CHAPTER 12
The Black–Scholes Model

12.1. The Black–Scholes option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

12.2. The standard deviation of the percentage price change in time δt is $\sigma\sqrt{\delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

12.3. The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

12.4. In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times0.25} - 50N(-0.2417)$$
$$= 50 \times 0.4634e^{-0.1\times0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

12.5. In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

12.6. The implied volatility is the volatility that makes the Black–Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

12.7. In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (12.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25}{\sqrt{2}}\right)$$

i.e.,

$$\phi(0.11875, 0.1768)$$

The expected value of the return is 11.875% per annum and the standard deviation is 17.68% per annum.

12.8. (a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5,\ 0.35\sqrt{0.5})$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.247)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables N(0.008) = 0.5032 so that the required probability is 0.4968. In general the required probability is $N(d_2)$. (See Problem 12.22).
(b) In this case the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

12.9. From equation 12.2:

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0 + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0 + (\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

12.10. The statement is misleading in that a certain sum of money, say \$1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.

The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

12.11. (a) At time t, the expected value of $\ln S_T$ is, from equation (12.2)

$$\ln S + (\mu - \frac{\sigma^2}{2})(T - t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore:

$$\ln S + (r - \frac{\sigma^2}{2})(T - t)$$

Using risk-neutral valuation the value of the security at time t is:

$$e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right]$$

(b) If:

$$f = e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right] - e^{-r(T-t)}(r - \frac{\sigma^2}{2})$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black Scholes differential equation is

$$e^{-r(T-t)}\left[r\ln S + r(r - \frac{\sigma^2}{2})(T-t) - (r - \frac{\sigma^2}{2}) + r - \frac{\sigma^2}{2}\right]$$

$$= re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$= rf$$

Hence equation (12.15) is satisfied.

12.12. This problem is related to Problem 11.10.

(a)If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b)The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

(c)The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

so that

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

12.13. In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$
$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)$$

$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$

$$= 5.06$$

or \$5.06.

12.14. In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$
$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times 0.5}N(0.0809) - 69N(-0.1666)$$

$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

or \$6.40.

12.15. Using the notation of Section 12.13, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2-t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

12.16. In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.
A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

12.17. (a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

$$= N'(d_2) \exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t})$$

$$= -\frac{\sigma}{2\sqrt{T-t}}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S}$$
$$= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

From the results in d) and e)

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$
$$+ rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$
$$= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]$$
$$= rc$$

This shows that the Black–Scholes formula for a call option does indeed satisfy the Black–Scholes differential equation

12.18. From the Black–Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$ this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black–Scholes equations are therefore consistent with put–call parity.

12.19. This problem naturally leads on to the material in Chapter 15 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

	Maturity (months)		
Strike Price ($)	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black–Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

12.20. Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.
It appears that Black's approach understates the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two alternatives implicitly assumed by the approach. These alternatives add value to the option. In fact Black's approach sometimes gives a higher value than the approach in Appendix 12B. This is because Appendix 12B applies the volatility to the stock price less the present value of the dividend whereas Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Thus part of the Black calculation assumes more stock price variability than Roll–Geske–Whaley. This issue is also discussed in Example 12.9.

12.21. With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times 0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1 - e^{-r(t_2-t_1)}\right] = 55(1 - e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1 - e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 12.13 that the option should never be exercised early.
The present value of the dividends is

$$1.5e^{-0.3333\times0.08} + 1.5e^{-0.8333\times0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times1.25} \times 0.3692 = 4.17$$

or \$4.17.

12.22. The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \phi[\ln S_0 + (r - \sigma^2/2)T, \sigma\sqrt{T}]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

12.23. If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2}S^{-2r/\sigma^2-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right) S^{-2r/\sigma^2-2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black–Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.