the probability distribution of the exchange rate has a less heavy left tail and a less heavy right tail than the lognormal distribution. We are in the opposite situation to that described for foreign currencies in Section 18.2. Both out-of-the-money and in-the-money calls and puts can be expected to have lower implied volatilities than at-the-money calls and puts. The pattern of implied volatilities is likely to be similar to Figure 18.7.

Problem 18.12

A deep-out-of-the-money option has a low value. Decreases in its volatility reduce its value. However, this reduction is small because the value can never go below zero. Increases in its volatility, on the other hand, can lead to significant percentage increases in the value of the option. The option does, therefore, have some of the same attributes as an option on volatility.

Problem 18.13

As explained in the chapter, put–call parity implies that European put and call options have the same implied volatility. If a call option has an implied volatility of 30% and a put option has an implied volatility of 33%, the call is priced too low relative to the put. The correct trading strategy is to buy the call, sell the put and short the stock. This does not depend on the lognormal assumption underlying Black–Scholes. Put–call parity is true for any set of assumptions.

Problem 18.14

Suppose that p is the probability of a favorable ruling. The expected prices of the company tomorrow is

$$75p + 50(1-p) = 50 + 25p$$

This must be the price of the company today. (We ignore the expected return to an investor over one day.) Hence

$$50 + 25p = 60$$

or $p = 0.4$.

If the ruling is favorable, the volatility , σ, will be 25%. Other option parameters are $S_0 = 75$, $r = 0.06$ and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 26.502.

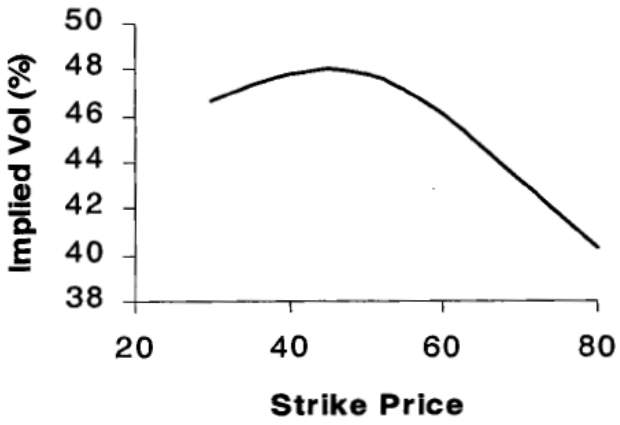

Figure 18.1 Implied Volatilities in Problem 18.14

If the ruling is unfavorable, the volatility, σ will be 40% Other option parameters are $S_0 = 50$, $r = 0.06$, and $T = 0.5$. For a value of K equal to 50, DerivaGem gives the value of a European call option price as 6.310.

The value today of a European call option with a strike price today is the weighted average of 26.502 and 6.310 or:

$$0.4 \times 26.502 + 0.6 \times 6.310 = 14.387$$

DerivaGem can be used to calculate the implied volatility when the option has this price. The parameter values are $S_0 = 60$, $K = 50$, $T = 0.5$, $r = 0.06$ and $c = 14.387$. The implied volatility is 47.76%.

These calculations can be repeated for other strike prices. The results are shown in the table below. The pattern of implied volatilities is shown in Figure 18.1.

Strike Price	Call Option Price Favorable Outcome	Call Option Price Unfavorable Outcome	Weighted Price	Implied Volatility (%)
30	45.887	21.001	30.955	46.67
40	36.182	12.437	21.935	47.78
50	26.502	6.310	14.387	47.76
60	17.171	2.826	8.564	46.05
70	9.334	1.161	4.430	43.22
80	4.159	0.451	1.934	40.36

Problem 18.15

An exchange rate behaves like a stock that provides a dividend yield equal to the foreign risk-free rate. Whereas the growth rate in a non-dividend-paying stock in a risk-neutral world is r, the growth rate in the exchange rate in a risk-neutral world is $r - r_f$. Exchange rates have low

systematic risks and so we can reasonably assume that this is also the growth rate in the real world. In this case the foreign risk-free rate equals the domestic risk-free rate ($r = r_f$). The expected growth rate in the exchange rate is therefore zero. If S_T is the exchange rate at time T, its probability distribution is given by equation(13.3) with $\mu = 0$:

$$\ln S_T \sim \phi(\ln S_0 - \sigma^2 T / 2, \sigma\sqrt{T})$$

Where S_0 is the exchange rate at time zero and σ is the volatility of the exchange rate. In this case $S_0 = 0.8000$ and $\sigma = 0.12$, and $T = 0.25$. So that

$$\ln S_T \sim \phi(\ln 0.8 - 0.12^2 \times 0.25/2, 0.12\sqrt{0.25})$$

or

$$\ln S_T \sim \phi(-0.2240, 0.06)$$

(a) $\ln 0.70 = -0.3567$. The probability that $S_T < 0.70$ is the same as the probability that $\ln S_T < -0.3567$. It is

$$N\left(\frac{-0.3567 + 0.2240}{0.06}\right) = N(-2.2117)$$

This is 1.35%.

(b)$\ln 0.75 = -0.2877$. The probability that $S_T < 0.75$ is the same as the probability that $\ln S_T < -0.2877$. It is

$$N\left(\frac{-0.2877 + 0.2240}{0.06}\right) = N(-1.0617)$$

This is 14.42%. The probability that the exchange rate is between 0.70 and 0.75 is therefore $14.42 - 1.35 = 13.07\%$.

(c)$\ln 0.80 = -0.2231$. The probability that $S_T < 0.80$ is the same as the probability that $\ln S_T < -0.2231$. It is

$$N\left(\frac{-0.2231 + 0.2240}{0.06}\right) = N(0.0150)$$

This is 50.60%. The probability that the exchange rate is between 0.75 and 0.80 is therefore $50.60 - 14.42 = 36.18\%$.

(d)$\ln 0.85 = -0.1625$. The probability that $S_T < 0.85$ is the same as the probability that $\ln S_T < -0.1625$. It is

$$N\left(\frac{-0.1625 + 0.2240}{0.06}\right) = N(1.0250)$$

This is 84.73%. The probability that the exchange rate is between 0.80 and 0.85 is therefore $84.73 - 50.60 = 34.13\%$.
(e) $\ln 0.90 = -0.1054$. The probability that $S_T < 0.90$ is the same as the probability that $\ln S_T < -0.1054$. It is

$$N\left(\frac{-0.1054 + 0.2240}{0.06}\right) = N(1.9767)$$

This is 97.60%. The probability that the exchange rate is between 0.85 and 0.90 is therefore $97.60 - 84.73 = 12.87\%$.
(f) The probability that the exchange rate is greater than 0.90 is $100 - 97.60 = 2.40\%$

The volatility smile encountered for foreign exchange options is shown in Figure 18.1 of the text and implies the probability distribution in Figure 18.2. Figure 18.2 suggests that we would expect the probabilities in (a), (c), (d), and (f) to be too low and the probabilities in (b) and (d) to be too high.

Problem 18.16

The difference between the two implied volatilities is consistent with Figure 18.3. For equities the volatility smile is downward sloping. A high strike price option has a lower implied volatility than a low strike price option. The reason is that traders consider that the probability of a larger downward movement in the stock price is higher than that predicted by the lognormal probability distribution. The implied distribution assumed by traders is shown in Figure 18.4.

To use DerivaGem to calculate the price of the first option, proceed as follows. Select Equity as the Underlying Type in the first worksheet. Select Analytic European as the Option Type. Input the stock price as 40, volatility as 35%, risk-free rate as 5%, time to exercise as 0.5 year, and exercise price as 30. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Do not select the implied volatility button. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 11.155. Change the volatility to 28% and the strike price to 50. Hit the *Enter* key and click on calculate. DerivaGem will show the price of the option as 0.725.
Put–call parity is

$$c + Ke^{-rT} = p + S_0$$

so that

$$p = c + Ke^{-rT} - S_0$$

For the first option, $c = 11.155$, $S_0 = 40$, $r = 0.054$, $K = 30$, and $T = 0.5$ so that

$$p = 11.155 + 30e^{-0.05\times 0.5} - 40 = 0.414$$

For the second option, $c = 0.725$, $S_0 = 40$, $r = 0.06$, $K = 50$, and $T = 0.5$ so that

$$p = 0.725 + 50e^{-0.05\times 0.5} - 40 = 9.490$$

To use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the

exercise price as 30. Input the price as 0.414 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 34.99%.
Similarly, to use DerivaGem to calculate the implied volatility of the first put option, input the stock price as 40, the risk-free rate as 5%, time to exercise as 0.5 year, and the exercise price as 50. Input the price as 9.490 in the second half of the Option Data table. Select the buttons for a put option and implied volatility. Hit the *Enter* key and click on calculate. DerivaGem will show the implied volatility as 27.99%.
These results are what we would expect. DerivaGem gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30. Similarly, it gives the implied volatility of a put with strike price 30 to be almost exactly the same as the implied volatility of a call with a strike price of 30.

Problem 18.17

When plain vanilla call and put options are being priced, traders do use the Black-Scholes model as an interpolation tool. They calculate implied volatilities for the options whose prices they can observe in the market. By interpolation between strike prices and between times to maturity, they estimate implied volatilities for other options. These implied volatilities are then substituted into Black-Scholes to calculate prices for these options. In practice much of the work in producing a table such as 18.2 in the over-the-counter market is done by brokers. Brokers often act as intermediaries between participants in the over-the-counter market and usually have more information on the trades taking place than any individual financial institution. The brokers provide a table such as 18.2 to their clients as a service.

Problem 18.18

Use the cubic spline interpolant to the data, we obtain the below table, so the implied volatility for an 8-month option with $K / S_0 = 1.04$ is 13.4053.

	K / S_0					
	0.90	0.95	1.00	1.04	1.05	1.10
1 month	14.2000	13.0000	12.0000	12.7688	13.1000	14.5000
3 month	14.0000	13.0000	12.0000	12.7792	13.1000	14.2000
6 month	14.1000	13.3000	12.5000	13.1384	13.4000	14.3000
8 month	14.2868	13.5519	12.8817	13.4053	13.6254	14.4701
1 year	14.7000	14.0000	13.5000	13.8416	14.0000	14.8000
2 year	15.0000	14.4000	14.0000	14.3544	14.5000	15.1000
3 year	14.8000	14.6000	14.4000	14.6160	14.7000	15.0000

Problem 18.19

In liquidation the company's stock price must be at least 300,000/100,000 = \$3. The company's stock price should therefore always be at least \$3. This means that the stock price distribution that has a less heavy left tail and heavier right tail than the lognormal distribution. An upward sloping volatility smile can be expected.

Problem 18.20

(a) If p is the risk-neutral probability of a positive outcome (stock price rises to \$24), we must have

$$24p + 18(1-p) = 20e^{0.08\times 0.0833}$$

so that $p = 0.356$

(b) The price of a call option with strike price K is $(24-K)pe^{-0.08\times 0.08333}$ when $K < 24$. Call options with strike prices of 19, 20, 21, 22, and 23 therefore have prices 1.766, 1.413, 1.060, 0.707, and 0.353, respectively.

(c) From DerivaGem the implied volatilities of the options with strike prices of 19, 20, 21, 22, and 23 are 49.8%, 58.7%, 61.7%, 60.2%, and 53.4%, respectively. The volatility smile is therefore a "frown" with the volatilities for deep-out-of-the-money and deep-in-the-money options being lower than those for close-to-the-money options.

(d) The price of a put option with strike price K is $(K-18)(1-p)e^{-0.08\times 0.08333}$. Put options with strike prices of 19, 20, 21, 22, and 23 therefore have prices of 0.640, 1.280, 1.920, 2.560, and 3.200. DerivaGem gives the implied volatilities as 49.81%, 58.68%, 61.69%, 60.21%, and 53.38%. Allowing for rounding errors these are the same as the implied volatilities for put options.

Problem 18.21

The calculations are shown in the following table. For example, when the strike price is 34, the price of a call option with a volatility of 10% is 5.926, and the price of a call option when the volatility is 30% is 6.312. When there is a 60% chance of the first volatility and 40% of the second, the price is $0.6\times 5.926 + 0.4\times 6.312 = 6.080$. The implied volatility given by this price is 23.21. The table shows that the uncertainty about volatility leads to a classic volatility smile similar to that in Figure 18.1 of the text. In general when volatility is stochastic with the stock price and volatility uncorrelated we get a pattern of implied volatilities similar to that observed for currency options.

Strike Price	Call Option Price 10% Volatility	Call Option Price 30% Volatility	Weighted Price	Implied Volatility (%)
34	5.926	6.312	6.080	23.21
36	3.962	4.749	4.277	21.03
38	2.128	3.423	2.646	18.88
40	0.788	2.362	1.418	18.00
42	0.177	1.560	0.730	18.80
44	0.023	0.988	0.409	20.61
46	0.002	0.601	0.242	22.43

Problem 18.22

The following table shows the percentage of daily returns greater than 1,2,3,4,5 and 6 standard deviations for each currency.

	> 1sd	> 2sd	> 3sd	> 4sd	> 5sd	> 6sd
AUD	24.8	5.3	1.3	0.2	0.1	0.0
BEF	24.3	5.7	1.3	0.6	0.2	0.0
CHF	26.1	4.2	1.3	0.6	0.1	0.0
DEM	23.9	5.0	1.4	0.6	0.1	0.0
DKK	26.7	5.8	1.3	0.3	0.0	0.0
ESP	28.2	5.1	0.9	0.3	0.1	0.0
FRF	26.0	5.4	1.4	0.2	0.0	0.0
GBP	23.9	6.4	1.1	0.4	0.1	0.0
ITL	25.4	6.6	1.1	0.2	0.0	0.0
NLG	25.6	5.7	1.7	0.2	0.0	0.0
SEK	28.2	5.2	1.0	0.0	0.0	0.0
Normal	31.7	4.6	0.3	0.0	0.0	0.0

Problem 18.23

The results are shown in the table below.

	> 3sd down	> 3sd up
TSE	0.88	0.22
S&P	0.55	0.44
FTSE	0.55	0.66
CAC	0.33	0.33
Nikkei	0.55	0.66
Total	0.57	0.46

Problem 18.24

Define c_1 and p_1 as the values of the call and the put when the volatility is σ_1. Define c_2 and p_2 as the values of the call and the put when the volatility is σ_2. From put–call parity

$$p_1 + S_0 e^{-qT} = c_1 + Ke^{-rT}$$

$$p_2 + S_0 e^{-qT} = c_2 + Ke^{-rT}$$

If follows that

$$p_1 - p_2 = c_1 - c_2$$

Problem 18.25

In this case, $S_0 = 1.0, r = r_f = 0.025, T = 0.5$. Assume that $g(K)$ is constant between $K = 0.7$ and $K = 0.8$, constant between $K = 0.8$ and $K = 0.9$, and so on. Define:

$$g(K) = g_1 \quad \text{for} \quad 0.7 \le K < 0.8$$

$$g(K) = g_2 \quad \text{for} \quad 0.8 \le K < 0.9$$

$$g(K) = g_3 \quad \text{for} \quad 0.9 \le K < 1.0$$

$$g(K) = g_4 \quad \text{for} \quad 1.0 \le K < 1.1$$

$$g(K) = g_5 \quad \text{for} \quad 1.1 \le K < 1.2$$

$$g(K) = g_6 \quad \text{for} \quad 1.2 \le K < 1.3$$

The value of g_1 can be calculated by interpolating to get the implied volatility for a 6-month option with strike price of 0.75 as 12.45%. This means that options with strike price of 0.7, 0.75 and 0.8 have implied volatility 13%, 12.45% and 12%, respectively. Their price are 0.2963, 0.2469 and 0.1976, respectively. Using equation (18A.1), with $K = 0.75$ and $\delta = 0.5$, gives

$$g_1 = e^{(0.025-0.025)\times 0.5} \frac{0.2963 + 0.1976 - 2 \times 0.2469}{0.05^2} = \quad 0.0316$$

Similar calculations show that

$$g_2 = 0.7001, g_3 = 4.3019, g_4 = 3.9255, g_5 = 0.6901 \ , g_6 = \quad 0.0941$$

About all the implied volatilities are 11.5%, we can obtain that

$$g_1 = 0.0236, g_2 = 0.9213, g_3 = \ 4.1723, g_4 = 3.7123, g_5 = 0.9494 \ , g_6 = 0.0927$$

Comparing the two distribution , we can obtain that the distribution for smile volatility have the heavier tails than the same volatility with 11.5%.

Problem 18.26.

Use the cubic spline interpolant to the data, we obtain the below table, so the implied volatility for an 11-month option with $K / S_0 = 0.98$ is 13.4759

	0.90	0.95	0.98	1.00	1.05	1.10
1 month	14.2000	13.0000	12.1952	12.0000	13.1000	14.5000
3 month	14.0000	13.0000	12.1984	12.0000	13.1000	14.2000
6 month	14.1000	13.3000	12.6568	12.5000	13.4000	14.3000
11 month	14.6049	13.9019	13.4759	13.3692	13.9173	14.7256
1 year	14.7000	14.0000	13.6008	13.5000	14.0000	14.8000
2 year	15.0000	14.4000	14.0672	14.0000	14.5000	15.1000
3 year	14.8000	14.6000	14.4320	14.4000	14.7000	15.0000