Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets

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ABSTRACT

We consider the optimal intertemporal consumption and investment policy of a CARA investor who faces fixed and proportional transaction costs when trading multiple risky assets. We show that when asset returns are uncorrelated, the optimal investment policy is to keep the dollar amount invested in each risky asset between two constant levels and upon reaching either of these thresholds, to trade to the corresponding optimal targets. An extensive analysis suggests that transaction cost is an important factor in affecting trading volume and that it can significantly diminish the importance of stock return predictability as reported in the literature.

This paper studies the optimal intertemporal consumption and investment policy of an investor with a constant absolute risk aversion (CARA) preference and an infinite horizon. The investor can trade in one risk-free asset and $n \geq 1$ risky assets. In contrast to the standard setting, the investor faces both fixed and proportional transaction costs in trading any of these risky assets. In the absence of transaction costs and when risky asset prices follow geometric Brownian motions, the optimal investment policy is to keep a constant dollar amount in each risky asset, as shown by Merton (1971). This trading strategy requires continuous trading in all the risky assets. In addition, the optimal consumption is affine in the total wealth. In the presence of transaction costs, however, trading continuously in a risky asset would incur infinite transaction costs. Therefore, risky assets are traded only infrequently in this case.

The literature on optimal consumption and investment with *multiple* risky assets subject to transaction costs is limited. Leland (2000) examines a multi-asset investment fund that is subject to transaction costs and capital gains taxes. Under the assumption that the fund has an exogenous target for each risky asset, he develops a relatively simple numerical procedure to compute the no-transaction region. Akian, Menaldi, and Sulem (1996) consider an optimal consumption and investment problem with proportional transaction costs for a constant relative risk aversion (CRRA) investor when asset returns are uncorrelated. They also use numerical simulations to compute the no-transaction region. Lynch and Tan (2002) numerically solve a similar problem when stock returns are predictable in a discrete time setting. Deelstra, Pham, and Touzi (2001) use the dual approach to obtain the sufficient conditions for the existence of a solution to the optimal investment problem for an investor who maximizes expected utility from her terminal wealth. Eastham and Hastings (1988) address the optimal consumption and portfolio choice problem with transaction costs and multiple stocks; however, they assume consumption can only be changed at the same time that stock holdings are changed. Bielecki and Pliska (2000) analyze a similar problem with a general transaction cost structure and risk-sensitive criteria but exclude intertemporal consumption. None of these models obtain any analytically explicit shape for the no-transaction region.

Our first contribution in this paper is to derive the optimal transaction policy in an explicit form when the risky asset returns are uncorrelated, up to some constants that can be solved numerically. In particular, it is shown that the optimal investment policy in each risky asset is for the investor to keep the dollar amount invested in the asset between two constant levels. Once the amount reaches one of these two thresholds, the investor trades to the corresponding optimal targets. To the best of our knowledge, this is the first paper to present such an explicit form of trading strategy in the case of multiple risky assets subject to fixed transaction costs.[1] The optimal trading strategy implies that the no-transaction and target boundaries have corners and only on reaching a corner does the investor trade in more than one risky asset. Since the corner is of measure zero relative to the no-transaction boundary, with probability one, the investor only trades in at most one risky asset at any point in time.

When there are only proportional transaction costs for a risky asset, we show that the optimal trading policy involves possibly an initial discrete change (jump) in the dollar amount invested in the asset, followed by trades in the minimal amount necessary to maintain the dollar amount within a constant interval.

The presence of fixed transaction costs implies that any optimal transaction involves a lump-sum trade. In the absence of proportional transaction costs, the optimal trading policy for each risky asset is to trade to the same target dollar amount as soon as the amount in a risky asset goes beyond a constant range. If there are also proportional transaction costs, the optimal investment policy then involves buying to a target amount as soon as the amount in the risky asset falls below a lower bound and selling to a different target amount as soon as the amount in the risky asset rises above an upper bound. Thus, the target amounts depend on the direction of a trade. These results generalize the no-transaction-cost case (the Merton case) where the optimal policy for a CARA investor is to maintain a constant dollar amount in a risky asset.

In the presence of transaction costs, the dependence of the optimal consumption on total wealth is also different from the standard results derived by Merton (1971). In particular, the optimal consumption is no longer affine in total wealth. Instead, it is affine only in the dollar amount invested in the risk-free asset but nonlinear in the dollar amounts in risky assets.

Our second contribution is that we conduct an extensive analysis of the optimal policy in the literature. We provide a simple way to compute the no-transaction and target boundaries. We analyze the impact of risk aversion, risk premium and volatility on the no-transaction region, the target amounts and the trading frequency. We also derive in closed-form the steady-state distribution of the amount invested in a risky asset and examine the steady-state average amount invested in the asset. With no explicit form of trading strategy derived, the existing literature provides only a very limited analysis of the trading strategy, rarely going beyond the computation of the no-transaction region and the target amounts. The explicit form of the boundaries (up to some numerically computed constants) allows us to conduct this extensive analysis, which enhances our understanding of the relationship between fundamental parameters and optimal investment policy in the presence of transaction costs, and also yields some interesting results.

First, we find that small transaction costs can induce large deviations from the no-transaction-cost case. For example, with five dollar fixed cost and one percent proportional cost (which includes the bid-ask spread), the investor would purchase additional units of a risky asset to reach the buy-target of $104,300 only when the actual amount fell below $93,500. On the other hand, only when the actual amount rose above $152,600, would the investor sell the risky asset to reach the sell-target of $138,300. In contrast, in the absence of transaction costs, the investor would trade continuously to keep a constant amount of $121,900 in the risky asset. This large deviation implies a very low frequency of trading. For example, with five dollar fixed cost and one percent proportional cost, the average time between sales would be about 1.2 years and the average time between purchases would be about 2.5 years. We show that trading more frequently than the optimal strategy would result in significant utility loss. This suggests that the gain from incorporating stock return predictability (see for example, Kandel and Stambaugh (1996)) would be significantly decreased if transaction costs were considered. Also, since transaction costs have dramatic effects on both trading frequency and trading size, to explain the observed trading volume, it seems that one must also consider transaction costs along with other standard factors considered in the literature such as information asymmetries and heterogeneous beliefs (e.g., Admati and Pfleiderer (1988) and Wang (1994)).[2]

Second, we show that conditional on positive investment in a risky asset, the steady-state average amount invested in the asset increases as the transaction cost increases. This result suggests that the presence of transaction costs makes the investor less risk averse overall. Intuitively, to compensate for the transaction costs, the investor overshoots by investing more than otherwise optimal in the risky asset. This finding in particular implies that after an *increase* in transaction costs, to induce an investor to hold the same average amount as before, one needs to *lower* the expected return of the risky asset ceteris paribus.

In addition, we find that as the return volatility of a risky asset rises, the no-transaction region narrows, the expected time to the next purchase after a trade decreases, but the expected time to the next sale after a trade increases. This finding seems counterintuitive because as the volatility increases the investor could be expected to widen the no-transaction region to decrease the trading frequency in order to save on transaction costs. However, saving transaction costs is not the investor's only concern. As volatility increases, so does risk, and hence, on average, the investor holds less in the risky asset. Over time then, the investor needs to sell the risky asset less frequently to increase current consumption, and actually buys more often to finance future consumption.

A large body of literature addresses the optimal transaction policy for an agent facing a proportional transaction cost in trading a *single* risky asset (see for example, Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Shreve and Soner (1994), Cuoco and Liu (2000), and Liu and Loewenstein (2002)). In contrast, this paper considers multiple risky assets with both proportional costs and fixed costs. Closely related models of optimal consumption and investment with fixed costs and one risky asset have been previously analyzed by Schroder (1995), Øksendal and Sulem (1999), and Korn (1998). These papers do not provide explicit forms for the no-transaction or target boundaries and they use numerical procedures to directly solve the Hamilton-Jacobi-Bellman partial differential equations (HJB PDE) with free boundaries. Lo, Mamaysky and Wang (2001) study the effect of fixed transaction costs on asset prices and find that even small fixed costs can give rise to a significant illiquidity discount on asset prices. This finding is in sharp contrast to the proportional transaction cost case considered by Constantinides (1986) and shows the importance of fixed transaction costs in a financial market.

Also related are papers that assume quasi-fixed transaction costs (see for example, Duffie and Sun (1990), Morton and Pliska (1995) , and Grossman and Laroque (1990)). While the assumption of quasi-fixed costs simplifies analysis (e.g., with power utility function, the homogeneity of the value function is preserved and hence the HJB PDE can be simplified into an ordinary differential equation (ODE)), the solution is at best an approximation for investors who face fixed costs such as those charged by brokers.

In a different context, Constantinides (1976) and Constantinides and Richard (1978) study the optimal cash management policy in the presence of fixed and proportional transaction costs. Cadenillas and Zapatero (1999) examine the optimal intervention of a central bank in the foreign exchange market where the bank directly controls the exchange rate but incurs fixed and proportional intervention costs. Korn (1997) investigates a one-dimensional optimal impulse control for a cost minimization problem when there are both fixed and proportional control costs.

Three main aspects of the model we present here make it more tractable and thus better able to yield an extensive analysis than other models in the literature. First, CARA preferences and the absence of borrowing constraints[3] imply the separability of optimal policies for the risk-free asset, risky assets and consumption. Second, the assumption of uncorrelated risky asset returns enables us to further break down the analysis of multiple risky assets into an analysis of individual assets. Third, the standard assumption of no transaction cost in liquidating the risk-free asset to buy the consumption good is also important. Without this feature, consumption would only occur at optimal stopping times, which would in turn require a more complicated analysis.

The case of uncorrelated asset returns is of practical interest. Uncorrelated assets are commonly recommended to achieve efficient diversification, and there exist asset classes with nearly zero correlations. Indeed, some investors (e.g., funds of funds) view themselves as facing a menu of uncorrelated assets. In addition, other investors may also find it beneficial to limit their trading to uncorrelated portfolios.

The remainder of the paper is organized as follows. Section I describes the model. Section II solves the investor's optimal consumption and investment problem in the absence of transaction costs, providing a benchmark for the subsequent analysis. Section III contains a heuristic derivation of the optimal policies in the presence of only proportional transaction costs. It also provides sufficient conditions under which the conjectured policies are indeed optimal. Section IV derives the optimal policy in the presence of only fixed transaction costs. Section V obtains the optimal policy in the presence of both fixed and proportional transaction costs. Section VI addresses the correlated asset case. Section VII contains an extensive analysis of the optimal policy. Section VIII concludes the paper and discusses some possible extensions. In Appendix A, we provide the proofs for the main results and in Appendix B, we provide the solution algorithms.

I. The Model

A. The Asset Market

Throughout this paper we assume a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}$. Uncertainty in the model is generated by a standard n-dimensional Brownian motion w (a $n \times 1$ column vector).

There are $n+1$ assets our investor can trade. The first asset ("the bond") is a money market account growing at a constant, continuously compounded rate of $r > 0$. The other n assets are risky (hereafter we will use "stocks" and "risky assets" interchangeably). The investor can buy stock i at the ask price of $S_i(t)$ and sell it at the bid price of $(1-\alpha_i)S_i(t)$, where $0 \leq \alpha_i < 1$ represents the proportional transaction cost rate.[4] In addition, the investor has to pay a fixed brokerage fee $F_i \geq 0$ for each transaction in either direction when trading stock i.[5] Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $F = (F_1, F_2, ..., F_n)$. For simplicity, we assume no dividend is paid by any stock. For $i = 1, 2, ..., n$, the ask price $S_i(t)$ is assumed to follow a geometric Brownian motion:

$$\frac{dS_{it}}{S_{it}} = \mu_i dt + \sigma_i dw_{it}, \tag{1}$$

where w_i is the ith element of the n-dimensional standard Brownian motion w, $\mu_i > r$, and $\sigma_i > 0$.[6]

B. The Investor's Problem

There is a single perishable consumption good (the numeraire). Following Merton (1971), we assume that the investor derives her utility from intertemporal consumption c of this good. We use $\mathcal{C}$ to denote the investor's admissible consumption space, which consists of progressively measurable consumption processes c_t such that $\int_0^t |c_s| ds < \infty$ for any $t \in [0, \infty)$. In addition, similar to Merton (1971), Vayanos (1998), and Lo, Mamaysky and Wang (2001), we assume that the investor has a CARA preference with time discounting, i.e., $u(c,t) = e^{-\delta t}(-e^{-\beta c})$ for some absolute risk aversion coefficient $\beta > 0$ and time discount parameter $\delta > 0$. We further assume that consumption withdrawals, stock trades and transaction cost payments are all made through the money market account.

Let x be the amount invested in the money market account, y_i be the amount in the ith stock, and $y = (y_1, y_2, ..., y_n)$. We then have the following dynamics for x_t and y_t:

$$dx_t = rx_t dt - c_t dt - \sum_{i=1}^{n} \Big(dI_{it} - (1-\alpha_i) dD_{it} + F_i 1_{\{dI_{it} + dD_{it} > 0\}} \Big), \tag{2}$$

$$dy_{it} = \mu_i y_{it} dt + \sigma_i y_{it} dw_{it} + dI_{it} - dD_{it}, \quad i = 1, 2, ..., n, \tag{3}$$

where the processes D_i and I_i represent the cumulative dollar amount of sales and purchases of the ith stock, respectively. These processes are nondecreasing, right-continuous and adapted, with $D(0) = I(0) = 0$, where $D = (D_1, D_2, ..., D_n)$ and $I = (I_1, I_2, ..., I_n)$. In addition, let

$$W_t = x_t + \sum_{i=1}^{n} [(1-\alpha_i) y_{it}^+ - y_{it}^- - F_i 1_{\{y_{it} \neq 0\}}] \tag{4}$$

denote the liquidated wealth at time t.

To rule out any arbitrage opportunity such as doubling strategies or "Ponzi schemes," similar to Lo, Mamaysky and Wang (2001), we restrict the set of trading policies to be such that[7]

$$\lim_{t \to \infty} E[e^{-\delta t - r\beta W_t}] = 0 \text{ and } E \int_0^T |y_t e^{-\delta t - r\beta W_t}|^2 dt < \infty, \forall T \in [0, \infty). \tag{5}$$

This set of trading policies is also the set within which the Merton solution (see next section) in the no-transaction-cost case is optimal.

We use $\Theta(x, y)$ to denote the set of admissible trading strategies (I, D, c) such that the implied x_t and y_t from equations (2) and (3) satisfy condition (5) starting from $x_0 = x$ and

$y_0 = y$. The investor's problem is then to choose admissible trading strategies I, D and c to maximize $E[\int_0^\infty u(c_t, t)dt]$. We define the value function at time t to be

$$v(x, y) = \sup_{(I,D,c)\in\Theta(x,y)} E[\int_t^\infty e^{-\delta(s-t)}(-e^{-\beta c_s})ds|\mathcal{F}_t, x_t = x, y_t = y]. \tag{6}$$

II. Optimal Policies with No Transaction Costs

For the purpose of comparison we present in this section the main results for the no-transaction-cost case (i.e., $\alpha = 0$ and $F = 0$) without proof (see Merton (1971)). In the absence of transaction costs, the cumulative purchase and sale processes of the stocks can be of infinite variation and in this case the liquidated wealth $W_t = x_t + y_t\bar{\mathbf{1}}$, where $\bar{\mathbf{1}}$ is an n-element column vector of 1's. The investor's problem can then be rewritten as

$$v(w) = \sup_{(y,c)} E[\int_0^\infty e^{-\delta t}(-e^{-\beta c_t})dt|W_0 = w]$$

subject to

$$dW_t = rW_t dt + \sum_{i=1}^n ((\mu_i - r)y_{it}dt + \sigma_i y_{it} dw_{it}) - c_t dt.$$

Theorem 1. *Suppose $\alpha = F = 0$. Let*

$$y_i^M = \frac{\mu_i - r}{r\beta\sigma_i^2}, \quad i = 1, 2, ..., n. \tag{7}$$

The optimal consumption and investment policies are

$$c_t^* = rW_t^* + \gamma, \quad y_{it}^* = y_i^M, \quad i = 1, 2, ..., n,$$

for all $t > 0$, respectively, where W_t^* is the optimal wealth process derived from following the above policies and

$$\gamma = \frac{\delta - r}{r\beta} + \sum_{i=1}^n \frac{(\mu_i - r)^2}{2r\beta\sigma_i^2}.$$

Moreover, the value function is

$$v(w) = -\frac{1}{r}e^{-r\beta w - \beta\gamma}.$$

Thus, without transaction costs, the optimal policy involves investing a constant dollar amount in each stock, and the optimal consumption is an affine function of total wealth. This investment policy requires continuous trading in every stock. We will show later that none of these results hold in the presence of transaction costs.

III. The Proportional Transaction Cost Case

We begin by addressing the case with only proportional transaction costs (i.e., $\alpha > 0$ and $F = 0$). In contrast to the no-transaction-cost case, the stock trading will now become infrequent. We provide a heuristic derivation of the optimal policy in this section. In the single-stock case, Davis and Norman (1990), Shreve and Soner (1994), and Liu and Loewenstein (2002) show that in the presence of proportional costs there exist a no-transaction region and exist a transaction region. Similarly, in the multiple stock case, we conjecture that there exist a transaction region wherein the investor trades at least one stock and a no-transaction region (NT) where she does not trade any stock. Inside NT, the value function must satisfy the HJB equation:

$$\max_c\left(\sum_{i=1}^{n}\left(\frac{1}{2}\sigma_i^2 y_i^2 v_{y_i y_i} + \mu_i y_i v_{y_i}\right) + rxv_x - cv_x - \delta v - e^{-\beta c}\right) = 0. \tag{8}$$

The optimal consumption is thus

$$c^* = -\frac{1}{\beta}\log\left(\frac{v_x}{\beta}\right),$$

which implies that (8) becomes

$$\sum_{i=1}^{n}\left(\frac{1}{2}\sigma_i^2 y_i^2 v_{y_i y_i} + \mu_i y_i v_{y_i}\right) + rxv_x + \frac{v_x}{\beta}\log\left(\frac{v_x}{\beta}\right) - \delta v - \frac{v_x}{\beta} = 0. \tag{9}$$

We conjecture that

$$v(x, y_1, y_2, ..., y_n) = -\frac{1}{r}e^{-r\beta x - \sum_{i=1}^{n}\varphi_i(r\beta y_i)}, \tag{10}$$

for some functions $\varphi_i : \mathbb{R} \to \mathbb{R}$.

For expositional convenience, we let $z_i = r\beta y_i$ be the scaled amount in the ith stock and ψ_i be the restriction of φ_i in the no-transaction region. Then equation (9) becomes

$$\sum_{i=1}^{n}\left(\frac{1}{2}\sigma_i^2 z_i^2 \psi_i'' - \frac{1}{2}\sigma_i^2 z_i^2 \psi_i'^2 + \mu_i z_i \psi_i' - r\psi_i\right) + (\delta - r) = 0. \tag{11}$$

For equation (11) to hold, it is clear that the following n ODEs must be satisfied:

$$\frac{1}{2}\sigma_i^2 z_i^2 \psi_i'' - \frac{1}{2}\sigma_i^2 z_i^2 \psi_i'^2 + \mu_i z_i \psi_i' - r\psi_i + \frac{\delta - r}{n} - \lambda_i = 0, \tag{12}$$

for some constants λ_i such that $\sum_{i=1}^{n} \lambda_i = 0$ and $i = 1, 2, ..., n$.

We note that the above ODE system is not only independent of the amount x in the money market account but also completely separable in z_i's. This observation suggests that if the boundary conditions are also separable in z_i's, then the optimal stock transaction policy in stock i would depend only on the amount in the stock, but not on the amount in the money market account or the amounts in other stocks. We will show later that this is indeed the case. We thus further conjecture that there exist two critical numbers, $\underline{y}_i$ and $\bar{y}_i$ with $\underline{y}_i < \bar{y}_i$, which characterize the optimal trading strategy for this stock. To be specific, we conjecture that the optimal policy is to buy enough to reach the buy boundary $\underline{y}_i$ if $y_{it} \leq \underline{y}_i$ and sell enough to reach the sell boundary $\bar{y}_i$ if $y_{it} \geq \bar{y}_i$. According to the proposed transaction policy, in a stock's transaction region the marginal (indirect) utility from the bond holding must be always equal to the marginal utility from the stock holding, net of transaction costs. Therefore, the differential equation in a transaction region where stock i is purchased can be written as

$$v_{y_i}(x, y_1, y_2, ..., y_i, ..., y_n) = v_x(x, y_1, y_2, ..., y_i, ..., y_n) \tag{13}$$

and similarly, in a transaction region where stock i is sold the differential equation must be

$$v_{y_i}(x, y_1, y_2, ..., y_i, ..., y_n) = (1 - \alpha_i)v_x(x, y_1, y_2, ..., y_i, ..., y_n). \tag{14}$$

In addition, the optimality of $\underline{y}_i$ and $\bar{y}_i$ implies that v is C^2 in all its arguments and in all regions (cf. Dumas (1991)).

Using equations (10), (13) and (14) and letting $\underline{z}_i = r\beta\underline{y}_i$ and $\bar{z}_i = r\beta\bar{y}_i$, we then obtain the following forms for φ_i in the transaction regions:

(i) if $z_i < \underline{z}_i$,

$$\varphi_i(z_i) = C_{i1} + z_i$$

and

(ii) if $z_i > \bar{z}_i$,

$$\varphi_i(z_i) = C_{i2} + (1 - \alpha_i)z_i,$$

where C_{i1} and C_{i2} are two constants to be determined. The proposed transaction policy and the C^2 property of the value function then imply the following six boundary conditions in terms of ψ_i:

$$\psi_i(\underline{z}_i) = C_{i1} + \underline{z}_i, \tag{15}$$

$$\psi_i'(\underline{z}_i) = 1, \tag{16}$$

$$\psi_i''(\underline{z}_i) = 0, \tag{17}$$

$$\psi_i(\bar{z}_i) = C_{i2} + (1-\alpha_i)\bar{z}_i, \tag{18}$$

$$\psi_i'(\bar{z}_i) = 1-\alpha_i \tag{19}$$

and

$$\psi_i''(\bar{z}_i) = 0. \tag{20}$$

Therefore, the boundary conditions (15)-(20) are indeed all independent of the holdings in the bond and separable in z_i's. Thus, the above conjectures about the form of the no-transaction region and the related optimal transaction policy are justified.

Next, consider a variation of the ODE (12) for stock i:

$$\frac{1}{2}\sigma_i^2 z_i^2 \psi_i'' - \frac{1}{2}\sigma_i^2 z_i^2 \psi_i'^2 + \mu_i z_i \psi_i' - r\psi_i + \frac{\delta - r}{n} - \lambda_i - \eta_i = 0, \tag{21}$$

where η_i is a constant. Suppose ψ_i, $\underline{z}_i$ and $\bar{z}_i$ are the solution to (12) subject to the boundary conditions (15)-(20), then $f_i(z_i) = \psi_i(z_i) - \eta_i/r$ and the same boundaries $\underline{z}_i$ and $\bar{z}_i$ are the solution to equation (21) subject to the corresponding six boundary conditions derived from replacing ψ_i with f_i in conditions (15)-(20). This result holds because $\underline{z}_i$ and $\bar{z}_i$ are independent of any constant term in ψ_i. This observation also applies to the cases considered in subsequent sections and implies in particular that the boundaries are independent of δ in all the cases considered in this paper. This shows that the undetermined λ_i in equation (12) does not affect the optimal boundaries $\underline{z}_i$ or $\bar{z}_i$. In addition, because of the condition $\sum_{i=1}^n \lambda_i = 0$ and the property of the solution, v is also independent of λ_i. Therefore, without loss of generality, we can set $\lambda_i = 0$ for all $i = 1, 2, ..., n$. Consequently, we have

$$\frac{1}{2}\sigma_i^2 z_i^2 \psi_i'' - \frac{1}{2}\sigma_i^2 z_i^2 \psi_i'^2 + \mu_i z_i \psi_i' - r\psi_i + \frac{\delta - r}{n} = 0, \tag{22}$$

for $i = 1, 2, ..., n$.

The above discussion suggests that when there are multiple risky assets subject to proportional costs and their returns are uncorrelated, we can compute the optimal boundaries separately for each stock. This greatly reduces the dimensionality of the computation problem, making it feasible to compute the optimal trading strategy for a large number of risky assets.

Define

$$\varphi_i(z_i) = \begin{cases} C_{i2} + (1-\alpha_i)z_i & \text{if } z_i \geq \bar{z}_i \\ \psi_i(z_i) & \text{if } \underline{z}_i < z_i < \bar{z}_i \\ C_{i1} + z_i & \text{if } z_i \leq \underline{z}_i. \end{cases} \tag{23}$$

We next provide a verification theorem which shows the validity of the above conjectured optimal policies and the form of the value function.

Theorem 2. *Assume $\alpha > 0$ and $F = 0$, and $\forall i \in \{1, 2, ..., n\}$, let φ_i be as defined in (23). Consider any stock i. Suppose there exist constants C_{i1}, C_{i2}, $\underline{z}_i$, and $\bar{z}_i$ such that ψ_i is a solution of ODE (22) subject to conditions (15)-(20) and in addition,*

$$1 - \alpha_i < \psi_i'(z_i) < 1, \quad \forall z_i \in (\underline{z}_i, \bar{z}_i). \tag{24}$$

Then ψ_i is the unique solution to ODE (22) subject to conditions (15)-(20) and (24), from which the corresponding optimal consumption policy is

$$c_t^* = r x_t^* + \frac{1}{\beta}\sum_{i=1}^{n} \varphi_i(r\beta y_{it}^*),$$

and the corresponding optimal risky asset trading policy is to transact the minimal amount necessary to maintain y_{it}^* between $\underline{y}_i$ and $\bar{y}_i$, where x_t^* and y_{it}^* are the bond holding and risky asset holding processes derived from following the above policies. Moreover, the value function is

$$v(x, y) = -\frac{1}{r} e^{-r\beta x - \sum_{i=1}^{n} \varphi_i(r\beta y_i)}.$$

Proof. The proof of this theorem is only a slight variation of the proof of Theorem 4 (see below) and is thus omitted.[8] □

If equation (22) has a closed form solution, then we would have a solution for ψ_i with two integration constants A_i and B_i. Using the above six boundary conditions, we would then solve for the six unknowns, $C_{i1}, C_{i2}, \underline{z}_i, \bar{z}_i, A_i$, and B_i. Unfortunately, equation (22) belongs to a special class of Abel differential equations whose closed form solution, if any, has not yet been obtained (see for example, Cheb-Terrab and Roche (1999)) except for the special case where $\mu_i = \frac{1}{2}\sigma_i^2$. However, the above free-boundary problem can be numerically solved quite easily using a simple algorithm (Algorithm 1) as explained in Appendix B.[9]

To facilitate understanding of the optimal policy, we provide numerical illustrations below. Since the optimal stock trading strategy is separable in individual stocks, most of the following numerical analysis will focus on the single stock case and for clarity, we will suppress all subscripts when there is only one stock considered in a figure. For all numerical illustrations, we use the following default values for the parameters unless otherwise stated: According to Ibbotson and Sinquefeld (1982), we set the excess return $\mu - r$ and the volatility σ at 5.9 percent and 22 percent, respectively; in addition, following Grossman and Laroque (1990), we set the real risk-free rate r at one percent and the time discount rate δ at 0.01; finally, Lo, Mamaysky and Wang (2001) examine cases in which β lies between 0.001 and 5.000, and we set it to the low end, 0.001, to emphasize the effect of transaction costs. Of course, this is by no means an attempt to calibrate our model for empirical analysis purposes.

Figure 1 displays the optimal no-transaction boundaries $\underline{z}$ and $\bar{z}$ as functions of the proportional transaction cost rate. Without transaction costs ($\alpha = 0$), the investor would always keep \$121,900 in the stock, as represented by the thin middle line. Note that this is the actual amount that is equal to the scaled amount in the figure divided by $r\beta$. In the presence of transaction costs, it is no longer optimal to always maintain a fixed amount in the stock. Instead, the investor allows the amount in the stock to fluctuate within a certain range. When $\alpha = 0.01$, for example, the investor will not adjust the amount she invests in the stock until it reaches the bounds of \$99,400 or \$144,700. Thus, the presence of transaction costs has a significant impact on the optimal trading strategy. It should also be noted that as the transaction cost rate increases, the buy boundary decreases and the sell boundary increases, making the investor trade less frequently.

Figure 1 Here →

IV. The Fixed Transaction Cost Case

When there are fixed transaction costs, the infinitesimal transaction policy proposed in the previous section is no longer optimal. In this case, all transactions involve lump-sum trades, because cost is independent of the size of a trade. In this section, we consider the case when the investor pays only fixed costs but not proportional transaction costs (i.e., $F > 0$ and $\alpha = 0$).

In the presence of only fixed costs, we conjecture that the optimal policy for any stock i is characterized by three (instead of two, as in the previous section) critical numbers: $\underline{y}_i$, y_i^*, and $\bar{y}_i$. When the amount in the stock reaches the buy boundary, $\underline{y}_i$, or the sell boundary, $\bar{y}_i$, it is optimal to trade to y_i^*. For the form of the value function, we conjecture that (10) is still valid.

In the no-transaction region, the HJB ODE system (22) in the previous section still holds. However, the conditions in the transaction regions (i.e., where $y_i \leq \underline{y}_i$ or $y_i \geq \bar{y}_i$) need to be changed.

According to the proposed transaction policy, we have

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i - (y_i^* - y_i), y_1, y_2, ..., y_i^*, ..., y_n) \tag{25}$$

for any $y_i \leq \underline{y}_i$ and

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i + (y_i - y_i^*), y_1, y_2, ..., y_i^*, ..., y_n) \tag{26}$$

for any $y_i \geq \bar{y}_i$. In addition, the optimality of y_i^* implies that

$$v_{y_i}(x, y_1, y_2, ..., y_i^*, ..., y_n) = v_x(x, y_1, y_2, ..., y_i^*, ..., y_n). \tag{27}$$

Let ψ_i be the restriction of φ_i in the no-transaction region, $\underline{z}_i = r\beta\underline{y}_i$, $z_i^* = r\beta y_i^*$, and $\bar{z}_i = r\beta\bar{y}_i$. To provide sufficient conditions for optimality, we focus on the case where the value function is C^1. Using equations (10), (25)-(27) and the C^1 property, we obtain the following seven boundary conditions:

$$\psi_i(\underline{z}_i) = C_{i1} + \underline{z}_i, \tag{28}$$

$$\psi_i'(\underline{z}_i) = 1, \tag{29}$$

$$\psi_i'(z_i^*) = 1, \tag{30}$$

$$\psi_i(\bar{z}_i) = C_{i2} + \bar{z}_i, \tag{31}$$

$$\psi_i'(\bar{z}_i) = 1, \tag{32}$$

$$\psi_i(z_i^*) = C_{i1} + r\beta F_i + z_i^* \tag{33}$$

and

$$\psi_i(z_i^*) = C_{i2} + r\beta F_i + z_i^*, \tag{34}$$

where C_{i1} and C_{i2} are two constants to be determined. Comparing equations (33) and (34), we have $C_{i1} = C_{i2}$. This result implies that for any stock i, we only need to solve six equations (as in the previous section) for six unknowns: $C_{i1}, \underline{z}_i, z_i^*, \bar{z}_i$, and two integration constants.

We note that, in contrast to the case with only proportional costs, in the presence of fixed costs the above free boundary problem is no longer β free. In particular, β enters the boundary conditions (33) and (34). However, given values of r, F_i, and β that are of economically meaningful magnitudes, $\underline{z}_i$, z_i^*, and $\bar{z}_i$ are generally not sensitive to changes in β.

The following theorem records results for the value function and the optimal trading strategy in this case.

Theorem 3. *Assume $F > 0$ and $\alpha = 0$, and $\forall i \in \{1, 2, ..., n\}$, let φ_i be as defined in (23). Consider any stock i. Suppose there exist constants C_{i1}, C_{i2}, $\underline{z}_i$, z_i^*, and $\bar{z}_i$ such that ψ_i is a solution of ODE (22) subject to conditions (28)-(34) and in addition,*

$$\psi_i'(z_i) > 1, \quad \forall z_i \in (\underline{z}_i, z_i^*), \tag{35}$$

and

$$0 < \psi_i'(z_i) < 1, \quad \forall z_i \in (z_i^*, \bar{z}_i). \tag{36}$$

Then ψ_i is the unique solution to ODE (22) subject to conditions (28)-(36), from which the corresponding optimal consumption policy is

$$c_t^* = rx_t^* + \frac{1}{\beta}\sum_{i=1}^{n} \varphi_i(r\beta y_{it}^*),$$

and the corresponding optimal risky asset trading policy is to transact to y_i^* *only when* $y_{it}^* \leq \underline{y}_i$ *or* $y_{it}^* \geq \bar{y}_i$*, where* x_t^* *and* y_{it}^* *are the bond holding and risky asset holding processes derived from following the above policies. Moreover, the value function is*

$$v(x,y) = -\frac{1}{r} e^{-r\beta x - \sum_{i=1}^{n} \varphi_i(r\beta y_i)}.$$

PROOF. This theorem is a special case of Theorem 4 (see below). □

Figure 2 displays the optimal no-transaction boundaries $\underline{z}$ and $\bar{z}$ and the optimal target z^* as functions of the fixed cost. In the presence of fixed transaction costs, it is no longer optimal for the investor to transact an infinitesimal amount to keep the amount in the stock within a specified range. When $F = \$5$, for example, the investor will allow the actual amount in the stock to fluctuate between \$105,200 and \$139,800. If the actual amount reaches \$105,200, the investor will buy \$16,600 worth of the stock. On the other hand, if the actual amount reaches \$139,800, the investor will sell \$18,000 worth of the stock. Thus, the presence of fixed transaction costs also has a significant impact on trading. The large size of the no-transaction region derives mainly from the low risk aversion we used in the numerical illustration. As the risk aversion β increases, the size of the no-transaction region shrinks, as will be shown later. In addition, it should be noted that as in the previous case, as transaction costs increase the buy boundary decreases and the sell boundary increases. However, the sensitivity of the optimal target y^* to changes in transaction costs is very small. It only decreases from \$121,900 to \$121,500 as the fixed cost increases from \$0 to \$30, making z^* indistinguishable from the Merton line in the figure. This finding is consistent with the intuition that roughly speaking, the investor is better off being around the Merton line, on average, even in the presence of transaction costs.

Figure 2 Here →

Based on the insensitivity of the target amount to fixed costs, to obtain the optimal boundaries, one can first fix z_i^* to be the Merton line, and then choose C_{i1} to satisfy all the conditions except (34). This one-dimensional search is straightforward.

To measure the relative effect of the proportional and fixed costs on the welfare of the investor, we define the equivalent fixed cost F for a given proportional cost α to be the fixed cost such that the investor is indifferent between facing only the fixed cost and facing only the proportional cost, i.e., the F such that $v(x,y;F) = v(x,y;\alpha)$. For a given α, if the fixed cost exceeds the equivalent F, then the investor prefers to face the proportional transaction cost. Otherwise, the investor prefers to face the fixed transaction cost. Figure 3 plots the equivalent fixed cost F against the proportional cost α for several risk aversion levels. For $\beta = 1$, the investor is indifferent between facing a proportional cost of five percent and facing a fixed cost of \$2. As the proportional cost increases, the equivalent fixed cost increases at an increasing rate. In addition, as the risk aversion decreases, the equivalent fixed cost increases significantly. For example, if $\beta = 0.1$, the equivalent fixed cost for a five percent proportional cost becomes as high as \$18. Intuitively, as the investor's risk aversion decreases, the amount the investor holds in a stock increases. Therefore, the relative impact of a given fixed cost becomes smaller.

Figure 3 Here

V. The Fixed and Proportional Cost Case

When the investor is subject to both fixed and proportional costs for each transaction, the problem becomes even more complicated. We conjecture that in this case, there exist four (instead of three, as in the previous section) critical numbers, $\underline{y}_i, \underline{y}_i^*, \bar{y}_i^*$, and $\bar{y}_i$ ($\underline{y}_i < \underline{y}_i^* < \bar{y}_i^* < \bar{y}_i$), characterizing the optimal trading strategy. Specifically, we conjecture that the optimal policy is to transact immediately to the buy-target $\underline{y}_i^*$ if $y_{it} \leq \underline{y}_i$ and to jump to the sell-target $\bar{y}_i^*$ if $y_{it} \geq \bar{y}_i$. In addition, the value function still satisfies the HJB ODE system (22) in the no-transaction region.

According to the proposed transaction policy, we must have

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i - (\underline{y}_i^* - y_i), y_1, y_2, ..., \underline{y}_i^*, ..., y_n)$$

for any $y_i \leq \underline{y}_i$, and

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i + (1 - \alpha_i)(y_i - \bar{y}_i^*), y_1, y_2, ..., \bar{y}_i^*, ..., y_n)$$

for any $y_i \geq \bar{y}_i$, where $i = 1, 2, ..., n$.

The optimality of $\underline{y}_i^*$ and $\bar{y}_i^*$ implies that

$$v_{y_i}(x, y_1, y_2, ..., \underline{y}_i^*, ..., y_n) = v_x(x, y_1, y_2, ..., \underline{y}_i^*, ..., y_n)$$

and

$$v_{y_i}(x, y_1, y_2, ..., \bar{y}_i^*, ..., y_n) = (1-\alpha_i) v_x(x, y_1, y_2, ..., \bar{y}_i^*, ..., y_n),$$

for any $i = 1, 2, ..., n$.

Plugging equation (10) into the boundary conditions and using the C^1 property of v, we obtain the following eight boundary conditions:

$$\psi_i(\underline{z}_i) = C_{i1} + \underline{z}_i, \tag{37}$$

$$\psi_i'(\underline{z}_i) = 1, \tag{38}$$

$$\psi_i'(\underline{z}_i^*) = 1, \tag{39}$$

$$\psi_i'(\bar{z}_i) = 1 - \alpha_i, \tag{40}$$

$$\psi_i'(\bar{z}_i^*) = 1 - \alpha_i, \tag{41}$$

$$\psi_i(\bar{z}_i) = C_{i2} + (1-\alpha_i)\bar{z}_i, \tag{42}$$

$$\psi_i(\underline{z}_i^*) = C_{i1} + r\beta F_i + \underline{z}_i^* \tag{43}$$

and

$$\psi_i(\bar{z}_i^*) = C_{i2} + r\beta F_i + (1-\alpha_i)\bar{z}_i^*, \tag{44}$$

for $i = 1, 2, ..., n$, where $\underline{z}_i = r\beta \underline{y}_i$, $\underline{z}_i^* = r\beta \underline{y}_i^*$, $\bar{z}_i^* = r\beta \bar{y}_i^*$, and $\bar{z}_i = r\beta \bar{y}_i$.

We then have the following result for the value function and the optimal trading strategy.

Theorem 4. *Assume $F > 0$ and $\alpha > 0$, and $\forall i \in \{1, 2, ..., n\}$, let φ_i be as defined in (23). Consider any stock i. Suppose there exist constants C_{i1}, C_{i2}, $\underline{z}_i$, $\underline{z}_i^*$, $\bar{z}_i^*$, and $\bar{z}_i$ such that ψ_i is a solution of ODE (22) subject to conditions (37)-(44), and in addition,*

$$\psi_i'(z_i) > 1, \quad \forall z_i \in (\underline{z}_i, \underline{z}_i^*), \tag{45}$$

$$1 - \alpha_i < \psi_i'(z_i) < 1, \quad \forall z_i \in (\underline{z}_i^*, \bar{z}_i^*) \tag{46}$$

and

$$0 < \psi_i'(z_i) < 1 - \alpha_i, \quad \forall z_i \in (\bar{z}^*, \bar{z}_i). \tag{47}$$

Then ψ_i is the unique solution to ODE (22) subject to conditions (37)-(47), from which the corresponding optimal consumption policy is

$$c_t^* = rx_t^* + \frac{1}{\beta}\sum_{i=1}^{n}\varphi_i(r\beta y_{it}^*), \tag{48}$$

and the corresponding optimal risky asset trading policy is to transact to $\underline{y}_i^*$ only when $y_{it}^* \le \underline{y}_i$, and transact to $\bar{y}_i^*$ only when $y_{it}^* \ge \bar{y}_i$, where x_t^* and y_{it}^* are the bond holding and risky asset holding processes derived from following the above policies. Moreover, the value function is

$$v(x,y) = -\frac{1}{r}e^{-r\beta x - \sum_{i=1}^{n}\varphi_i(r\beta y_i)}.$$

Proof. See Appendix A. □

To help us compute the optimal boundaries and understand the boundary behavior, we present the following proposition that provides some bounds on the optimal boundaries.

Proposition 1. *For any $i = 1, 2, ..., n$, if $\underline{y}_i(\alpha_i, F_i)$ and $\bar{y}_i(\alpha_i, F_i)$ are, respectively, the optimal buy and sell boundaries as specified in Theorem 4 for given α_i and F_i, with $\alpha_i + F_i > 0$, then*

$$\underline{y}_i(\alpha_i, F_i) < y_i^M \quad \text{and} \quad \bar{y}_i(\alpha_i, F_i) > \frac{y_i^M}{1-\alpha_i}, \tag{49}$$

where y_i^M is the Merton line for stock i as defined in (7). In addition, for $F_i > 0$, we have

$$\underline{y}_i(\alpha_i, F_i) < \underline{y}_i(\alpha_i, 0) \quad \text{and} \quad \bar{y}_i(\alpha_i, F_i) > \bar{y}_i(\alpha_i, 0). \tag{50}$$

Proof. See Appendix A. □

Proposition 1 shows that the buy and sell boundaries always bracket the Merton line. In addition, as $\alpha_i \uparrow 1$, the sell boundary goes to infinity and thus cannot be bounded from above. Moreover, the boundaries with fixed costs always bracket the corresponding boundaries with no fixed costs. This proposition makes the computation of the optimal boundaries more efficient by providing better initial values for the boundaries and the direction of changes as transaction costs change.

According to Theorem 4, we need to find $\underline{z}_i, \underline{z}_i^*, \bar{z}_i^*, \bar{z}_i, C_{i1}$, and C_{i2} such that ψ_i solves ODE (22) and satisfies conditions (37)-(44).[10] Appendix B presents an algorithm that effectively reduces the problem to a two-dimensional search procedure.

Figure 4 shows the typical shape of $\varphi'(z)$ within the no-transaction region. Clearly, it satisfies conditions (45)-(47) in the above theorem. This figure also shows that the value function is C^2 almost everywhere except at $\underline{z}$ and $\bar{z}$, where it is only C^1. In addition, $\varphi(z)$ is first convex, then turns into a concave function, then changes back into a convex function. This implies that the value function v is not globally concave. This is because a convex combination of two policies does not always outperform these two policies due to the presence of fixed costs.

Figure 4 Here $\longrightarrow$

Figure 5 shows the no-transaction and transaction regions when there are two stocks subject to both fixed and proportional costs. The interior of "ABCD" represents the no-transaction region; "abcd" and its extensions inside "ABCD" are the target boundaries. There are eight transaction regions. The arrow lines represent the transaction directions in these transaction regions. For example, in the "Sell 1 Buy 2" region (the quadrant starting at point "C"), the investor sells stock 1 and buys stock 2 to reach the target point "c." Similarly, in the "NT 1 Sell 2" region, the investor sells stock 2 but does not trade in stock 1 to reach the target point on the segment "ad." After the initial trade, the investor always stays in "ABCD." In addition, only when she reaches one of the four corners, "A", "B", "C" or "D," does she trade simultaneously in more than one stock. This event is obviously of probability zero because the set of these corners is of measure zero relative to the no-transaction boundary, and z_{1t} and z_{2t} follow geometric Brownian motions inside "ABCD." In general, when there are n stocks, the investor trades in more than one stock only when these stocks *simultaneously* reach their respective transaction boundaries. This implies that when there are multiple risky assets, with probability one, the investor only trades in at most one stock at any point in time.

Figure 5 Here $\longrightarrow$

This figure is in contrast to that of Morton and Pliska (1995) whose numerical computation shows that the no-transaction region approximates an ellipse. It is generally suspected that the no-transaction region boundary should be an ellipse and thus differentiable everywhere. We show, however, that this is not true in our case. In particular, the boundary of the no-transaction region in our model is not an ellipse, but rather does have "corners" (in general, a set with dimension $n-2$), and thus is not differentiable at these points. The assumption of uncorrelated returns is not the reason for this difference. In the presence of correlations among the stock returns, we conjecture the no-transaction boundaries would also have corners as long as the correlations were not perfect. The only difference would be that the no-transaction boundaries would be skewed one way or the other depending on the signs of the correlations (see next section for an example).[11] Moreover, the assumption of a CARA preference is not critical either. For other utility functions such as a CRRA preference, the no-transaction and target boundaries would also have these non-smooth points. Intuitively, these "corners" arise because, to the investor, one stock is not a perfect substitute for another.

Figure 6 plots the optimal boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ as functions of the fixed cost for $\alpha = 0.01$. In the presence of both fixed and proportional transaction costs, it is no longer optimal to trade to the same boundary as was suggested in the previous section. If $F = \$5$, for example, the investor would buy \$10,800 worth of the stock to reach the buy-target of \$104,300 when the actual amount of the investment decreases to \$93,500. If, on the other hand, the market goes up and the actual amount of the investment increases to \$152,600, the investor would sell \$14,300 worth of the stock to reach the sell-target of \$138,300. In addition, as the fixed cost decreases toward zero, $\underline{z}$ and $\underline{z}^*$ ($\bar{z}$ and $\bar{z}^*$) approaches the $\underline{z}$ ($\bar{z}$) for the case with only proportional costs. Furthermore, as the fixed cost F increases, $\underline{z}^*$ and $\bar{z}^*$ converge to z^* in the fixed cost case. This convergence occurs because as F becomes much larger than the proportional cost, α, the impact of transaction costs originates more and more from the fixed costs.

Figure 6 Here $\longrightarrow$

Figure 7 shows the optimal boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ as functions of the proportional cost rate for $F = \$5$. If $\alpha = 0.05$, for example, the investor will buy \$8,200 worth of stock when the actual amount of the investment reaches \$79,600. If the market goes up and the actual amount increases to \$171,900, the investor will sell \$13,500 worth of stock. As the proportional transaction cost increases, both the size of a purchase after reaching the buy boundary $\underline{z}$ and the size of a sale after reaching the sell boundary $\bar{z}$ decrease. In addition, as the proportional cost approaches zero, $\underline{z}^*$ and $\bar{z}^*$ approach the z^* for the case with only fixed costs.

Figure 7 Here $\longrightarrow$

VI. Fixed and Proportional Costs with Correlated Asset Returns

In this section, we extend the analysis in the previous sections to the case with correlated asset returns. We assume that the asset prices still evolve as in (1). However, we allow the correlations among the asset returns to be nonzero, i.e., $w_i(t)$ and $w_j(t)$ may have nonzero correlation. We denote the correlation between asset i return and asset j return as ρ_{ij}, with $\rho_{ii} = 1, \forall i = 1, 2, ..., n$. While we extend the logic of the previous section to conjecture the optimal policies in this case, we cannot make the formal statement analogous to Theorem 4.

Inside NT, the value function must satisfy the HJB equation:

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\rho_{ij}\sigma_i\sigma_j y_i y_j v_{y_i y_j}) + \sum_{i=1}^{n}(\mu_i y_i v_{y_i}) + rxv_x + \frac{v_x}{\beta}\log(\frac{v_x}{\beta}) - \delta v - \frac{v_x}{\beta} = 0. \tag{51}$$

We conjecture that

$$v(x, y_1, y_2, ..., y_n) = -\frac{1}{r}e^{-r\beta x - \varphi(r\beta y_1, ..., r\beta y_n)}, \tag{52}$$

for some function $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Let ψ be the restriction of φ in the no-transaction region. Then equation (51) becomes

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}[\rho_{ij}\sigma_i\sigma_j z_i z_j(\psi_{z_i z_j} - \psi_{z_i}\psi_{z_j})] + \sum_{i=1}^{n}(\mu_i z_i \psi_{z_i}) - r\psi + (\delta - r) = 0. \tag{53}$$

We conjecture that in this case, there exist four critical *functions* (instead of numbers, as in the previous section), $\underline{y}_i(y_{-i}), \underline{y}_i^*(y_{-i}), \bar{y}_i^*(y_{-i})$, and $\bar{y}_i(y_{-i})$, where $y_{-i} = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_n)$, defining the no-transaction region and the optimal target boundaries. Accordingly, we must have $\forall i = 1, 2, ..., n$,

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i - (\underline{y}_i^*(y_{-i}) - y_i), y_1, y_2, ..., \underline{y}_i^*(y_{-i}), ..., y_n)$$

for any $y_i \leq \underline{y}_i(y_{-i})$, and

$$v(x, y_1, y_2, ..., y_i, ..., y_n) = v(x - F_i + (1 - \alpha_i)(y_i - \bar{y}_i^*(y_{-i})), y_1, y_2, ..., \bar{y}_i^*(y_{-i}), ..., y_n)$$

for any $y_i \geq \bar{y}_i(y_{-i})$.

The optimality of $\underline{y}_i^*(y_{-i})$ and $\bar{y}_i^*(y_{-i})$ implies that

$$v_{y_i}(x, y_1, y_2, ..., \underline{y}_i^*(y_{-i}), ..., y_n) = v_x(x, y_1, y_2, ..., \underline{y}_i^*(y_{-i}), ..., y_n)$$

and

$$v_{y_i}(x, y_1, y_2, ..., \bar{y}_i^*(y_{-i}), ..., y_n) = (1 - \alpha_i)v_x(x, y_1, y_2, ..., \bar{y}_i^*(y_{-i}), ..., y_n),$$

for any $i = 1, 2, ..., n$.

Plugging equation (52) into the boundary conditions and using the C^1 property of v, we obtain the following eight boundary conditions:

$$\psi(z_1, ..., \underline{z}_i(z_{-i}), ..., z_n) = C_{i1}(z_{-i}) + \underline{z}_i(z_{-i}), \tag{54}$$

$$\psi_{z_i}(z_1, ..., \underline{z}_i(z_{-i}), ..., z_n) = 1, \tag{55}$$

$$\psi_{z_i}(z_1, ..., \underline{z}_i^*(z_{-i}), ..., z_n) = 1, \tag{56}$$

$$\psi_{z_i}(z_1, ..., \bar{z}_i(z_{-i}), ..., z_n) = 1 - \alpha_i, \tag{57}$$

$$\psi_{z_i}(z_1, ..., \bar{z}_i^*(z_{-i}), ..., z_n) = 1 - \alpha_i, \tag{58}$$

$$\psi(z_1, ..., \bar{z}_i(z_{-i}), ..., z_n) = C_{i2}(z_{-i}) + (1 - \alpha_i)\bar{z}_i(z_{-i}), \tag{59}$$

$$\psi(z_1, ..., \underline{z}_i^*(z_{-i}), ..., z_n) = C_{i1}(z_{-i}) + r\beta F_i + \underline{z}_i^*(z_{-i}) \tag{60}$$

and

$$\psi(z_1, ..., \bar{z}_i^*(z_{-i}), ..., z_n) = C_{i2}(z_{-i}) + r\beta F_i + (1 - \alpha_i)\bar{z}_i^*(z_{-i}), \tag{61}$$

for $i = 1, 2, ..., n$, where $z_{-i} = (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n)$, $\underline{z}_i = r\beta\underline{y}_i$, $\underline{z}_i^* = r\beta\underline{y}_i^*$, $\bar{z}_i^* = r\beta\bar{y}_i^*$, and $\bar{z}_i = r\beta\bar{y}_i$. We then need to solve for $\underline{z}_i, \underline{z}_i^*, \bar{z}_i^*, \bar{z}_i, C_{i1}$ and C_{i2} for all i, which are all functions of z_{-i}. This n-dimensional nonlinear PDE with $4n$ free boundaries is difficult to solve even numerically, especially when n is large. To get some idea on how correlation affects the no-transaction region and to see if the uncorrelated return case provides some useful insights into the correlated case, we use Algorithm 3 described in Appendix B, which is essentially the Projection Method introduced by Judd (1999), to numerically solve the two-asset case with a correlation of $\rho_{12} = 0.1$. Similar to Leland (2000), we assume that the four no-transaction boundaries and the four target boundaries (see Figure 5) are all straight lines.[12] Although this linearity has already significantly simplified the computation, we still need to optimally choose $16 + (m+1)(m+2)/2$ constants to minimize a test function, where m is the order of the series solution in the Projection Method. When the correlation is small, an order of two is generally sufficient ($m = 2$), which means we need to minimize over 22 constants. As n and m grow, the number of constants we need to minimize over grows quickly. In general, one needs to minimize over $n2^{n+1} + \sum_{j=0}^{m} \frac{(n+j-1)!}{j!(n-1)!}$ (which is equal to 796 when $n = 6, m = 2$) constants. This illustrates the extreme difficulty of computing the optimal boundaries in the correlated return case when n is large. Fortunately, when the correlation is small, as Figure 8 suggests, the solution to the uncorrelated return case provides a reasonable estimate of the optimal boundaries.

Figure 8 Here →

In Figure 8, we present the no-transaction region and the target boundaries for a two-stock example with ten percent correlation.[13] The dashed lines show the boundaries when the correlation is zero. This figure suggests that the boundaries for the uncorrelated return case are close to those of the correlated case. In addition, all the boundaries are negatively sloped. This is because in the presence of positive correlation, the two stocks have substitution effects for each other. Furthermore, compared to the boundaries in the uncorrelated return case, all the boundaries in the correlated return case move southwest respectively. This suggests that in the presence of positive correlation, one tends to invest less in each stock. This is because the diversification benefit of a stock is smaller when its return is correlated with another stock.

VII. Analysis of the Optimal Policy

One of the main reasons for investing in multiple risky assets is to reduce portfolio risk through diversification. There are asset classes that have nearly zero correlations and for diversification purposes investors may find it efficient to limit their trading to these uncorrelated asset classes. This suggests that from an economic point of view the uncorrelated return case is an important case to study. Therefore, in this section, we provide some further analysis of the optimal trading strategy in the uncorrelated return case. As shown in Sections III-V, analysis of the optimal policy for multiple stocks can be decomposed into analysis for individual stocks in this case. We thus (without loss of generality) pick one of the stocks and conduct the analysis on this stock. The location of free boundaries, the sensitivity of these boundaries to changes in risk aversion and volatility, the frequency of transaction, and the optimal size of a purchase and a sale are all examples of questions we will address in this section.

A. Optimal Boundaries

A.1 Changes in Risk Aversion

Figure 9 plots the optimal boundaries $\underline{y}$, $\underline{y}^*$, $\bar{y}^*$, and $\bar{y}$ (the actual amount, instead of the scaled amount) against the absolute risk aversion coefficient β. As the risk aversion increases, $\underline{y}$, $\underline{y}^*$, $\bar{y}^*$, and $\bar{y}$ all decrease. The amount of each purchase and sale also decreases. The target amounts $\underline{y}^*$ and $\bar{y}^*$ quickly converge to the Merton line. The pattern of the boundaries also suggests that, on average, the amount invested in the stock decreases as the investor becomes more risk averse.

Figure 9 Here →

A.2 Changes in Volatility

Figure 10 plots the optimal boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ against the stock return volatility, σ. As the volatility increases, $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ all decrease. In contrast to the intuition that as volatility increases, to save transaction costs, the investor would widen the no-transaction region, the NT region actually shrinks (see Subsection B for the implication on the frequency of trading). Furthermore, both $\underline{z}^*$ and $\bar{z}^*$ move closer to the Merton line, but the amount of each transaction is not very sensitive to changes in the volatility.

Figure 10 Here →

A.3 Changes in Expected Return

Figure 11 plots the optimal boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ against the expected stock return μ. As the expected return increases, $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ all increase. Both $\underline{z}$ and $\underline{z}^*$ increase at a lower rate than the Merton line, while $\bar{z}$ and $\bar{z}^*$ increase at a higher rate. This implies that the no-transaction region widens as the expected return rises. In addition, the size of each purchase and sale also increases.

Figure 11 Here →

B. Frequency of Trading

To better understand the optimal transaction policy in the stock, we now analyze the stochastic behavior of the investment in the stock in this subsection. Within the no-transaction region, the (scaled) amount in stock z evolves as follows:

$$dz_t = \mu z_t dt + \sigma z_t dw_t.$$

Now let $z_0 = z \in (\underline{z}, \bar{z})$ be fixed,

$$\tau = \inf\Big\{t \geq 0 : z_t \notin (\underline{z}, \bar{z})\Big\}$$

denote the time of the next transaction,

$$P_z(\tau < \infty) = P\Big(\tau < \infty \mid z_0 = z\Big)$$

denote the conditional probability that τ is finite, and

$$\mathrm{E}_z[\tau] = \mathrm{E}\Big[\tau \mid z_0 = z\Big]$$

denote the conditional expectation of τ.

The following proposition states that with positive probability the investor will transact in the stock, and that the expected time to the next transaction is always finite.

Proposition 2. *If $0 < \underline{z} < \bar{z} < \infty$, then $P_z(\tau < \infty) = 1$ and $\mathrm{E}_z[\tau] < \infty$ for all $z \in (\underline{z}, \bar{z})$. Moreover, both boundaries of the no-transaction region, $\underline{z}$ and $\bar{z}$, can be reached with positive probability.*

Proof. This follows immediately from the propositions in Section 5.5 of Karatzas and Shreve (1988). □

Since for the case in which $0 < \underline{z} < \bar{z} < \infty$ both boundaries can be reached in finite expected time, we can compute a set of measures of trading frequency; e.g., expected time to the next trade, expected time to the next sale after a purchase, and so on. Let

$$\tau_s = \inf\Big\{t \geq 0 : z_t = \bar{z}\Big\} \quad \text{and} \quad \tau_b = \inf\Big\{t \geq 0 : z_t = \underline{z}\Big\}$$

represent the first time z_t reaches the sell boundary $\bar{z}$ and the buy boundary $\underline{z}$ of the no-transaction region, respectively. Let

$$\mathrm{E}_z[\tau_s] = \mathrm{E}\Big[\tau_s \mid z_0 = z\Big] \quad \text{and} \quad \mathrm{E}_z[\tau_b] = \mathrm{E}\Big[\tau_b \mid z_0 = z\Big]$$

denote the conditional expectations of τ_s and τ_b, respectively.

Letting $T(z) = \mathrm{E}_z[\tau_s]$ and applying Itô's lemma, we find that T satisfies the following differential equation (cf. Karlin and Taylor (1981, p.192)):

$$\frac{1}{2}\sigma^2 z^2 T'' + \mu z T' + 1 = 0. \tag{62}$$

For the boundary conditions, first we note that obviously $T(\bar{z}) = 0$. Since the transaction policy is to jump to $\underline{z}^*$ from $\underline{z}$ as soon as $\underline{z}$ is reached, we must have the second boundary condition $T(\underline{z}) = T(\underline{z}^*)$. Solving the above ODE (62) subject to these two boundary conditions and following a similar procedure for $\mathrm{E}_z[\tau_b]$, we have the following result.

Proposition 3. *Suppose* $0 < \underline{z} < \bar{z} < \infty$. *Then*

$$E_z[\tau_s] = \begin{cases} \frac{\log(\bar{z}/z)(\underline{z}^{*k} - \underline{z}^k) - \log(\underline{z}^*/\underline{z})(\bar{z}^k - z^k)}{(\mu - \frac{1}{2}\sigma^2)(\underline{z}^{*k} - \underline{z}^k)} & \text{if } \mu \neq \frac{1}{2}\sigma^2 \\ \frac{1}{\sigma^2}\log(\frac{\bar{z}}{z})\log(\frac{\bar{z}z}{\underline{z}^*\underline{z}}) & \text{if } \mu = \frac{1}{2}\sigma^2 \end{cases}$$

and

$$E_z[\tau_b] = \begin{cases} \frac{\log(\underline{z}/z)(\bar{z}^{*k} - \bar{z}^k) - \log(\bar{z}^*/\bar{z})(\underline{z}^k - z^k)}{(\mu - \frac{1}{2}\sigma^2)(\bar{z}^{*k} - \bar{z}^k)} & \text{if } \mu \neq \frac{1}{2}\sigma^2 \\ \frac{1}{\sigma^2}\log(\frac{\underline{z}}{z})\log(\frac{\underline{z}z}{\bar{z}^*\bar{z}}) & \text{if } \mu = \frac{1}{2}\sigma^2, \end{cases}$$

where

$$k = 1 - \frac{2\mu}{\sigma^2}. \tag{63}$$

Figure 12 plots the expected time to the next sale after a sale and the expected time to the next purchase after a purchase against the proportional transaction cost rate. When $\alpha = 0.01$, on average, it takes about 1.2 years from sell to sell and about 2.5 years from buy to buy. As the transaction costs increase, the transaction frequency decreases and the difference between the expected time from buy to buy and the expected time from sell to sell also becomes greater.

Figure 12 Here

A wealth of literature exists on stock return predictability (e.g., Kandel and Stambaugh (1996), Barberis (2000), Xia (2001)). Generally, it is found that incorporating predictability would significantly increase the welfare of an investor, even in the presence of parameter uncertainty. However, most of these studies do not take transaction costs into account. The large deviation of trading policy in the presence of transaction costs from optimal policy in the absence of transaction costs implies, as found in the above analysis, a very low frequency of trading. This infrequency of trading seems to suggest that the gain from incorporating predictability would be significantly decreased if transaction costs were considered. We will return to this point later. This finding of low trading frequency in the presence of transaction costs also has some implications for models of trading volume. Since transaction costs have dramatic effects on both trading frequency and trading size, to explain the observed trading volume, it seems that one has to consider transaction costs in addition to other standard factors considered in the literature (e.g., Admati and Pfleiderer (1988) and Wang (1994)) such as information asymmetry and heterogeneous beliefs.

Figures 13, 14, and 15 plot the expected time to the next sale after a sale and the expected time to the next purchase after a purchase against the absolute risk aversion coefficient β, stock return volatility σ and the expected return μ, respectively. As the investor becomes more risk averse, the frequency of trading decreases. The expected time between purchases increases faster than the expected time between sales. It should be noted, however, that although the no-transaction region narrows as β increases, the trading frequency decreases. This inverse correlation suggests that trading frequency not only depends on the width of the no-transaction region but also on the location of the NT region.

Figure 13
Figure 14
Figure 15
Here
$\longrightarrow$

As the stock return volatility increases, while the expected time between sales increases, the expected time between purchases decreases. This finding seems counterintuitive, because as the volatility increases it seems probable that the investor would widen the no-transaction region to decrease the trading frequency so as to save on the transaction costs. However, the investor's response is more sophisticated than simply saving transaction costs. As the volatility increases, the risk increases. So, on average, the investor holds less in the stock (as suggested by Figure 10 and can be verified using the measure developed in the next subsection). Over time then, the investor needs to sell less frequently to finance current consumption and actually needs to buy stock more often to finance future consumption.

As the expected stock return increases, both the expected time between purchases and the expected time between sales decrease, but the expected time between sales decreases faster. Again, it should be noted that the frequency of transaction is not determined only by the width of the no-transaction region (Figure 11 shows that the region widens as μ grows). With the risk premium increasing from five percent to nine percent, the expected time between sales reduces from 1.4 years to about seven months.

C. Average Amount Invested in Stock

In this subsection, we compute a measure of the average amount the investor would optimally hold in the risky asset. When $0 < \underline{z} < \bar{z} < \infty$, the expected time to reach either boundary is finite; therefore, it follows that z is a positively recurrent process. Let k be as defined in equation (63) and

$$G(x,\xi) = \begin{cases} 2(x^k - \underline{z}^k)(\bar{z}^k - \xi^k)\xi^{-1-k}/[\sigma^2(\bar{z}^k - \underline{z}^k)] & \text{if } \underline{z} \le x \le \xi \le \bar{z} \\ 2(\bar{z}^k - x^k)(\xi^k - \underline{z}^k)\xi^{-1-k}/[\sigma^2(\bar{z}^k - \underline{z}^k)] & \text{if } \underline{z} \le \xi \le x \le \bar{z} \end{cases}$$

be the Green function of z inside the no-transaction region. We focus on the case with both fixed and proportional costs. Then

$$f(z) = \frac{(\bar{z}^k - \bar{z}^{*k})G(\underline{z}^*, z) + (\underline{z}^{*k} - \underline{z}^k)G(\bar{z}^*, z)}{\int_{\underline{z}}^{\bar{z}}((\bar{z}^k - \bar{z}^{*k})G(\underline{z}^*, \eta) + (\underline{z}^{*k} - \underline{z}^k)G(\bar{z}^*, \eta))d\eta}$$

is the stationary (or steady-state) probability density function (cf. Karlin and Taylor (1981), p. 381).

Figure 16 shows the typical shape of the stationary density function. As expected, significant mass falls around the optimal targets $\underline{z}^*$ and $\bar{z}^*$, because these are the points to which the investor must return after reaching the transaction boundaries. Since $\mu > 0$, there is greater mass around $\bar{z}^*$ than around $\underline{z}^*$.

Figure 16 Here $\rightarrow$

Using the stationary distribution, we can compute the average amount invested in the stock in the steady state (as t approaches ∞). Figure 17 shows the steady-state average amount invested in the stock as a function of the proportional transaction cost rate α. Surprisingly, the average amount invested in the stock increases as the transaction costs increase. As transaction costs increase, to save on such costs, the investor widens the no-transaction region. The tension occurs between investing more on average versus transacting more often to keep a lower average but paying higher transaction costs. In this case, saving transaction costs is dominant. Figure 18 shows the steady-state average amount invested in the stock as a function of the fixed transaction cost F. Again, as transaction costs increase, the average amount increases. However, the increase in the average amount as the fixed cost increases from \$0 to \$30 is small compared to that shown in Figure 17, because the fixed cost is small compared to the actual transaction size of \$22,000 when $F = \$30$. That the steady state average amount invested in the stock increases as transaction costs increase suggests that to induce an investor to hold the same average amount as before, one needs to make the stock less attractive, for example by *lowering* the expected return of the stock. Figure 19 shows the expected returns that induce the investor to hold the same steady state average amount as that in the absence of transaction costs as a function of the proportional cost rate α when the fixed cost is \$5. Consistent with the above analysis, the expected return of the stock that implies the same average amount in stock is inversely related to the transaction costs. In addition, this relationship is almost linear in this range of the transaction costs.

Figure 17
Figure 18
Figure 19
Here

As already shown, in the presence of transaction costs, optimal trading occurs infrequently. To measure how much the investor loses from trading at a higher frequency than the optimal one and to be consistent with the convention of using extra risk premium to measure utility gain from incorporating predictability, we compute the extra risk premium required to compensate the investor for trading more frequently than the optimal trading strategy. Specifically, we suppose that the investor shrinks the optimal buy boundary and sell boundary symmetrically about the mid-point of the no-transaction region, but then chooses optimally the buy and sell targets. This change would imply an increase in trading frequency and a loss of utility. Figure 20 plots the extra risk premium required against the average time between transactions in the steady state. This figure shows that with a monthly trading frequency, the investor would need about 25 basis points extra premium. With a daily trading frequency, the extra premium required would be as high as 300 basis points. These numbers seem to suggest that the importance of predictability as reported in the literature would be significantly diminished if transaction costs were taken into account.

Figure 20 Here →

VIII. Conclusions and Extensions

In this paper we consider the optimal intertemporal consumption and investment policy of an infinite-horizon CARA investor, who faces both fixed and proportional transaction costs in trading multiple risky assets. We find that in the presence of even small transaction costs, trading in the risky assets becomes infrequent and increasing trading frequency beyond the optimal frequency results in significant utility loss. These findings suggest that transaction costs are an important factor in affecting trading volume, and the importance of stock return predictability as reported in the literature would be significantly diminished if transaction costs were taken into account. In addition, we find that conditional on investment, as transaction costs increase, the average amount invested in each risky asset increases.

Compared to the existing literature, this paper provides a simple model that makes it feasible to compute the optimal trading strategies when there are large number of risky assets subject to both fixed and proportional transaction costs. Incorporating more realistic market features such as stochastic investment opportunities, portfolio constraints, exogenous income, correlated asset returns, and incomplete information would be economically interesting but mathematically challenging for future research.

Appendix A

In the first part of this appendix, we provide a proof of Theorem 4. The proofs of Theorem 2 and Theorem 3 are special cases and are thus omitted. Since the investor's problem involves continuous consumption and discrete stock transactions at stopping times, this optimal control problem belongs to the class of *combined stochastic control* as studied by Brekke and Øksendal (1998).[14] In contrast to Brekke and Øksendal (1998), the investor in this model has an infinite horizon. The proof in this appendix is a variation of the proofs in Brekke and Øksendal (1998) and Korn (1998). We first introduce some notation and terminology, then provide a modified version of the verification theorems of Brekke and Øksendal (1998) and Korn (1998) and finally show that the conditions provided in Theorem 4 satisfy the conditions in this verification theorem.

Definition 1 *An impulse control* $\chi = \{(\tau_j, \zeta^j), j \in \mathbb{N}\}$ *is a sequence of trading times* τ_j *and trading amounts* $\zeta^j = dI_{\tau_j} - dD_{\tau_j} \in \mathbb{R}^n$ *such that* $\forall j \in \mathbb{N}$*,*

1. $0 \le \tau_j \le \tau_{j+1}$ *a.s.,*

2. τ_j *is a stopping time and* ζ^j *is* $\mathcal{F}_{\tau_j}$ *measurable, and*

3. $P(\lim_{n\to\infty} \tau_n \le K) = 0, \forall K \ge 0,$

where $\mathbb{N}$ *denotes the set of natural numbers, and* I *and* D *are the cumulative purchase and sale processes respectively.*

Definition 2 *For a given impulse control* χ *and a consumption policy* c*, the pair* $\xi = (\chi, c)$ *is called a combined stochastic control.*

Definition 3 *A combined stochastic control* $\xi = (\chi, c)$ *is admissible if the implied processes* (I, D) *and* c *form an admissible strategy as defined in the text; i.e., the implied* x_t *and* y_t *from (2) and (3) satisfy (5). Let* $\mathcal{W}$ *denote the set of admissible combined stochastic controls.*

Next, we let $\mathcal{H}$ denote the space of all measurable functions $h : \mathbb{R}^{n+1} \to \mathbb{R}$. We define the maximum operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{M}h(x, y) \equiv \sup_{\zeta \in \mathbb{R}^n \setminus \{0\}} h\Big(x - \sum_{i=1}^{n} (F_i 1_{\{\zeta_i \neq 0\}} + \zeta_i^+ - (1 - \alpha_i)\zeta_i^-), y + \zeta\Big),$$

where ζ_i is the ith element of ζ. For each $(x, y) \in \mathbb{R}^{n+1}$, let $\hat{\zeta}^h(x, y)$ be such that

$$\mathcal{M}h(x, y) = h\Big(x - \sum_{i=1}^{n} (F_i 1_{\{\hat{\zeta}_i^h(x,y) \neq 0\}} + \hat{\zeta}_i^h(x, y)^+ - (1 - \alpha_i)\hat{\zeta}_i^h(x, y)^-), y + \hat{\zeta}^h(x, y)\Big). \quad \text{(A1)}$$

For a given consumption policy c, we next define the differential operator $\mathcal{L}^c$ by

$$\mathcal{L}^c g(x, y) \equiv \frac{1}{2}\sigma^2 y^2 g_{yy} + \mu y g_y + r x g_x - c g_x - \delta g,$$

for all functions $g : \mathbb{R}^{n+1} \to \mathbb{R}$ for which the derivatives involved exist at (x, y). We now provide a lemma which serves as a verification theorem for solving the investor's problem. It provides sufficient conditions under which a combined stochastic control $\xi = (\chi, c)$ solves the investor's optimal consumption and investment problem, and a given function V is the value function.

Lemma 1. *(Verification Theorem)*

(a). Suppose there exists a C^1 function $V : \mathbb{R}^{n+1} \to \mathbb{R}$, which is C^2 except over a Lebesgue measure zero subset of $\mathbb{R}^{n+1}$, such that

1. $\mathcal{L}^c V(x, y) + u(c) \leq 0, \forall c \in \mathcal{C}$;

2. $V(x, y) \geq \mathcal{M}V(x, y)$;

3. $$\forall T \in [0, \infty), \quad E \int_0^T |e^{-\delta s} y_s V_Y(x_s, y_s)|^2 ds < \infty \quad \text{(A2)}$$

   and

   $$\lim_{T \to \infty} E[e^{-\delta T} V(x_T, y_T)] = 0, \quad \text{(A3)}$$

   for any process (x_t, y_t) corresponding to an admissible combined stochastic control, where V_Y is the $n \times n$ diagonal matrix with V_{y_i} $(i = 1, 2, ..., n)$ as its diagonal elements and $|\cdot|$ represents the Euclidian norm; and, *4.* $\{e^{-\delta t}V(x_t, y_t)\}_{t\geq 0}$ *is uniformly integrable.*

Then

$$V(x,y) \geq v^{\xi}(x,y), \forall \xi \in \mathcal{W}, (x,y) \in I\!R^{n+1}, \tag{A4}$$

where $v^{\xi}(x,y)$ *is the value function from following* ξ*.*

(b). Define

$$NT = \{(x,y) : V(x,y) > \mathcal{M}V(x,y)\}.$$

Suppose in addition to the conditions in part (a), there exists a function $\hat{c} : NT \to I\!R$ *such that*

$$\mathcal{L}^{\hat{c}(x,y)}V(x,y) + u(\hat{c}(x,y)) = 0, \tag{A5}$$

for all $(x,y) \in NT$*. Define the impulse control*

$$\hat{\chi} \equiv (\hat{\tau}_1, \hat{\tau}_2, ...; \hat{\zeta}^1, \hat{\zeta}^2, ...)$$

inductively as follows: $\hat{\tau}_0 = 0$ *and* $\forall k = 0, 1, 2, ...,$

$$\hat{\tau}_{k+1} = \inf\{t > \hat{\tau}_k : (x_t^{(k)}, y_t^{(k)}) \notin NT\}$$

and

$$\hat{\zeta}^{k+1} = \hat{\zeta}^V(x_t^{(k)}, y_t^{(k)}),$$

where $(x_t^{(k)}, y_t^{(k)})$ *is the result of applying the combined stochastic control*

$$\hat{\xi}_k \equiv ((\hat{\tau}_1, ..., \hat{\tau}_k; \hat{\zeta}^1, ..., \hat{\zeta}^k), \hat{c})$$

and $\hat{\zeta}^V$ *is as defined in (A1) for* V*. If* $\hat{\xi} \equiv (\hat{\chi}, \hat{c})$ *is admissible, then*

$$V(x,y) = v(x,y)$$

and the combined stochastic control $\xi^* = \hat{\xi}$ *is optimal, where* $v(x,y)$ *is the value function defined in (6).*

Proof. (a) Assuming that V satisfies the conditions in part (a), we let $\xi = (\chi, c) \in \mathcal{W}$ be any admissible combined stochastic control, where

$$\chi = (\tau_1, \tau_2, ...; \zeta^1, \zeta^2, ...).$$

Let $T \in [0, \infty)$ be fixed. For all $k \geq 0$, define

$$\theta_k = \tau_k \wedge T,$$

with $\tau_0 = 0$ and let $(x_t, y_t) = (x_t^\xi, y_t^\xi)$. We can then write for every $n \in \mathbb{N}$,

$$\begin{aligned}
&e^{-\delta\theta_n} V(x_{\theta_n}, y_{\theta_n}) - V(x, y) \\
&= \sum_{i=1}^{n} [e^{-\delta\theta_i} V(x_{\theta_i^-}, y_{\theta_i^-}) - e^{-\delta\theta_{i-1}} V(x_{\theta_{i-1}}, y_{\theta_{i-1}})] \\
&+ \sum_{i=1}^{n} 1_{\{\tau_i < T\}} e^{-\delta\theta_i} [V(x_{\theta_i}, y_{\theta_i}) - V(x_{\theta_i^-}, y_{\theta_i^-})].
\end{aligned} \tag{A6}$$

Since y_t is a continuous semi-martingale in the stochastic interval $[\theta_k, \theta_{k+1})$ and V is C^2 except over a Lebesgue measure zero subset of $\mathbb{R}^{n+1}$ and C^1 in $\mathbb{R}^{n+1}$, Lemma (45.9) (a generalized version of Itô's lemma) of Rogers and Williams (2000) applies (see also Korn (1997)). Therefore, $\forall i \in \mathbb{N}$, we have

$$\begin{aligned}
&e^{-\delta\theta_i} V(x_{\theta_i^-}, y_{\theta_i^-}) - e^{-\delta\theta_{i-1}} V(x_{\theta_{i-1}}, y_{\theta_{i-1}}) \\
&= \int_{\theta_{i-1}}^{\theta_i} e^{-\delta s} \mathcal{L}^c V(x_s, y_s) ds + \int_{\theta_{i-1}}^{\theta_i} e^{-\delta s} y_s V_Y(x_s, y_s) \sigma dw_s,
\end{aligned} \tag{A7}$$

where σ is a $n \times n$ diagonal matrix with σ_i $(i = 1, 2, ..., n)$ as its elements. By condition 1, we have

$$\begin{aligned}
&e^{-\delta\theta_i} V(x_{\theta_i^-}, y_{\theta_i^-}) - e^{-\delta\theta_{i-1}} V(x_{\theta_{i-1}}, y_{\theta_{i-1}}) \\
&\leq - \int_{\theta_{i-1}}^{\theta_i} u(c_s, s) ds + \int_{\theta_{i-1}}^{\theta_i} e^{-\delta s} y_s V_Y(x_s, y_s) \sigma dw_s.
\end{aligned} \tag{A8}$$

By condition 2, we have

$$V(x_{\theta_i}, y_{\theta_i}) - V(x_{\theta_i^-}, y_{\theta_i^-}) \leq 0. \tag{A9}$$

Combining (A6)-(A9) and taking expectations, we then get

$$V(x, y) \geq E[e^{-\delta\theta_n} V(x_{\theta_n}, y_{\theta_n}) + \sum_{i=0}^{n} \Big(\int_{\theta_{i-1}}^{\theta_i} u(c_s, s) ds - \int_{\theta_{i-1}}^{\theta_i} e^{-\delta s} y_s V_Y(x_s, y_s) \sigma dw_s \Big)]. \tag{A10}$$

By (A2), for any fixed n, we have

$$E[\int_0^{\theta_n} e^{-\delta s} y_s V_Y(x_s, y_s) \sigma dw_s] = 0.$$

From condition 3 in Definition 1 and condition 4 in this lemma, we have

$$\lim_{n\to\infty} E[e^{-\delta\theta_n}V(x_{\theta_n}, y_{\theta_n})] = E[e^{-\delta T}V(x_T, y_T)].$$

Therefore, taking the limit as $n \to \infty$ in (A10) and using the monotone convergence theorem, we have

$$V(x,y) \geq E[e^{-\delta T}V(x_T, y_T)] + E[\int_0^T u(c_s, s)ds].$$

Taking the limit as $T \to \infty$ and using (A3) and the monotone convergence theorem, we obtain

$$V(x,y) \geq E[\int_0^\infty u(c_s, s)ds],$$

for all $\xi \in \mathcal{W}$ and thus $V(x,y) \geq v^\xi(x,y)$.

(b) By (A5), we have equality (rather than inequality) in (A8). Given the definition of $\hat{\xi}$ we also have equality in (A9). Combining this with (A4), we then get

$$V(x,y) \geq \sup_{\xi\in\mathcal{W}} v^\xi(x,y) \geq v^{\hat{\xi}}(x,y) = V(x,y).$$

Hence $V(x,y) = v(x,y)$ and $\xi^* = \hat{\xi}$ is optimal. □

Since one of the conditions in the above verification theorem is that $\hat{\xi}$ is an admissible combined stochastic control, we next show that the combined stochastic control implied by the consumption policy and trading strategy specified in Theorem 4 is indeed admissible.

Lemma 2. *Let $\hat{\xi} = (\hat{\chi}, \hat{c})$ represent the stock trading strategy specified in Theorem 4. Then $\hat{\xi}$ is an admissible combined stochastic control.*

Proof. Let τ_j, $j \in \mathbb{N}$ denote the time when the investor trades according to the policy specified in Theorem 4. Since the prescribed stock trading strategy is to trade stock i whenever y_{it} is outside $(\underline{y}_i, \bar{y}_i)$, the trading time is clearly a stopping time with $0 \leq \tau_j \leq \tau_{j+1}$ a.s., $\forall j \in \mathbb{N}$. For all $j \in \mathbb{N}$, define

$$\hat{\zeta}_i^j = \begin{cases} \underline{y}_i^* - y_{i\tau_j} & \text{if } y_{i\tau_j} \leq \underline{y}_i \\ \bar{y}_i^* - y_{i\tau_j} & \text{if } y_{i\tau_j} \geq \bar{y}_i \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\hat{\zeta}^j$ is adapted to $\mathcal{F}_{\tau_j}$. Because $\forall t \in (0,\infty)$, $P\{z_{it} \in [\underline{z}_i, \bar{z}_i]\} = 1$, and z_{it} is positively recurrent by Proposition 2 for $i = 1, 2, ..., n$, we find that $P(\lim_{m\to\infty} \tau_m \leq K) = 0, \forall K \geq 0$ and thus condition 3 in Definition 1 is also satisfied. To complete the proof we now show that (5) is also satisfied. For all $t > 0$ and $m \in \mathbb{N}$, by (2) and (48), we have

$$\begin{aligned} r\beta x_{t\wedge\tau_m} &= r\beta x_0 - \sum_{i=1}^{n}\int_0^{t\wedge\tau_m} r\psi_i(z_{is})ds \\ &+ \sum_{i=1}^{n}\sum_{j=0}^{m} 1_{\{\tau_j<t\}}(-r\beta F_i 1_{\{|\hat{\zeta}_i^j|>0\}} - r\beta\hat{\zeta}_i^{j+} + (1-\alpha_i)r\beta\hat{\zeta}_i^{j-}), \end{aligned}$$

where $\hat{\zeta}_i^j$ is the ith element of $\hat{\zeta}^j$. By (37) and (42)-(44), we find that

$$-r\beta F_i 1_{\{|\hat{\zeta}_i^j|>0\}} - r\beta\hat{\zeta}_i^{j+} + (1-\alpha_i)r\beta\hat{\zeta}_i^{j-} = \psi_i(z_{i\tau_{j-}}) - \psi_i(z_{i\tau_j}),$$

where at time $\tau_j > 0$: if it is a purchase in ith stock, then $z_{i\tau_{j-}} = \underline{z}_i$ and $z_{i\tau_j} = \underline{z}_i^*$; if it is a sale then $z_{i\tau_{j-}} = \bar{z}_i$ and $z_{i\tau_j} = \bar{z}_i^*$; if there is no trade in the stock (i.e., $\hat{\zeta}_i^j = 0$) then $z_{i\tau_{j-}} = z_{i\tau_j}$. We then have

$$\begin{aligned} \sum_{j=0}^{m} 1_{\{\tau_j<t\}}(\psi_i(z_{i\tau_{j-}}) - \psi_i(z_{i\tau_j})) &= \psi_i(z_{i,0}) - \psi_i(z_{i,t\wedge\tau_m}) \\ &+ \sum_{j=1}^{m}[\psi_i(z_{i,t\wedge\tau_{j-}}) - \psi_i(z_{i,t\wedge\tau_{j-1}})]. \end{aligned}$$

Since ψ_i is a solution of (22) subject to (37)-(44) and (22) satisfies the conditions of Corollary 4.1 of Hartman (1964), $\varphi_i(z_i)$ as defined in (23) is C^2 except at $\{\underline{z}_i, \bar{z}_i\}$ (a Lebesgue measure zero set) and C^1 at these points. Using the generalized version of Itô's lemma, we then obtain

$$\begin{aligned} \psi_i(z_{i,t\wedge\tau_{j-}}) - \psi_i(z_{i,t\wedge\tau_{j-1}}) &= \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_{j-}} (\frac{1}{2}\sigma_i^2 z_{is}^2\psi_i'' - \frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 + \mu_i z_{is}\psi_i')ds \\ &+ \int_{t\wedge\tau_{j-1}}^{t\wedge\tau_{j-}} [\frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 ds + \sigma_i z_{is}\psi_i' dw_{is}]. \end{aligned}$$

Therefore

$$\begin{aligned} r\beta x_{t\wedge\tau_m} &= r\beta x_0 + \sum_{i=1}^{n}\Bigg(\psi_i(z_{i,0}) - \psi_i(z_{i,t\wedge\tau_m}) \\ &+ \int_0^{t\wedge\tau_m}(\frac{1}{2}\sigma_i^2 z_{is}^2\psi_i'' - \frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 + \mu_i z_{is}\psi_i' - r\psi_i)ds \\ &+ \int_0^{t\wedge\tau_m}[\frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 ds + \sigma_i z_{is}\psi_i' dw_{is}]\Bigg). \end{aligned}$$

By (22), we then have

$$\begin{aligned} r\beta x_{t\wedge\tau_m} &= r\beta x_0 + \sum_{i=1}^{n}(\psi_i(z_{i,0}) - \psi_i(z_{i,t\wedge\tau_m})) - (\delta - r)(t\wedge\tau_m) \\ &+ \sum_{i=1}^{n}\int_0^{t\wedge\tau_m}[\frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 ds + \sigma_i z_{is}\psi_i' dw_{is}]. \end{aligned} \tag{A11}$$

Taking the limit as $m \to \infty$ on both sides of (A11), by condition 3 in Definition 1 shown above, we get

$$r\beta x_t = r\beta x_0 + \sum_{i=1}^{n}(\psi_i(z_{i,0}) - \psi_i(z_{i,t})) - (\delta - r)t + \sum_{i=1}^{n}\int_0^t[\frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 ds + \sigma_i z_{is}\psi_i' dw_{is}].$$

By (4) we have $\forall t \in [0,\infty)$,

$$e^{-\delta t - r\beta W_t} = e^{-rt - r\beta x_0 - \sum_{i=1}^{n}[\psi_i(z_{i,0}) - \psi_i(z_{i,t}) + (1-\alpha_i)z_{it}^+ - z_{it}^- - F_i 1_{\{z_{it}\neq 0\}}]}N(t),$$

where

$$N(t) = e^{-\sum_{i=1}^{n}\int_0^t[\frac{1}{2}\sigma_i^2 z_{is}^2(\psi_i')^2 ds + \sigma_i z_{is}\psi_i' dw_{is}]}.$$

Since $\forall t \in [0,\infty)$, z_{it}, $\psi_i(z_{it})$, and $\psi_i'(z_{it})$ are all bounded and $E[N(t)] = 1$, we obtain,

$$0 \le \lim_{t\to\infty} E[e^{-\delta t - r\beta W_t}] \le \lim_{t\to\infty}[Ke^{-rt}E(N(t))] = 0,$$

where K is some finite constant. This shows that the first part of (5) holds, i.e.,

$$\lim_{t\to\infty} E[e^{-\delta t - r\beta W_t}] = 0.$$

In addition, since $\forall t \in [0,\infty)$, z_{it} and $\psi_i(z_{it})$ are both bounded, we have for some finite constant K_1,

$$e^{-2\delta t - 2r\beta W_t} = e^{-2rt - 2r\beta x_0 - 2\sum_{i=1}^{n}[\psi_i(z_{i,0}) - \psi_i(z_{i,t}) + (1-\alpha_i)z_{it}^+ - z_{it}^- - F_i 1_{\{z_{it}\neq 0\}}]}N(t)^2 < K_1 e^{-2rt}N(t)^2.$$

Since $\forall t \in [0,\infty)$, z_{it} and $\psi_i'(z_{it})$ are also bounded, we obtain $E[N(t)^2] < e^{K_2 t}$, for some finite constant K_2. Therefore, we have $\forall i = 1, 2, ..., n$ and $T \in [0,\infty)$,

$$E[\int_0^T |y_{it}e^{-\delta t - r\beta W_t}|^2 dt] = \frac{1}{r\beta}E[\int_0^T |z_{it}e^{-\delta t - r\beta W_t}|^2 dt] < \infty.$$

This shows that the second part of (5) is also satisfied. $\square$

We are now ready to prove Theorem 4.

Proof of Theorem 4. In this proof, we show that all the conditions in the verification theorem Lemma 1 are satisfied. First, by Lemma 2, the combined stochastic control proposed in Theorem 4 is admissible. Also, as explained in the proof of Lemma 2, the φ_i's are C^2 except over a Lebesgue measure zero subset of $\mathbb{R}$ and C^1 in $\mathbb{R}$ and thus $v(x, y)$ are C^2 except over a Lebesgue measure zero subset of $\mathbb{R}^{n+1}$ and C^1 in $\mathbb{R}^{n+1}$. Next, we show that (A2) and (A3) hold with the proposed value function.

First recall that

$$v(x, y_1, ..., y_n) = -\frac{1}{r} e^{-r\beta x - \sum_{i=1}^n \varphi_i(r\beta y_i)}$$

and

$$v_{y_i}(x, y_1, ..., y_n) = \beta \varphi_i' e^{-r\beta x - \sum_{i=1}^n \varphi_i(r\beta y_i)}, \forall i = 1, 2, ..., n.$$

By (4) we have $\forall t \in [0, \infty)$,

$$0 \leq e^{-\delta t - r\beta x_t - r\beta \sum_{i=1}^n (y_{it}^+ - y_{it}^-)} \leq e^{-\delta t - r\beta x_t - r\beta \sum_{i=1}^n [(1-\alpha_i) y_{it}^+ - y_{it}^- - F_i 1_{\{y_{it} \neq 0\}}]} = e^{-\delta t - r\beta W_t}. \tag{A12}$$

Taking the expectation and the limit, (5) then directly implies that

$$\lim_{t \to \infty} E[e^{-\delta t - r\beta(x_t + \sum_{i=1}^n y_{it})}] = 0. \tag{A13}$$

Since $\forall z_i < \underline{z}_i$, $\varphi_i(z_i) = C_{i1} + z_i$ and $\forall z_i, \varphi_i'(z_i) > 0$ according to the conditions in the theorem, we then have

$$0 \leq e^{-\delta T - r\beta x_T - \sum_{i=1}^n \varphi_i(r\beta y_{iT})} \leq e^{-\delta T - r\beta(x_T + \sum_{i=1}^n y_{iT}) - \sum_{i=1}^n C_{i1}}, \tag{A14}$$

and thus taking the expectation and the limit as $T \to \infty$ we have (A3) by (A13); i.e.,

$$\lim_{T \to \infty} E[e^{-\delta T} v(x_T, y_T)] = \lim_{T \to \infty} -\frac{1}{r} E[e^{-\delta T - r\beta x_T - \sum_{i=1}^n \varphi_i(r\beta y_{iT})}] = 0. \tag{A15}$$

The above expression implies that for any fixed $t \geq 0$,

$$E[|e^{-\delta t} v(x_t, y_t)|] < \infty \tag{A16}$$

and thus $e^{-\delta t} v(x_t, y_t)$ is in L^1. In addition, (A15) also implies that $e^{-\delta t} v(x_t, y_t)$ converges to 0 in L^1. By Theorem 13.7 in Williams (1994), we have that condition 4 in Lemma 1 holds.

For all $T \in [0, \infty)$, we then have for some finite constants $K_1, K_2 > 0$,

$$\begin{aligned} E\int_0^T |e^{-\delta t} y_t v_y(x_t, y_t)|^2 dt &= E\int_0^T [r\beta(\sum_{i=1}^n y_{it}\varphi_i')e^{-\delta t} v(x_t, y_t)]^2 dt \\ &< E\int_0^T K_1 \sum_{i=1}^n y_{it}^2 |e^{-\delta t} v(x_t, y_t)|^2 dt \\ &< E\int_0^T K_2 \sum_{i=1}^n |y_{it} e^{-\delta t - r\beta W_t}|^2 dt \\ &< \infty, \end{aligned}$$

where the first inequality stems from the fact that φ_i' is bounded, the second inequality follows from (A12) and (A14) and the last inequality follows from (5). Therefore (A2) also holds. Next, defining

$$\begin{aligned} \mathcal{G}v(x,y) &\equiv \frac{1}{2}\sigma^2 y^2 v_{yy} + \mu y v_y + r x v_x + \frac{v_x}{\beta}\log(\frac{v_x}{\beta}) - \delta v - \frac{v_x}{\beta} \\ &= \Big(\sum_{i=1}^n (\frac{1}{2}\sigma_i^2 z_i^2 \varphi_i'' - \frac{1}{2}\sigma_i^2 z_i^2 \varphi_i'^2 + \mu_i z_i \varphi_i' - r\varphi_i) + (\delta - r)\Big)|v(x,y)|, \end{aligned}$$

we then have for an arbitrary consumption policy c,

$$\mathcal{L}^c v(x,y) + u(c) \le \max_c(\mathcal{L}^c v(x,y) + u(c)) = \mathcal{L}^{c^*} v(x,y) + u(c^*) = \mathcal{G}v(x,y),$$

where the first equality follows from the optimality of c^* defined in (48) for a given $v(x,y)$, which is straightforward to verify. By (22) and summing up over i, we then have $\mathcal{G}v(x,y) = 0$ in NT and thus (A5) is satisfied in NT with $\hat{c} = c^*$. By (38) and (45), we must have $\psi_i''(\underline{z}_i) > 0$. By (40) and (47), we must have $\psi_i''(\bar{z}_i) > 0$. By (22) and plugging in (37), (38), (40) and (42) , we then have

$$-\frac{1}{2}\sigma_i^2 \underline{z}_i^2 + (\mu_i - r)\underline{z}_i - rC_{i1} + \frac{\delta - r}{n} \le 0 \tag{A17}$$

and

$$-\frac{1}{2}\sigma_i^2((1-\alpha_i)\bar{z}_i)^2 + (\mu_i - r)(1-\alpha_i)\bar{z}_i - rC_{i2} + \frac{\delta - r}{n} \le 0, \tag{A18}$$

for $i = 1, 2, ..., n$. Equations (A17) and (A18) then, respectively, imply that $\forall z_i \le \underline{z}_i$,

$$-\frac{1}{2}\sigma_i^2 z_i^2 + (\mu_i - r)z_i - rC_{i1} + \frac{\delta - r}{n} \le 0$$

and $\forall z_i \ge \bar{z}_i$,

$$-\frac{1}{2}\sigma_i^2((1-\alpha_i)z_i)^2 + (\mu_i - r)(1-\alpha_i)z_i - rC_{i2} + \frac{\delta - r}{n} \le 0,$$

for $i = 1, 2, ..., n$. Summing over i and noting the fact that $\varphi_i''(z_i) = 0$ outside NT, we then have $\mathcal{L}^c v(x, y) + u(c) \leq \mathcal{G}v(x, y) \leq 0$ outside the NT region. Therefore, condition 1 in part (a) also holds. Next, we show that condition 2 in part (a) is true.

First,

$$\mathcal{M}v(x, y) = -\frac{1}{r} e^{-r\beta x - \sum_{i=1}^{n} \nu_i(y_i)}, \tag{A19}$$

where

$$\nu_i(y_i) \equiv \sup_{\zeta_i} (\varphi_i(r\beta(y_i + \zeta_i)) + (1 - \alpha_i) r\beta\zeta_i^- - r\beta\zeta_i^+ - r\beta F_i 1_{\{\zeta_i \neq 0\}}),$$

where $\zeta_i \neq 0$ for at least one i. Conditional on a trade, by (45)-(47), we have

$$\nu_i(y_i) = \begin{cases} \psi_i(r\beta \underline{y}_i^*) - r\beta(\underline{y}_i^* - y_i) - r\beta F_i & \text{if } y_i < \underline{y}_i^* \\ \psi_i(r\beta y_i) - r\beta F_i & \text{if } \underline{y}_i^* \leq y_i \leq \bar{y}_i^* \\ \psi_i(r\beta \bar{y}_i^*) + r\beta(1 - \alpha_i)(y_i - \bar{y}_i^*) - r\beta F_i & \text{if } y_i > \bar{y}_i^*. \end{cases} \tag{A20}$$

By (43) and (44), we find that (A20) becomes

$$\nu_i(y_i) = \begin{cases} C_{i1} + r\beta y_i & \text{if } y_i < \underline{y}_i^* \\ \psi_i(r\beta y_i) - r\beta F_i & \text{if } \underline{y}_i^* \leq y_i \leq \bar{y}_i^* \\ C_{i2} + (1 - \alpha_i) r\beta y_i & \text{if } y_i > \bar{y}_i^*. \end{cases} \tag{A21}$$

By (45) and $\psi_i(r\beta \underline{y}_i) = C_{i1} + r\beta \underline{y}_i$, we have $\forall y_i \in (\underline{y}_i, \underline{y}_i^*), \psi_i(r\beta y_i) > C_{i1} + r\beta y_i$. Similarly, by (47) and $\psi_i(r\beta \bar{y}_i) = C_{i2} + (1 - \alpha_i) r\beta \bar{y}_i$, we have $\forall y_i \in (\bar{y}_i^*, \bar{y}_i), \psi_i(r\beta y_i) > C_{i2} + (1 - \alpha_i) r\beta y_i$. Combined with (A19) and (A21), this implies that in NT

$$v(x, y) > \mathcal{M}v(x, y).$$

For any stock i that is in the buy region of this stock, i.e., $y_i \leq \underline{y}_i$, $\varphi_i(r\beta y_i) = C_{i1} + r\beta y_i$ by (23). Similarly, for any stock i that is in the sell region of this stock, i.e., $y_i \geq \bar{y}_i$, $\varphi_i(r\beta y_i) = C_{i2} + (1 - \alpha_i) r\beta y_i$. Thus, outside NT, we have

$$v(x, y) = \mathcal{M}v(x, y).$$

Therefore condition 2 in part (a) also holds. Finally, we show that if there is a solution to (22) subject to conditions (37)-(44), then it is unique. We prove by contradiction.

Suppose there are two different optimal combined controls ξ and $\hat{\xi}$. Clearly, the value functions associated with these two controls must be identical in $\mathbb{R} \times \mathbb{R}^n$ for both to be optimal. The separability of the value function in φ_i then implies that $\forall i = 1, 2, ..., n$, $\varphi_i(\cdot)$ is identical to $\hat{\varphi}_i(\cdot)$ (and thus $C_{i1} = \hat{C}_{i1}$ and $C_{i2} = \hat{C}_{i2}$), where $\varphi_i(\cdot)$ and $\hat{\varphi}_i(\cdot)$ are associated with ξ and $\hat{\xi}$, respectively. Since the optimal consumption policy is completely determined by the value function, it must be also identical for any given x_t and y_t. Therefore, the difference in ξ and $\hat{\xi}$ must come from the difference in the optimal stock trading policy (for at least one stock). Without loss of generality, we suppose for stock k between 1 and n, there are two different optimal policies $\{\underline{z}_k, \underline{z}_k^*, \bar{z}_k^*, \bar{z}_k\}$ and $\{\hat{\underline{z}}_k, \hat{\underline{z}}_k^*, \hat{\bar{z}}_k^*, \hat{\bar{z}}_k\}$. Without loss of generality, we suppose $\underline{z}_k > a > \hat{\underline{z}}_k$, where a is a constant such that $\hat{\underline{z}}_k < a < \hat{\underline{z}}_k^*$. By (37), we have $\psi_k'(a) = 1$. On the other hand, by (45), we have $\hat{\psi}_k'(a) > 1$, which contradicts the fact that $\varphi_k(\cdot)$ is identical to $\hat{\varphi}_k(\cdot)$. Therefore, the solution of the conjectured form is unique. This completes the proof of Theorem 4. $\square$

Proof of Proposition 1. Differentiating (22) once, we obtain

$$\frac{1}{2}\sigma_i^2 z_i^2 \psi_i''' + (\sigma_i^2 z_i - \sigma_i^2 z_i^2 \psi_i' + \mu_i z_i)\psi_i'' - \sigma_i^2 z_i \psi_i'^2 + (\mu_i - r)\psi_i' = 0.$$

By (38) and (39), we have

$$\psi_i'(\underline{z}_i) = \psi_i'(\underline{z}_i^*) = 1.$$

This implies that there must exist a $\hat{z}_i \in (\underline{z}_i, \underline{z}_i^*)$ such that $\psi_i''(\hat{z}_i) = 0$ and $\psi_i'''(\hat{z}_i) < 0$. Otherwise, at any point $\tilde{z}_i$ such that $\psi_i''(\tilde{z}_i) = 0$ we would have $\psi_i'(\tilde{z}_i) < 1$, contradicting (45). We therefore have

$$-\sigma_i^2 \hat{z}_i \psi_i'^2 + (\mu_i - r)\psi_i' > 0.$$

This implies

$$\frac{\hat{z}_i}{r\beta} < \frac{\mu_i - r}{r\beta\sigma_i^2\psi_i'} < \frac{\mu_i - r}{r\beta\sigma_i^2} = y_i^M.$$

Since $\underline{z}_i < \hat{z}_i$ and $\underline{y}_i(\alpha_i, F_i) = \frac{\underline{z}_i}{r\beta}$, the first inequality in (49) must hold. Similarly, by (40), (41) and (47), the second inequality in (49) must also hold. Next, we show that (50) holds.

We let $\tilde{\underline{z}}_i = \underline{z}_i(\alpha_i, F_i)$ and $\underline{z}_i = \underline{z}_i(\alpha_i, 0)$. Similar to (15)-(17), we have $\psi_i(\underline{z}_i) = C_{i1} + \underline{z}_i$, $\psi_i'(\underline{z}_i) = 1$, and $\psi_i''(\underline{z}_i) = 0$. By (22), this implies that

$$-\frac{1}{2}\sigma_i^2 \underline{z}_i^2 + (\mu_i - r)\underline{z}_i - rC_{i1} + \frac{\delta - r}{n} = 0. \tag{A22}$$

By (37) and (38), we have $\tilde{\psi}_i'(\underline{\tilde{z}}_i) = 1$ and $\tilde{\psi}_i(\underline{\tilde{z}}_i) = \tilde{C}_{i1} + \underline{\tilde{z}}_i$. By (45), we then have $\tilde{\psi}_i''(\underline{\tilde{z}}_i) > 0$. By (22), we then have

$$-\frac{1}{2}\sigma_i^2\underline{\tilde{z}}_i^2 + (\mu_i - r)\underline{\tilde{z}}_i - r\tilde{C}_{i1} + \frac{\delta - r}{n} < 0. \tag{A23}$$

Given a $z_i < \min(\underline{z}_i, \underline{\tilde{z}}_i)$, we have $\tilde{\psi}_i(z_i) = \tilde{C}_{i1} + z_i$ and $\psi_i(z_i) = C_{i1} + z_i$ by the boundary conditions. Because an increase in the fixed cost from zero to $F_i > 0$ decreases the value function for any given z_i, we then must have $C_{i1} > \tilde{C}_{i1}$. Combining this observation with (A22) and (A23), we then have

$$\begin{aligned} 0 &< -\frac{1}{2}\sigma_i^2(\underline{z}_i^2 - \underline{\tilde{z}}_i^2) + (\mu_i - r)(\underline{z}_i - \underline{\tilde{z}}_i) - r(C_{i1} - \tilde{C}_{i1}) \\ &< (-\frac{1}{2}\sigma_i^2(\underline{z}_i + \underline{\tilde{z}}_i) + (\mu_i - r))(\underline{z}_i - \underline{\tilde{z}}_i). \end{aligned} \tag{A24}$$

By the first inequality of (49), we have $\underline{z}_i < \frac{\mu_i - r}{\sigma_i^2}$ and $\underline{\tilde{z}}_i < \frac{\mu_i - r}{\sigma_i^2}$. Inequality (A24) then implies the first inequality of (50). Similarly, using the boundary conditions at the sell boundary and the second inequality of (49), we find that the second inequality of (50) holds. $\square$

Appendix B

In this appendix, we provide the solution algorithms for solving the free-boundary problems.

Algorithm 1: When there are only proportional costs.

1. Define a test function $q : \mathbb{R}_+ \to \mathbb{R}_+$ as follows: for a given candidate $\underline{z}_i$, solve the ODE (22) subject to equation (16) and
$$\psi(\underline{z}_i) = \frac{-\frac{1}{2}\sigma_i^2 \underline{z}_i^2 + \mu_i \underline{z}_i + (\delta - r)}{r},$$
which is obtained from equation (22) evaluated at $\underline{z}_i$ using equations (16) and (17); then solve equation (20) for $\bar{z}$. If there is no $\bar{z}$ satisfying equation (20), set q equal to an arbitrarily large positive number, such as ten. If there is a $\bar{z}$ satisfying equation (20), set q equal to $(1 - \alpha_i - \psi_i'(\bar{z}_i))^2$.

2. Use a standard minimization algorithm to find the optimal $\underline{z}_i \in [0, r\beta y_i^M]$ that minimizes q.[15]

Algorithm 2: When there are both fixed and proportional costs.

1. Define a test function $q : \mathbb{R}_+^2 \to \mathbb{R}_+$ as follows: for a candidate $\underline{z}_i$ and a candidate d for $\psi_i''(\underline{z}_i)$, solve the ODE (22) subject to condition (38) and
$$\psi(\underline{z}_i) = \frac{\frac{1}{2}\sigma_i^2 \underline{z}_i^2 d - \frac{1}{2}\sigma_i^2 \underline{z}_i^2 + \mu_i \underline{z}_i + (\delta - r)}{r},$$
which is obtained from ODE (22) evaluated at $\underline{z}_i$ using condition (38); then solve conditions (39)-(41) for $\underline{z}_i^*$, $\bar{z}_i^*$ and $\bar{z}_i$ respectively. If there is no solution for $\underline{z}_i^*$, $\bar{z}_i^*$ or $\bar{z}_i$, set q equal to an arbitrarily large positive number, such as ten. Otherwise, set $C_{i1} = \psi_i(\underline{z}_i) - \underline{z}_i$, $C_{i2} = \psi_i(\bar{z}_i) - (1 - \alpha_i)\bar{z}_i$, and q equal to $[\psi_i(\underline{z}_i^*) - (C_{i1} + r\beta F_i + \underline{z}_i^*)]^2 + [\psi_i(\bar{z}_i^*) - (C_{i2} + r\beta F_i + (1 - \alpha_i)\bar{z}_i^*)]^2$.

2. Use a standard minimization algorithm to find the optimal $\underline{z}_i \in [0, r\beta y_i^M]$ and $d > 0$ that minimize q, whose minimum theoretical value is zero at the optimal solution.

Algorithm 3: When asset returns are correlated.

To save space, we only describe the algorithm for the two-stock case. For the general case of n stocks, the procedure is similar. This algorithm is an application of the Projection Method proposed by Judd (1999) to our problem. We thus only provide the main steps in applying this method here. For details and its theoretical foundation, we refer readers to Judd (1999). Let $m = 0$ and $\hat{z}_i, i = 1, 2, ..., 16$, denote the coordinates of the eight corners of the no-transaction and target boundaries (e.g., points "A","B", "C", "D", "a", "b", "c", and "d" in Figure 5).

1. Set $m = m + 1$. Let the approximation function be

$$\tilde{\psi}_m(z_1, z_2) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} a_{ij} H_i(z_1) H_j(z_2),$$

   where the $H_i(.)$ is the Hermite function of order i and coefficients a_{ij} are to be determined.

2. Integrate the left hand side of the PDE (53) over the no-transaction region NT using $\tilde{\psi}_m(z_1, z_2)$ in place of $\psi(z_1, z_2)$. Denote this value as d_1.

3. Next, reduce the four boundary conditions (54) and (59)-(61) to two conditions by eliminating C_{i1} and C_{i2}. Use $\tilde{\psi}_m(z_1, z_2)$ in place of $\psi(z_1, z_2)$ to compute the difference between the left hand side value and the right hand side value of each of the resulting six boundary conditions ((55)-(58) plus the newly obtained two conditions) for each stock. Denote these differences (six for each stock) as $d_j, j = 1, 2, ..., 12$.

4. Then define the sum-of-squares test function $q_m = \sum_{i=1}^{13} d_i^2$.

5. Finally, use standard optimization procedure to minimize q_m over a_{ij}, $i + j \le m$, $0 \le i, j$ and $\hat{z}_i$, $i = 1, 2, ..., 16$. Note that q_m is a polynomial function of a_{ij}'s.

6. Repeat Steps 1-5 until both q_m and $q_{m+1} - q_m$ are smaller than a preset approximation error tolerance level.

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Footnotes

1. In contrast to Leland (2000), the form and the magnitude of the targets in this paper are endogenously derived.

2. I thank the referee for pointing out the relevance of transaction costs to the predictability and trading volume literature.

3. We do impose portfolio constraints to rule out any arbitrage opportunities.

4. It should also be noted that it is without loss of generality to represent the proportional transaction cost this way instead of having proportional costs for both sales and purchases, because one can always normalize the latter representation to obtain the former.

5. It is straightforward to extend this analysis to the case where the fixed cost for a purchase is different from the one for a sale.

6. When $\mu_i < r$, the investor shorts the stock. This analysis is symmetric to the case analyzed in this paper. The fact that only one element of the Brownian motion appears in each stock return equation implies that the stock returns are assumed to be uncorrelated. Some discussion of the correlated return case will be provided later.

7. Mathematically speaking, the second part of condition (5) is to ensure that $\int_0^T y_t e^{-\delta t - r\beta W_t} dw_t$ is a martingale, which is necessary for the Merton solution to be optimal in the no-transaction-cost case. As shown by Cox and Huang (1989), the optimal policies with nonnegative wealth and consumption constraints converge to the policies without these constraints as the initial wealth of the investor increases. We thus do not impose these constraints to simplify the analysis but focus accordingly on investors with large initial wealth such as mutual funds and hedge funds.

8. Interested readers may also see Shreve and Soner (1994) and Theorem VIII.4.1 in Fleming and Soner (1993) for similar proofs for the CRRA case with one stock. Although we have not been able to prove that condition (24) in this theorem and similar conditions in Theorems 2, 3, and 4 are automatically satisfied by the corresponding

φ_i, we strongly suspect that this is indeed the case from checking these conditions in all the cases we examined.

9. Although we cannot show the existence of a solution of the corresponding conjectured forms in Theorems 2-4, the numerical algorithms in Appendix B have always successfully found one in every numerical case considered in this paper.

10. Both C_{i1} and C_{i2} can be easily eliminated to reduce the number of equations to six. We choose not to do so to preserve clarity.

11. In other words, the optimal dollar amount range for a stock would not be constant but rather would depend on the amounts in other stocks.

12. We find that relaxing this assumption to allow all the boundaries to be piecewise linear does not yield any noticeable changes in the optimal boundaries.

13. Other numerical examples we investigated yield similar qualitative results.

14. I thank the referee for pointing this out.

15. According to Theorems 2-4, the minimum of q is theoretically zero. Alternative numerical procedures proposed in an earlier version of this paper also work well and obtain the same solutions. However, this procedure and Algorithm 2 offer the advantage that they need virtually no intervention on the starting points and are thus more robust for a wide range of parameters.

Figure Captions

Figure 1. Boundaries as functions of the proportional cost. The graph plots the no-transaction boundaries $\underline{z}$ and $\bar{z}$ against proportional cost α for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, fixed cost $F = 0$, and absolution risk aversion coefficient $\beta = 0.001$. The thin middle line is the Merton line.

Figure 2. Boundaries as functions of the fixed cost. The graph plots the optimal boundaries $\underline{z}$, z^*, and $\bar{z}$ against fixed cost F for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 3. Equivalent fixed costs as functions of the proportional cost. The graph plots the equivalent fixed cost F against proportional cost α for absolute risk aversion coefficients $\beta = 0.01, \beta = 0.1$, $\beta = 1$, and other parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, and return volatility $\sigma = 0.22$.

Figure 4. First derivative of φ. The graph plots $\varphi'(z)$ against z for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolute risk aversion coefficient $\beta = 0.001$.

Figure 5. No-transaction and transaction regions for two stocks. The graph shows the no-transaction and transaction regions when there are two stocks subject to both fixed and proportional costs for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, stock 1 expected return $\mu_1 = 0.069$, stock 2 expected return $\mu_2 = 0.10$, stock return volatilities $\sigma_1 = \sigma_2 = 0.22$, proportional costs $\alpha_1 = \alpha_2 = 0.01$, fixed costs $F_1 = F_2 = 5$, and absolute risk aversion coefficient $\beta = 0.001$.

Figure 6. Boundaries as functions of the fixed cost. The graph plots the boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ against fixed cost F for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0$, and absolution risk aversion coefficient $\beta = 0.001$. The thin middle line is the Merton line.

Figure 7. Boundaries as functions of the proportional cost. The graph plots the boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ against proportional cost α for the following parameters: time discount rate $\delta = 0.01$,

risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$. The thin middle line is the Merton line.

Figure 8. No-transaction and target boundaries for two correlated stocks. The graph shows the no-transaction and target boundaries when there are two correlated stocks subject to both fixed and proportional costs for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, stock expected returns $\mu_1 = \mu_2 = 0.069$, stock return volatilities $\sigma_1 = \sigma_2 = 0.22$, proportional costs $\alpha_1 = \alpha_2 = 0.01$, fixed costs $F_1 = F_2 = 5$, absolute risk aversion coefficient $\beta = 0.001$, and return correlation $\rho_{12} = 0.1$. The dashed lines are corresponding boundaries for the uncorrelated return case.

Figure 9. Boundaries as functions of the absolute risk aversion coefficient. The graph plots the optimal boundaries $\underline{y}$, $\underline{y}^*$, $\bar{y}^*$, and $\bar{y}$ against absolute risk aversion coefficient β for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, and fixed cost $F = 5$.

Figure 10. Boundaries as functions of the return volatility. The graph plots the optimal boundaries $\underline{z}$, $\underline{z}^*$,$\bar{z}^*$, and $\bar{z}$ against return volatility σ for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 11. Boundaries as functions of the expected return. The graph plots the optimal boundaries $\underline{z}$, $\underline{z}^*$, $\bar{z}^*$, and $\bar{z}$ against expected return μ for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 12. Expected time to the next transaction as functions of the proportional cost. The graph plots the expected time to the next transaction $E_z[\tau_s]$ and $E_z[\tau_b]$ starting from $\bar{z}^*$ and $\underline{z}^*$ respectively against proportional cost α for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 13. Expected time to the next transaction as functions of the absolute risk aversion coefficient. The graph plots the expected time to the next transaction $E_z[\tau_s]$ and $E_z[\tau_b]$ starting from $\bar{z}^*$ and $\underline{z}^*$ respectively against absolute risk aversion coefficient β for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, and fixed cost $F = 5$.

Figure 14. Expected time to the next transaction as functions of the return volatility. The graph plots the expected time to the next transaction $E_z[\tau_s]$ and $E_z[\tau_b]$ starting from $\bar{z}^*$ and $\underline{z}^*$ respectively against return volatility σ for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$..

Figure 15. Expected time to the next transaction as functions of the expected return. The graph plots the expected time to the next transaction $E_z[\tau_s]$ and $E_z[\tau_b]$ starting from $\bar{z}^*$ and $\underline{z}^*$ respectively against expected return μ for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 16. The stationary density function of the amount in a stock. The graph plots the stationary density function $f(z)$ for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

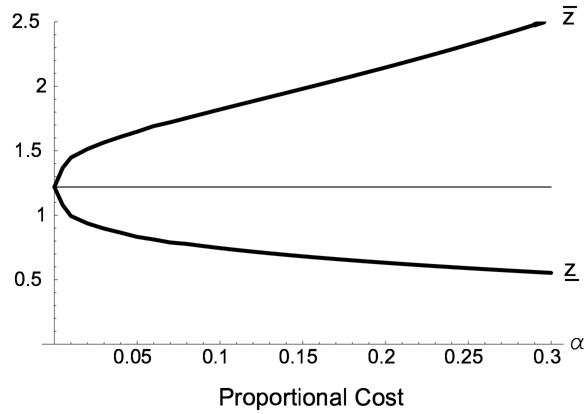
Figure 17. The average amount in a stock as a function of the proportional cost. The graph plots the average amount in stock against proportional cost α for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 18. The average amount in stock against fixed cost. The graph plots the average amount in a stock as a function of the fixed cost F for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, expected return $\mu = 0.069$, return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 19. The expected return that implies the same average amount in a stock as a function of the proportional cost. The graph plots an expected return that implies the same average amount in a stock against proportional cost α for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$, return volatility $\sigma = 0.22$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.

Figure 20. The required extra risk premium as a function of the average time between transactions. The graph plots the required extra risk premium against the average time between transactions for the following parameters: time discount rate $\delta = 0.01$, risk free rate $r = 0.01$,

return volatility $\sigma = 0.22$, proportional cost $\alpha = 0.01$, fixed cost $F = 5$, and absolution risk aversion coefficient $\beta = 0.001$.



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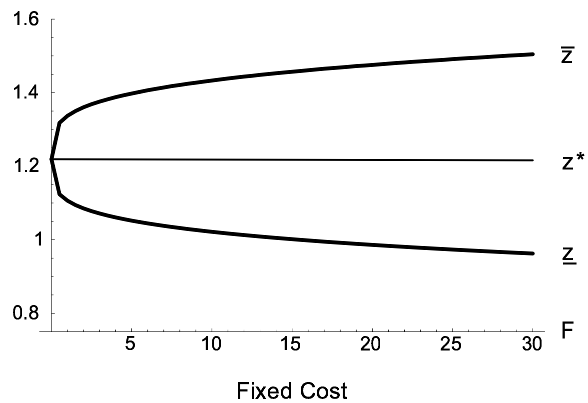

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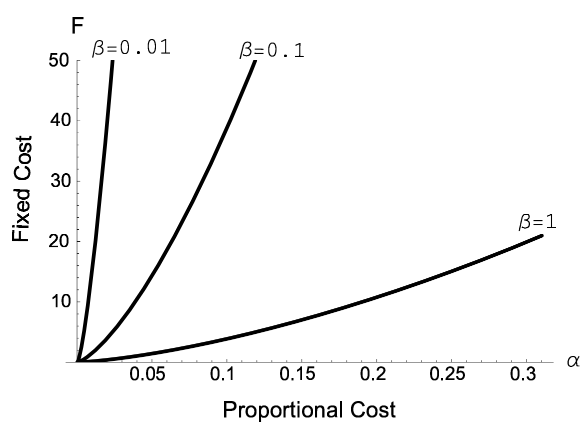

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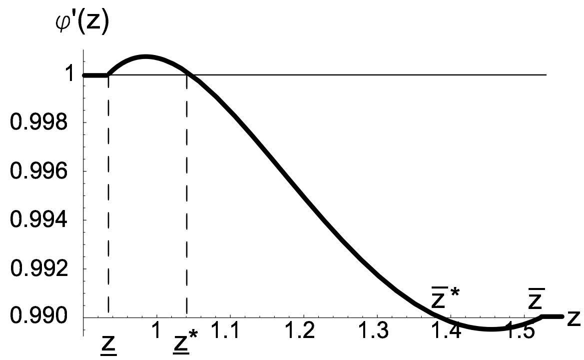

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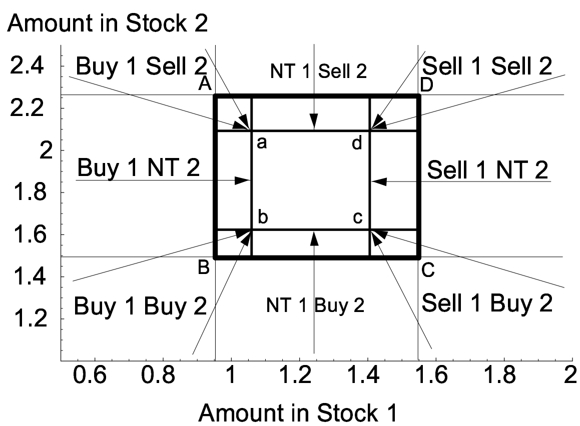

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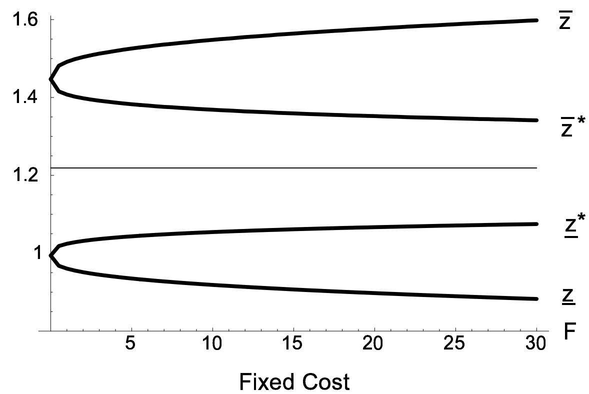

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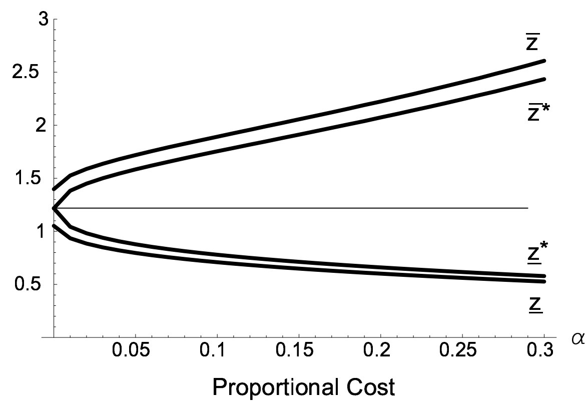

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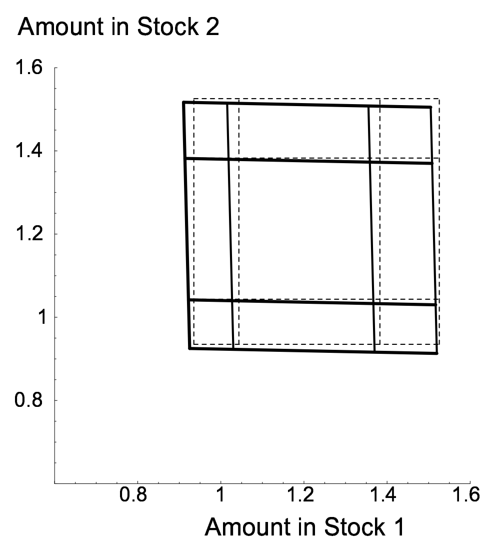

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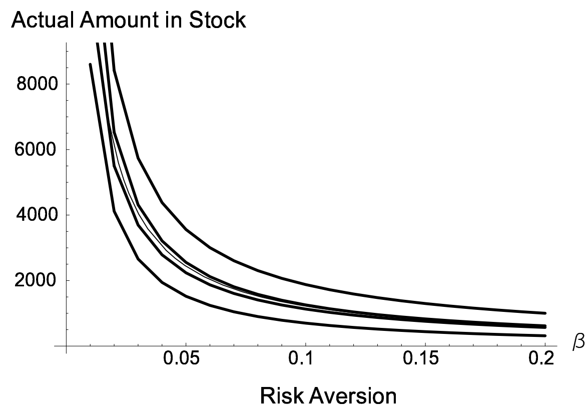

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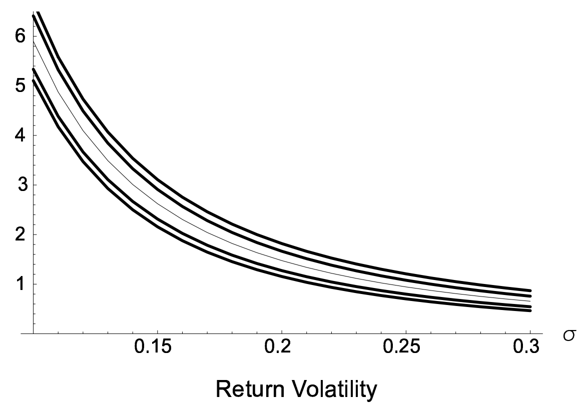

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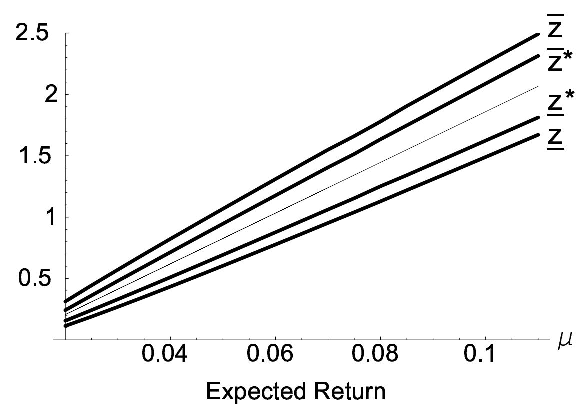

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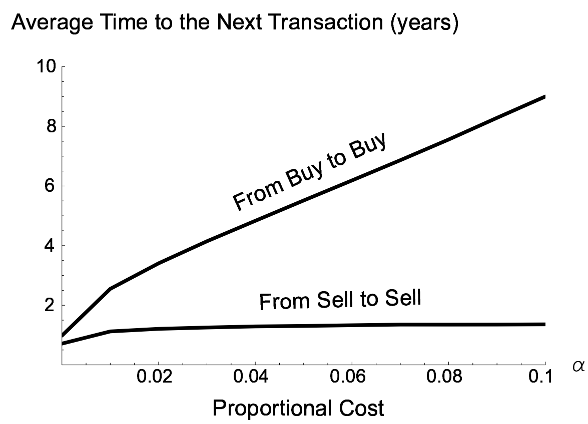

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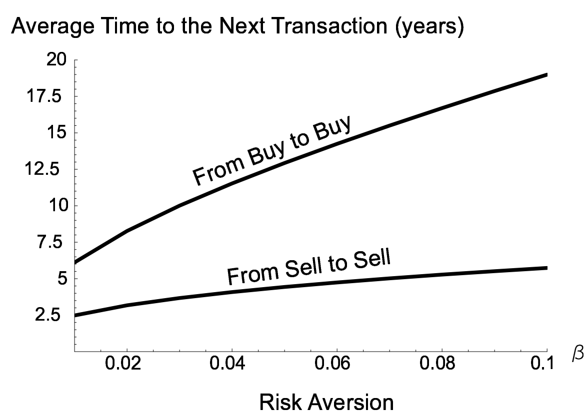

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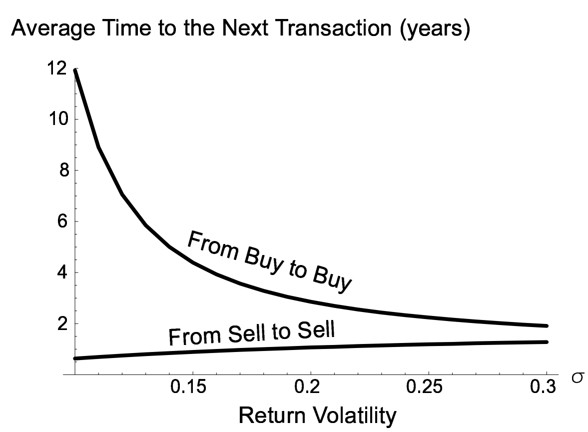

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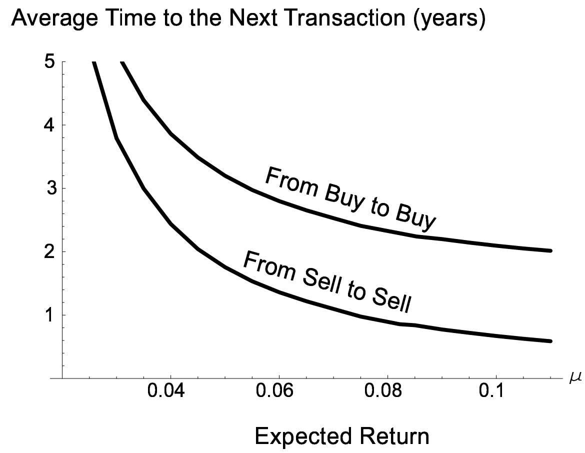

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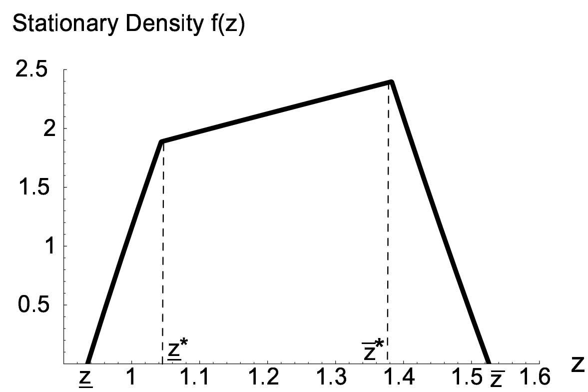

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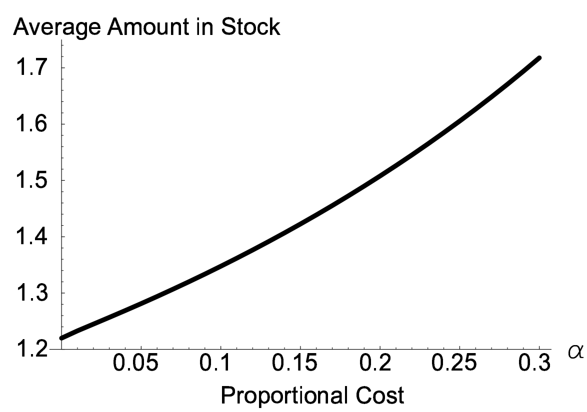

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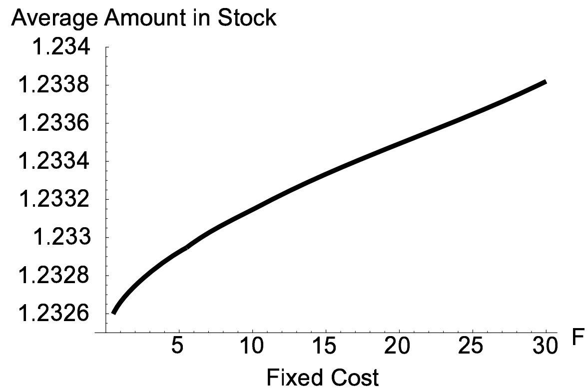

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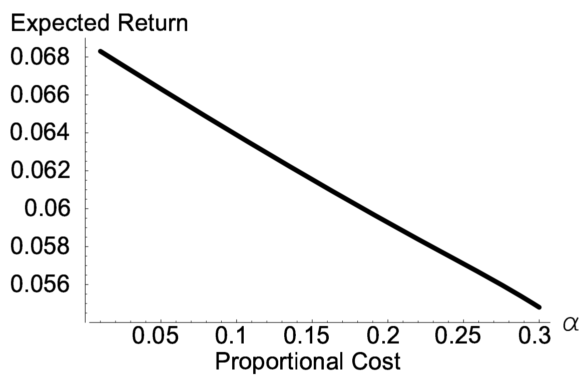

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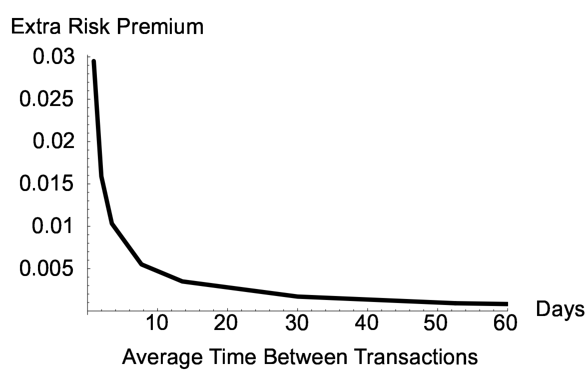

Figure 20.