

Estimation and Test of a Simple Model of Intertemporal Capital Asset Pricing

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ABSTRACT

A simple valuation model with time-varying investment opportunities is developed and estimated. The model assumes that the investment opportunity set is completely described by the real interest rate and the maximum Sharpe ratio, which follow correlated Ornstein-Uhlenbeck processes. The model parameters and time series of the state variables are estimated using U.S. Treasury bond yields and expected inflation from January 1952 to December 2000, and as predicted, the estimated maximum Sharpe ratio is related to the equity premium. In cross-sectional asset pricing tests, both state variables have significant risk premia, which is consistent with Merton's ICAPM.

In a continuous-time diffusion setting, the instantaneous investment opportunity set is completely described by the real interest rate and the slope of the capital market line, or Sharpe ratio, as in the classic Sharpe-Lintner Capital Asset Pricing Model (CAPM). The slope of the capital market line depends in turn on the risk premium and volatility of the market return, and there is now strong evidence of time variation both in the equity risk premium and in market volatility, implying variation in the market Sharpe ratio, as well as in the real interest rate. Kandel and Stambaugh (1990), Whitelaw (1997), and Perez-Quiros and Timmermann (2000) have all found significant cyclical variation in the market Sharpe ratio. Other studies that identify significant predictors of the equity risk premium include: Lintner (1975), the interest rate; Campbell and Shiller (1988) and Fama and French (1988), the market dividend yield; Fama and French (1989), the term spread and the junk bond yield spread; Kothari and Shanken (1997), the book-to-market ratio.

The Intertemporal Capital Asset Pricing Model (ICAPM) of Merton (1973) suggests that when there is stochastic variation in investment opportunities, it is likely that there will be risk premia associated with innovations in the state variables that describe the investment opportunities. However, despite this evidence of time variation in investment opportunities, and despite the lack of empirical success of the classic single period CAPM and its consumption based variant, there has been relatively little effort to test models based on Merton's classic framework, a significant exception being Campbell (1993). One reason for this lack of emphasis on the ICAPM may have been the tendency to lump the ICAPM and Ross' (1976) Arbitrage Pricing Theory (APT) together as simply different examples of factor pricing models.¹ Yet this ignores the distinguishing characteristic of the ICAPM - that the

priced “factors” are not just any set of factors that are correlated with returns, but are the innovations in state variables *that predict future returns*.² In this paper we estimate a simple ICAPM that allows for time-variation in the real interest rate and slope of the capital market line, and evaluate the ability of the model to account for the returns on portfolios sorted according to size and book-to-market (B/M) ratio, as well as according to industry.

In the simple ICAPM that we develop, time variation in the instantaneous investment opportunity set is fully described by the dynamics of the real interest rate and the maximum Sharpe ratio. We assume that these two variables follow correlated Ornstein-Uhlenbeck processes so that their current values are sufficient statistics for all future investment opportunities. As a result, they are the only state variables that are priced in an ICAPM setting. This is demonstrated formally by Nielsen and Vassalou (2003), who show that investors hedge only against stochastic changes in the slope and the intercept of the instantaneous capital market line. Using the martingale pricing approach, we then show how a claim to a future cash flow is valued.³ With additional assumptions about the stochastic process for the price level, the model is adapted in a simple fashion to the pricing of default-free nominal bonds, and the model parameters, as well as the time series of the real interest rate and the Sharpe ratio, are estimated via a Kalman filter from data on U.S. Treasury bond yields and inflation expectations for the period from January 1952 to December 2000. It is shown that, as predicted by the model, the Sharpe ratio estimate is significantly related to the “ex-post” equity market Sharpe ratio, which is measured by the ratio of the excess return on the market index to an estimate of volatility obtained from an EGARCH model.

In order to determine whether the simple ICAPM can account for the cross-section of

returns on the 25 Fama-French size and B/M sorted portfolios for the period from January 1952 to December 2000, we follow a two-stage cross-sectional regression procedure. In the first stage, we regress the 25 portfolio returns on the market excess return, and the estimated innovations in the state variables, the real interest rate and the maximum Sharpe ratio, to obtain the three betas. In the second stage, the sample mean excess returns are regressed on the estimated betas to obtain the risk premia for market risk, the real interest rate risk, and the maximum Sharpe ratio risk. A χ^2 test of the pricing restrictions implied by the model cannot reject them for the whole sample period, although the model is rejected using data from the second half of the period. Moreover, the simple ICAPM explains well the returns on both of the Fama-French arbitrage portfolios.⁴ The pricing error for *SMB* is only -0.07% per month, and for *HML* it is 0.02% per month, and neither of these pricing errors is significant. On the other hand, when similar tests are conducted for the Fama-French three-factor model and the CAPM, both models are strongly rejected for the whole sample period, as well as for both halves of the sample. As a robustness check, we also test the three models using the one-step Generalized Method of Moments Discount Factor (GMM/DF) approach. Again, the simple ICAPM is not rejected while the other two models are strongly rejected.

Motivated by the data-snooping concerns expressed by Lo and MacKinlay (1990), we also report the results of tests using 30 industrial portfolios instead of the size and book-to-market sorted portfolios. In this case, none of the three models is rejected for any of the sample periods. However, for both the simple ICAPM and the three-factor model, the point estimates of the risk premia are quite different when the 30 industrial portfolios

are used. Therefore, we combine all 55 portfolios in a single estimation. The ICAPM risk premia remain significant in the combined sample while the estimated premia associated with the Fama-French arbitrage portfolios become economically and statistically insignificant. However, the pricing restrictions imposed by all three models are rejected. We conjecture that the rejection of the simple ICAPM is due to the omission of significant state variables that forecast future values of the interest rate and maximum Sharpe ratio.

The remainder of the paper is organized as follows. In Section I we construct a simple valuation model that allows for a stochastic interest rate and Sharpe ratio and specialize the model to the ICAPM. In Section II we describe the data and estimation of the valuation model and the state variables. The main empirical results are reported and discussed in Section III, and Section IV concludes.

I. Valuation with Stochastic Investment Opportunities

The value of a claim to a future cash flow depends both on the characteristics of the cash flow itself, its expected value, time to realization, and risk, and on the macroeconomic environment as represented by interest rates and risk premia. Holding the risk characteristics of the cash flow constant, unanticipated changes in claim value will be driven by changes in interest rates and risk premia, as well as by changes in the expected value of the cash flow. Most extant valuation models place primary emphasis on the role of cash flow-related risk. However, Campbell and Ammer (1993) estimate that only about 15% of the variance of aggregate stock returns is attributable to news about future dividends. Their results further suggest that news about real interest rates plays a relatively minor role, leaving about 70%

of the total variance of stock returns to be explained by news about future excess returns or risk premia. Fama and French (1993) show that there is considerable common variation between bond and stock returns, which also suggests that changes in interest rates and risk premia are important determinants of stock returns. In this section we construct an explicit model for the valuation of stochastic cash flows that takes account of stochastic variation in interest rates and risk premia.

Let V denote the value of a non-dividend paying asset. The absence of arbitrage opportunities implies the existence of a pricing kernel, a random variable, m , such that $E[d(mV)] = 0$.⁵ This condition implies that the expected return on the asset can be written as:

$$E \left[\frac{dV}{V} \right] = -E \left[\frac{dm}{m} \right] - \text{Cov} \left(\frac{dm}{m}, \frac{dV}{V} \right). \quad (1)$$

Assume that the dynamics of the pricing kernel can be written as a diffusion process:

$$\frac{dm}{m} = -r(X)dt - \eta(X)dz_m, \quad (2)$$

where X is a vector of variables that follow a vector Markov diffusion process:

$$dX = \mu_X dt + \sigma_X dz_X. \quad (3)$$

Then equations (1) and (2) imply that the expected return on the asset is given by

$$\mathbb{E} \left[\frac{dV}{V} \right] \equiv \mu_V dt = r(X)dt + \eta(X)\rho_{Vm}\sigma_V dt, \quad (4)$$

where $\rho_{Vm}dt = dz_V dz_m$, and σ_V is the volatility of the return on the asset. It follows, first, that $r(X)$ is the risk-free rate, since it is the return on an asset with $\sigma_V = 0$, and secondly, that $\eta(X)$ is the risk premium per unit of covariance with the pricing kernel. Equation (4) implies that the Sharpe ratio for any asset, V , is given by $S_V \equiv (\mu_V - r)/\sigma_V = \eta\rho_{Vm}$. Recognizing that ρ_{Vm} is a correlation coefficient, it follows that η is the maximum Sharpe ratio for any asset in the market - it is the slope of the capital market line, or “market” Sharpe ratio. An investor’s *instantaneous* investment opportunities then are fully described by the vector of the instantaneously riskless rate and the Sharpe ratio of the capital market line, $(r, \eta)'$.

In order to construct a tractable valuation model, we shall simplify, by identifying the vector X with $(r, \eta)'$, and assuming that r and η follow simple correlated Ornstein-Uhlenbeck processes.⁶ Then, the dynamics of the investment opportunity set are fully captured by

$$\frac{dm}{m} = -r dt - \eta dz_m \quad (5.1)$$

$$dr = \kappa_r(\bar{r} - r)dt + \sigma_r dz_r \quad (5.2)$$

$$d\eta = \kappa_\eta(\bar{\eta} - \eta)dt + \sigma_\eta dz_\eta. \quad (5.3)$$

Although model (5) is not a structural model since it does not start from the specification

of the primitives - the tastes, beliefs, and opportunities of investors⁷ it provides a simple basis for consideration of the essential feature of the ICAPM, the pricing of risk associated with variation in investment opportunities, since it allows for variation in the instantaneous investment opportunity set while limiting the number of state variables to be considered to the two that are required to describe that set. The strong assumption in model (5) is that r and η follow a joint Markov process.

The structure (5) implies that the riskless interest rate is stochastic and that all risk premia are proportional to the stochastic Sharpe ratio η . To analyze the valuation implications of the system (5), consider a claim to a (real) cash flow, x , which is due at time T . Let the expectation at time t of the cash flow be given by $y(t) \equiv E[x|\Lambda_t]$, where Λ_t is the information available at time t , and $y(t)$ follows a driftless geometric Brownian motion with constant volatility, σ_y :⁸

$$\frac{dy}{y} = \sigma_y dz_y. \tag{6}$$

Letting ρ_{ij} denote the correlation between dz_i and dz_j , the value of the claim to the cash flow is given in the following theorem.

THEOREM 1: Value of a claim on a single real cash flow.

In an economy in which the investment opportunity set is described by (5), the value at time t of a claim to a real cash flow x at time $T \equiv t + \tau$, whose expectation, y , follows the

stochastic process (6), is given by:

$$V(y, \tau, r, \eta) = \mathbb{E}_t^Q \left[x_T \exp^{-\int_t^T r(s)ds} \right] = \mathbb{E}_t^Q \left[y_T \exp^{-\int_t^T r(s)ds} \right] = yv(\tau, r, \eta), \quad (7)$$

where Q denotes the risk neutral probability measure, and

$$v(\tau, r, \eta) = \exp[A(\tau) - B(\tau)r - D(\tau)\eta], \quad (8)$$

with $A(\tau)$, $B(\tau)$ and $D(\tau)$ defined in Appendix A.

Theorem 1 implies that, $v(\tau, r, \eta)$, the value per unit of expected payoff of the claim, is a function of the maturity, τ , of the cash flow. It also depends on the covariance of the cash flow with the pricing kernel, $\phi_y \equiv \sigma_m \sigma_y \rho_{ym}$, through $A(\tau)$ and $D(\tau)$. Finally, it depends on the current values of the two state variables that fully describe the investment opportunity set, r and η . An increase in the interest rate, r , unambiguously decreases the normalized claim value, $v(\tau, r, \eta)$. An increase in the Sharpe ratio, η , also decreases $v(\tau, r, \eta)$ if $\rho_{ym} > 0$, which corresponds to positive cash flow systematic risk, and $\rho_{rm} < 0$, which corresponds to a positive interest rate risk premium. The sensitivities of $v(\tau, r, \eta)$ to r and η , $B(\tau)$, and $D(\tau)$, are closely related to the mean reversion parameters κ_r and κ_η .

Applying Ito's lemma, the theorem implies that the return on the claim can be written as:

$$\frac{dV}{V} = \mu(r, \eta, \tau)dt + \frac{dy}{y} - B(\tau)\sigma_r dz_r - D(\tau)\sigma_\eta dz_\eta, \quad (9)$$

where

$$\mu \equiv \mu(r, \eta, \tau) = r + (D_\tau(\tau) + \kappa_\eta D(\tau))\eta.$$

Thus, the risk premium of the claim, $\mu - r$, is proportional to the state variable η , and the constant of proportionality depends on the claim's cash flow maturity. The return on the claim is determined by the innovations in the two state variables, r and η , as well as in the cash flow expectation, y . Note that the innovations in r and η are systematic factors that affect the returns on all securities, while $\frac{dy}{y}$ is security-specific. Moreover, the loadings on the systematic risks, $B(\tau)$ and $D(\tau)$, are functions of the cash flow maturity.

Although equation (9) is for claims to a single cash flow, a similar equation can be easily derived for securities with multiple cash flows, which are equivalent to portfolios of single cash flow claims. Therefore equation (9) implies that securities (cash flow claims) with different cash flow maturities typically have different risk exposures to the systematic state variables, so that if there are risk premia associated with these state variables as the ICAPM (see equation (14) below) implies, the model may have the potential to explain cross-sectional portfolio returns, if the portfolios are constructed so that they have different cash flow maturities. Growth and value firms obviously have different cash flow maturities, and since Perez-Quiros and Timmermann (2000) show that portfolios of large and small firms have different sensitivities to credit conditions, we should expect them also to have different loadings on r . In Section III, we shall empirically examine the ICAPM defined in equation (14) below using 25 size and book-to-market sorted portfolios.

The value of a real discount bond is obtained as a special case of Theorem 1 by imposing $x \equiv y \equiv 1$ and $\sigma_y = 0$. The resulting expression generalizes the Vasicek (1977) model for the

price of a (real) discount bond to the case in which the risk premium, as well as the interest rate, is stochastic. In order to value *nominal* bonds, it is necessary to specify the stochastic process for the price level, P ; this is assumed to follow the diffusion

$$\frac{dP}{P} = \pi dt + \sigma_P dz_P, \quad (10)$$

where the volatility of inflation, σ_P , is constant, while the expected rate of inflation, π , follows an Ornstein-Uhlenbeck process:

$$d\pi = \kappa_\pi(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi. \quad (11)$$

Then, noting that the real payoff of the nominal bond is $1/P_T$, the nominal price of a zero coupon bond with a face value of \$1 and maturity of τ , $N(P, r, \pi, \eta, \tau)$, and the corresponding real price, $n(P, r, \pi, \eta, \tau)$, are given in the following theorem.

THEOREM 2: Value of a nominal zero-coupon bond.

If the stochastic process for the price level P is as described by (10) and (11), the nominal and the real prices of a zero-coupon bond with a face value of \$1 and maturity τ are given by

$$N(P, r, \pi, \eta, \tau) \equiv Pn(r, \pi, \eta, \tau) = \exp[\widehat{A}(\tau) - B(\tau)r - C(\tau)\pi - \widehat{D}(\tau)\eta], \quad (12)$$

where $\widehat{A}(\tau)$, $B(\tau)$, $C(\tau)$, and $\widehat{D}(\tau)$ are given in Appendix A.

Equation (12) implies that the nominal yield on a bond of given maturity is a linear

function of the state variables, r , π , and η :

$$-\frac{\ln N}{\tau} = -\frac{\hat{A}(\tau)}{\tau} + \frac{B(\tau)}{\tau}r + \frac{C(\tau)}{\tau}\pi + \frac{\hat{D}(\tau)}{\tau}\eta. \quad (13)$$

This equation shows that our valuation model, when applied to bonds, is a special case of the (essentially) affine term structure model considered in Duffee (2002) and is also closely related to the class of (complete) affine structures discussed by Duffee and Kan (1996) and by Dai and Singleton (2000).

While the reduced-form model (5) explicitly allows for time-variation in the investment opportunity set, it is not equivalent to Merton's ICAPM without further specification of the covariance characteristics of the pricing kernel: For example, the model will be equivalent to the simple static CAPM if the innovation in the pricing kernel is perfectly correlated with the return on the market portfolio. In order to investigate the cross-sectional asset pricing implication of the model (5), we consider a specific version of the ICAPM by specializing the pricing model (5) so that the innovation in the pricing kernel is an exact linear function of the total wealth portfolio return and the innovations in r and η :

$$\frac{dm}{m} = -r dt - \omega \eta \zeta' dz, \quad (14)$$

where $\zeta' = (\zeta_M, \zeta_\eta, \zeta_r)'$, $dz = (dz_M, dz_\eta, dz_r)'$, $\omega \equiv (\zeta' \Omega \zeta)^{-1/2}$, and $\Omega dt = (dz)(dz)'$, where M denotes the portfolio for total wealth.

II. Data and Estimation

The primary data set consists of monthly observations of the yields on eight synthetic constant maturity zero-coupon U.S. Treasury bonds with maturities of 3 and 6 months, and 1, 2, 3, 4, 5, and 10 years for the period from January 1952 to December 2000. Panel A of Table I reports summary statistics for the bond yield data. The sample mean of the bond yields increases slightly with maturity, while the standard deviation remains relatively constant across maturities. The returns on 25 size and book-to-market sorted value-weighted portfolios and the nominal short interest rate for the same period are used for the initial cross-sectional pricing tests. The CRSP value-weighted market portfolio is used to proxy for the portfolio of total wealth. The tests are repeated using the returns on 30 industrial portfolios constructed by Fama and French (1997); these portfolios are re-balanced at the end of June each year using 4-digit SIC codes.⁹

Insert Table I about here

In principle, it is possible to estimate the parameters of the system (5) by the standard maximum likelihood method from the yields on three different bonds using equation (13) for the nominal bond yield. However, the choice of bonds to use in the estimation is arbitrary, and there is no guarantee that the estimates will be consistent with the yields of other bonds. Therefore, in order to minimize the consequences of possible model misspecification and measurement errors in the fitted bond yield data, we allow for errors in the pricing of individual bonds. We use a Kalman filter to estimate the time series of the unobservable state variables r , π , and η , and their dynamics, from data on yields of eight bonds with different

maturities. Details of the estimation are presented in Appendix B. In summary, there are three transition equations for the unobserved state variables, and there are n observation equations based on the yields at time t , $y_{\tau_j,t}$, on bonds with maturities τ_j , $j = 1, \dots, n$.

The transition equations are the discrete time versions of equations (5.2), (5.3), and (11), the equations that describe the dynamics of the state variables, r , η , and π .

The observation equations are derived from equation (13) by the addition of measurement errors, ϵ_{τ_j} :

$$y_{\tau_j,t} \equiv -\frac{\ln N(t, t + \tau_j)}{\tau_j} = -\frac{\widehat{A}(t, \tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j}r_t + \frac{C(\tau_j)}{\tau_j}\pi_t + \frac{\widehat{D}(\tau)}{\tau}\eta_t + \epsilon_{\tau_j}(t). \quad (15)$$

The measurement errors, $\epsilon_{\tau_j}(t)$, are assumed to be serially and cross-sectionally uncorrelated, and to be uncorrelated with the innovations in the transition equations. To reduce the number of parameters to be estimated, the measurement error variance was assumed to be of the form $\sigma^2(\epsilon_{\tau_j}) = \sigma_b^2/\tau_j$, where σ_b is a parameter to be estimated. This is equivalent to the assumption that the measurement error variance of the log price of the bonds is independent of the maturity.¹⁰

The final observation equation uses the Livingston Survey data on the expected rate of inflation over the next six months,¹¹ π_{liv} , as a signal of the instantaneous expected rate of inflation π :

$$\pi_{\text{liv}} = \pi + \epsilon_{\text{liv}}. \quad (16)$$

Although equation (16) is not required to estimate the model, it helps to identify r and π ,

which enter the bond yield observation equation (15) in a symmetric way. To gauge how the additional observation equation (16) affects the state variable estimates, we also estimated the model without equation (16), using only the bond yield data. Finally, we estimated the model using the realized CPI inflation,¹² $\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}}$, as a noisy signal of expected inflation. In this case, the observation equation for the instantaneous expected inflation rate π is changed from equation (16) to $\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}} = \pi\Delta t + \epsilon_P(t)$. The realized rate of inflation has a mean (standard deviation) of 3.85% (1.16%) while the corresponding figures for the Livingston survey data are 3.15% (0.68%). The effect on the parameter estimates of the additional observation equation and the choice of expected inflation signal is discussed below.

III. Empirical Results

In Section III.A we show that the time series of nominal bond yields and inflation provide strong evidence of time variation in both the real interest rate and the Sharpe ratio, and that these estimates are associated with the nominal risk-free rate and the equity premium. In Section III.B we estimate the risk premia associated with innovations in r and η and consider the ability of the model to explain the cross-section of returns first, on 25 size and book-to-market sorted equity portfolios, second on 30 industry portfolios, and finally on 55 combined portfolios.

A. State Variable Estimates

In this section we report Kalman filter estimates of the parameters of the stochastic process for the state variables, r , π , and η , as well as the estimated time series of the state

variables when the Livingston expected inflation series is used in the estimation. In order to identify the process for the Sharpe ratio, η , it is necessary to impose a restriction that determines the overall favorableness of investment opportunities, since equation (4) shows that the structure of risk premia is invariant up to a scalar multiplication of η and the vector of inverse security correlations with the pricing kernel with the typical element, $1/\rho_{Vm}$. The restriction we impose is that $\bar{\eta} = 0.7$. This value was chosen after fitting an EGARCH model to the market excess return and using the resulting times series of volatility estimates to calculate a time series of realized equity market Sharpe ratios. This series has a mean of 0.57.¹³ Since η is the maximum Sharpe ratio of the economy, we set $\bar{\eta}$ to 0.7 to allow for the fact that the equity market portfolio is not mean-variance efficient. This normalization affects only estimates of the scale of η and correlations of other variables with η . Finally, to improve the efficiency of estimation, $\bar{\pi}$ was set equal to the sample mean of CPI inflation, \bar{r} was set equal to the difference between the sample mean of the three-month nominal interest rate and $\bar{\pi}$, and the volatility of unexpected inflation, σ_P , was set equal to 1.16%, the sample volatility of CPI inflation. As a result of predetermining these parameter values, the standard errors of all other parameters reported in Panel C of Table I are understated.

Since the bond yield data were constructed by estimating a cubic spline for the spot yield curve from the prices of coupon bonds, the bond yields are measured with error. For estimation purposes, the variance of the yield measurement error was assumed to be inversely proportional to maturity τ . The estimated measurement error parameter, σ_b , implies that the standard deviation of the measurement error varies from 16 basis points for the three-month maturity to 5 basis points for the ten-year maturity, which is comparable to previous

estimates.¹⁴

The estimated volatility of expected inflation, σ_π , is around 0.75% per year, while κ_π is close to zero, so that the expected rate of inflation rate follows almost a random walk.¹⁵ The estimated volatility of the real interest rate process, σ_r , is 1.11% per year; on the other hand, the sample volatility of the innovations to the estimated r series is 0.4% per year, which compares to Campbell and Viceira's (2001) estimate of 0.5% per year for a slightly shorter sample period. The estimated mean reversion intensity for the interest rate, κ_r , is 0.074 per year, which implies a half life of about nine years. The volatility of the Sharpe ratio process, σ_η , is 0.31 per year which compares with the imposed long run mean value of 0.70; the mean reversion intensity for the Sharpe ratio is 0.047, which implies a half life of more than 14 years. Thus the real interest rate and the Sharpe ratio are persistent processes. This means that they are important state variables from the point of view of an investor with a long horizon so that we should expect to see significant risk premia associated with these variables.

To place the volatility of the real interest rate and the Sharpe ratio in perspective, suppose that the volatility of the market return, σ_M , is constant at 15%. Then if the correlation between the market and the pricing kernel is, say, 0.7, which corresponds to the point estimate from equation (19) below, the estimated volatility of the expected market return that is due to the volatility of the Sharpe ratio is $0.15 \times 0.31 \times 0.7 = 3.33\%$, which is three times as great as the estimated volatility attributable to the real interest rate (1.11%), so that the variation in η is of much more importance for the variation in the expected return on the equity market than is the variation in r . This is consistent with the finding of Campbell

and Ammer (1993) that stock returns are strongly affected by “news about future excess stock returns,” (p.3) while real interest rates have relatively little impact. Note, however, that the estimated correlation between the innovations in r and in η is -0.53 , so that a significant part of the effect of the innovations in these two variables on the expected market return is offsetting. The Wald statistic for the null hypothesis that $\sigma_\eta = \kappa_\eta = 0$ (so that η is a constant) is highly significant, providing strong evidence, given the pricing model, that the Sharpe ratio is time-varying. Finally, the t -statistics on $\rho_{\eta m}$ and $\rho_{r m}$ strongly reject the null hypothesis that the opportunity set state variables, r and η , are unpriced. However, this is not in itself evidence in favor of the ICAPM, because it is possible that the risk premia are due to the correlation of these variables with the market portfolio as in the classical CAPM; we shall investigate this further below. The t -statistics on $\rho_{P m}$ and $\rho_{\pi m}$ are either not significant or are only marginally significant: Thus there does not appear to be a risk premium associated with inflation.

Figure 1 plots the time series of the estimated instantaneous real interest rate r . It reaches a maximum of 6.91% in mid-1984, and a minimum of -2.37% in mid-1980: It is positive for most of the sample period and its average value is 2.5%. The real interest rate tends to fall during recessionary periods, which are represented by the shaded areas in the figures, and to rise following cyclical troughs.¹⁶ Figure 2 plots the estimated series for π along with the corresponding Livingston estimates of the expected rate of inflation. The estimated π series tracks the Livingston inflation well, both of them capturing the inflationary episodes of 1974 to 1975 and 1980 to 1981.

Insert Figure 1 about here

Insert Figure 2 about here

When the model is estimated either using bond yields only without inflation data, or using both CPI inflation data and bond yields, the estimates of the state variables, r , π , and η , are highly correlated with the estimates obtained using bond yields and the Livingston inflation data. The correlations between the levels of (innovations in) the Livingston data-based series of state variable estimates and the series based only on bond yields, and the series obtained using the CPI data are, respectively, for r , 0.93 and 0.92 (0.96 and 0.97); for π , 0.97 and 0.98 (0.77 and 0.83); and for η , 0.95 and 0.95 (0.87 and 0.81). Moreover, the time series of estimates of π that are obtained using either bond yield data alone, or bond yields and CPI inflation, are highly correlated with the Livingston inflation series. However, the estimated sample means of r and π that are obtained using the Livingston data in equation (16), 2.47% and 3.38%, are more reasonable than those obtained if no inflation data is used (4.54% and 1.15%), or if the realized rate of inflation is used (3.50% and 2.26%). For this reason, we concentrate on the Livingston data-based estimates. The estimates of the parameters of the joint stochastic process for r , π , and η are similar across the three estimators.

Since the real interest rate series in Figure 1 appears very volatile, while the expected inflation series in Figure 2 tracks the Livingston survey quite well, it is of interest to check how well the instantaneous *nominal* interest rate implied by the model tracks the one-month

nominal interest rate, R_f , which is taken from French's Web site. The model-implied instantaneous nominal risk-free rate is given by

$$R = r + \pi - \sigma_P \rho_{Pm} \eta - \sigma_P^2. \tag{17}$$

Since the estimate of ρ_{Pm} reported in Table I is not significantly different from zero and $\sigma_P^2 \approx 0$, we can approximate the model-implied nominal interest rate, R , by $r + \pi$. Figure 3 plots $R \approx r + \pi$ along with the empirical one-month rate R_f . The two series are almost coincident. Given that the π series tracks the Livingston expected inflation rates well, this implies that our estimated real interest rate series, r , is approximately equal to the real interest rate implied by the one-month interest rate and the Livingston survey data.

Insert Figure 3 about here

Figure 4 plots the estimated Sharpe ratio, which shows considerable variation over time, reaching a maximum of 2.02 in April 1985 and a minimum of -1.64 in March 1980.¹⁷ Recessions are generally associated with an increasing Sharpe ratio. Whitelaw (1997) and Perez-Quiros and Timmermann (2000) have found similar cyclical patterns in the Sharpe ratio in the equity market, and Fama and French (1989) have also documented common variation in expected returns on bonds and stocks that is related to business conditions. The correlation between the estimated levels of r and η is about 0.1, but there is strong negative correlation between the innovations in these two variables (-0.53). Cochrane and

Piazzesi (2002) have recently constructed a bond excess return forecasting factor as a linear combination of several different bond yields, without imposing any model restrictions. Like our estimate of η , their forecasting factor has high volatility and exhibits a strong cyclical pattern. Therefore, their forecasting factor corresponds to our η in an unrestricted model and its sample correlation with our estimate of η is 0.65.¹⁸ The similarity is not surprising, in view of the fact that our model has three state variables and Litterman and Scheinkman (1991) find that three principal components capture well over 90% of the variation in bond yields.

Insert Figure 4 about here

Equation (15) implies that the difference in yields of bonds with different maturities, the “term spread,” is a linear function of the state variables r , π , and η . We find that our estimated state variables explain almost 99% of the variance of the term spread as measured by the difference between the yields on one-year and ten-year zero-coupon Treasury bonds; this is further confirmation of the model fit. To the extent that the state variable innovations are factors in pricing equity returns as our theory predicts and the empirical results reported below confirm, the innovation to the term spread will also inherit some of their power in pricing equity returns.

The pricing kernel represented by equation (5) implies that the risk premia on all assets vary together with η . In particular, equation (4) implies that the equity market risk premium

is related to η by

$$\mu_M - r = \eta \rho_{Mm} \sigma_M,$$

which implies that the equity market Sharpe ratio can be written as

$$SH_M \equiv \frac{\mu_M - r}{\sigma_M} = \eta \rho_{Mm}. \quad (18)$$

The realized market Sharpe ratio for each month, SH_M^R , was constructed by dividing the market excess return for that month by the EGARCH fitted volatility. To test equation (18), SH_M^R was regressed on η_{-1} . The estimated equation is:

$$SH_M^R = 0.156 + 0.707\eta_{-1}, \quad \bar{R}^2 \quad (19)$$

(0.71) (2.64) 1.2%

where the Newey-West adjusted t -ratios are in the parentheses. Thus, as the model predicts, the Sharpe ratio that we have estimated using data on bond yields and inflation has significant information for normalized excess returns on the stock market. The estimated coefficient of η_{-1} implies that $\rho_{Mm} = 0.71$,¹⁹ and the lack of significance of the intercept is consistent with the theoretical specification (18). The adjusted R^2 increases from 1.2% to 4% if the realized average Sharpe ratio over the next 12 months is the dependent variable. Note that the filtered estimate of the Sharpe ratio is a linear function of the bond yields; this provides an economic rationale for the empirical success of yield-based predictors for expected equity returns, such as nominal interest rates and yield spreads found by Stambaugh

(1988) and Fama and French (1989), among others.

However, the parsimony of the simple ICAPM that we have proposed comes at the cost of possible model mis-specification. In particular, the assumption that r , π , and η follow a correlated O-U process is a strong one. Examination of the estimated innovations to r , π , and η reveals that they are autocorrelated and non-normal. The autocorrelation suggests that there may be additional, potentially priced, state variables, X , that affect the dynamics of r and η . We shall discuss this issue further below.

The significant values of ρ_{rm} (-0.85) and $\rho_{\eta m}$ (0.96) reported in Table I imply that there are indeed risk premia associated with innovations in r and η . In order to test whether premia associated with innovations in the investment opportunity set state variables, r and η , provide incremental explanatory power for equity portfolio returns relative to the CAPM, we calculate the innovations in the state variable (sv) estimates, Δr , $\Delta \pi$ and $\Delta \eta$, using the parameter values reported in Table I:

$$\Delta sv = sv - \bar{sv} (1 - e^{-\kappa_{sv}/12}) - sv(-1)e^{-\kappa_{sv}/12}, \quad sv = r, \pi, \eta.$$

In the next section, we consider whether the estimated risk premia associated with these innovations can account for the observed returns on size and B/M sorted portfolios and on industry portfolios.

B. Cross-sectional Pricing

Under the simple ICAPM described in Section II, the pricing kernel is a linear function of the excess return on the market portfolio and the innovations in the state variables r and

η so that the unconditional risk premium on asset i may be written as

$$\begin{aligned}\mu_i - r &= \bar{\eta}(\beta_{iM}\rho_{mM}\sigma_M + \beta_{ir}\rho_{rm}\sigma_r + \beta_{i\eta}\rho_{\eta m}\sigma_\eta) \\ &= \beta_{iM}\lambda_M + \beta_{ir}\lambda_r + \beta_{i\eta}\lambda_\eta,\end{aligned}\tag{20}$$

where the λ 's are the unconditional premia for market risk, real interest rate risk, and Sharpe ratio risk, and β_{iM} , etc., are coefficients from the regression of asset returns on market excess returns and state variable innovations:

$$R_i - R_f = a_i + \beta_{iM}(R_M - R_f) + \beta_{i\eta}\Delta\eta + \beta_{ir}\Delta r + \epsilon_i.\tag{21}$$

We can directly test the implications of the ICAPM for stock returns by using the two-stage cross-sectional regression procedure described in chapter 12 of Cochrane (2001). Initially, we use as test returns the monthly excess returns on the 25 Fama-French size and book-to-market sorted portfolios for the period from January 1952 to December 2000. In the first stage, we regress the 25 portfolio returns on the market excess return, and the estimated innovations in the real interest rate and the maximum Sharpe ratio to obtain the three betas, as described in equation (21). In the second stage, we estimate the cross-sectional regression:

$$\overline{R_i - R_f} = \hat{\beta}_{iM}\lambda_M + \hat{\beta}_{ir}\lambda_r + \hat{\beta}_{i\eta}\lambda_\eta + u_i, \quad i = 1, \dots, 25,$$

which corresponds to (20), with μ_i replaced by portfolio i 's sample mean \bar{R}_i and betas replaced by their estimates from the first stage.

The model mispricing for portfolio i , $\hat{\alpha}_i$, is then defined by:

$$\hat{\alpha}_i \equiv \overline{R_i - R_f} - \left(\hat{\beta}_{iM} \hat{\lambda}_M + \hat{\beta}_{ir} \hat{\lambda}_r + \hat{\beta}_{i\eta} \hat{\lambda}_\eta \right).$$

The variance-covariance matrix of the pricing errors, $\Sigma \equiv \text{Cov}(\hat{\alpha})$, is calculated using Shanken's (1992) correction. The model restrictions are tested using the quadratic form $\hat{\alpha}' \Sigma^{-1} \hat{\alpha}$, which is distributed as $\chi^2(22)$ under the null hypothesis. For comparison, we also estimate and test the CAPM and the Fama-French three-factor model using the same approach.

The estimated risk premia and the χ^2 test statistics for the three models are reported in Table II along with the sample means of the three Fama-French portfolio returns. For the whole sample period, the χ^2 test statistic does not reject the pricing restrictions implied by the simple ICAPM (p -value $\approx 20\%$), while both the CAPM and the Fama-French three-factor model are strongly rejected (p -values $< 1\%$). The CAPM and Fama-French three-factor models are also rejected using the Gibbons-Ross-Shanken (1989) (GRS) test for the joint significance of the a 's in OLS regressions of portfolio excess returns on the market excess return and (for the three-factor model) the returns on *SMB* and *HML*.

Insert Table II about here

For the ICAPM, both the real interest rate risk and the Sharpe ratio risk are significantly priced; moreover, the signs of the risk premia are consistent with the estimates of the corre-

lations of the innovations in these variables with the pricing kernel reported in Table I. For the Fama-French three-factor model, the factor *HML* carries a significant 0.4% per month risk premium, which is close to the mean sample return on the *HML* portfolio of 0.36% per month; the risk premium for the *SMB* factor is only 0.08% per month and not significant, and the mean return on the *SMB* portfolio for the sample period is only 0.10% per month and is not significant either. The estimated monthly market risk premium is 0.56% from the ICAPM, 0.58% from the FF three-factor model, and 0.70% from the CAPM, as compared to the sample mean of the market excess return of 0.62% per month.

The estimated loadings of the portfolio returns on the market return and the state variable innovations are reported in Table III. There is wide dispersion across the portfolios in their exposures to innovations in the state variables, and these have large effects on estimated risk premiums. Small firms have higher values of β_η and β_r than big firms - the average value of β_r for the small firm portfolios exceeds that of the large firm portfolios by 1.40; using the value of λ_r in Panel A of Table 2, this translates into an annualized excess risk premium for small firms over big firms due to *r* risk of *minus* 7.2% per year. The difference between the average value of β_η for the small and large firm portfolios is 5.0×10^{-2} , which translates into an annualized risk premium difference of 8.2%. Combining the two risk premia differences yields an average excess risk premium for small firms relative to large firms due to η and *r* risk of 1.0%. High B/M firms have higher values of β_η and *lower* values of β_r than low B/M firms. The average difference in β_η (β_r) between the high and low B/M portfolios is 2.33×10^{-2} (-0.42), which, using the risk premium values in Panel A of Table II, translates into annualized risk premium differences due to η and *r* risk of 3.8% and 2.2%, respectively,

so that the combined effect is 6.0% per year.

Insert Table III about here

The mispricing, $\hat{\alpha}$, for each of the three models is reported in Table IV. The individual α 's are about the same magnitude under the ICAPM as those under the FF three-factor model, with an average absolute mispricing at around 10 basis points per month (1.2% per year) across the 25 portfolios, so that the theoretically motivated ICAPM performs as well as the FF three-factor model in this regard. Interestingly, there is considerable overlap between the portfolios whose $\hat{\alpha}$'s have significant t -statistics under the ICAPM and the three-factor model. Small, high book-to-market portfolios are significantly mispriced by both models, as are small and big low book-to-market portfolios; however, the t -statistics are more significant for the three-factor model, and there are other portfolios that are well-priced by the ICAPM but are mispriced by the three-factor model. The difference between the t -statistics (and the χ^2 statistics) for the FF three-factor model and the ICAPM is associated with the smaller residual variance in the time-series regressions of portfolio returns on the three FF factors than in the corresponding regressions on the three ICAPM variables. The test results for the two halves of the sample period are reported in Panels B and C of Table II: The CAPM and the three-factor model are rejected in both subperiods, while the ICAPM is rejected only for the second subperiod. The estimated risk premia for r and η , and for the Fama-French arbitrage portfolios, are qualitatively similar across the two subperiods.

Insert Table IV about here

When the two special portfolios from the Fama-French three-factor model, *SMB* and *HML*, are added as additional test assets in the cross-sectional test, the ICAPM is still not rejected, and the mispricings for *SMB* and *HML* are -0.07% and 0.02% per month, respectively, and are statistically insignificant. Thus the ICAPM is able to account for the mean excess returns on the two Fama-French risk factors themselves.

The standard errors used to compute the t -ratios reported in Table II are calculated using the Shanken (1992) correction to account for errors in the first-stage beta estimates. However, the beta estimates from the first stage will be biased to the degree that the factors Δr and $\Delta \eta$ are estimated with error. Then the second-stage parameter estimates are biased due to the errors in the estimated betas. Therefore, as a robustness check, we also estimate and test the three models using the one-step generalized method of moments discount factor (GMM/DF) approach. A discrete time version of the pricing model (5) implies that $E[\psi(\tilde{R} - R_f)] = 0$ where ψ can be written as a linear function of the three ICAPM factors:

$$\psi = 1 + b_M R_M + b_r \Delta r + b_\eta \Delta \eta.$$

Define $g_T(b) = \frac{1}{T} \sum_{t=1}^T \psi_t (R_t - R_f \mathbf{1}_{25})$ where $b = (b_1, b_2, b_3)'$, R_t is a (1×25) vector of portfolio returns, and $\mathbf{1}_{25}$ is a vector of ones with length 25. The GMM/DF approach is carried out by minimizing the quadratic form $g_T(b) W g_T(b)'$ with respect to b , where the weighting matrix W is set to be the inverse of the variance covariance matrix of the pricing

errors, S^{-1} . The same approach is also applied to the FF three-factor model and the CAPM. Under the null hypothesis of the ICAPM, $TJ_T \equiv Tg_T(\hat{b})S^{-1}g_T(\hat{b})'$ is distributed as $\chi^2(22)$.

Again for the whole sample period, the simple ICAPM is not rejected (p -value = 6.04%) while the three-factor model and the CAPM are rejected at significance levels of $< 1\%$. The estimated coefficients of the ICAPM pricing kernel are (t -ratios in parentheses): $\hat{b}_1 = -4.40$ (3.1), $\hat{b}_2 = -69.48$ (1.4), and $\hat{b}_3 = -7.51$ (3.8). When the tests are repeated for the two halves of the sample period, the ICAPM is not rejected for the first half sample with a p -value of 11.2% but is rejected at better than the 1% significance level for the second subperiod. The estimated coefficients of the pricing kernel are (t -ratios in parentheses): $\hat{b}_1 = -3.07$ (1.9), $\hat{b}_2 = -77.14$ (1.3), and $\hat{b}_3 = -7.06$ (3.2) for the first subperiod, and $\hat{b}_1 = -6.89$ (3.8), $\hat{b}_2 = -57.02$ (1.6), and $\hat{b}_3 = -4.35$ (2.9) for the second subperiod. The FF three-factor model and the CAPM are rejected for both subperiods at better than the 1% significance level. Thus, the GMM results confirm the cross-sectional pricing results.

Lo and MacKinlay (1990) advise caution in drawing inferences from samples of characteristic sorted data such as the size and book-to-market sorted portfolios. Therefore, as a robustness check, we repeat the analysis using the returns on the 30 Fama-French industrial portfolios. The results are reported in Table V, which contains the test statistics and the estimated factor risk premia for the whole sample period and two subperiods, and in Table VI, which contains the pricing errors for the whole sample period.

Insert Table V about here

Insert Table VI about here

Now, none of the three models is rejected by the two-stage cross-sectional test, either for the whole period or for the subperiods, and the pricing errors of the models are similar. The estimates of the risk premium associated with the market factor of 0.64% to 0.75% per month are similar to those obtained with the 25 size and B/M portfolios, and are broadly consistent with the sample mean excess return on the market portfolio of 0.62% per month. However, the signs of the risk premium estimates associated with η for the ICAPM, and with both *SMB* and *HML* for the three-factor model, are reversed when the industry data are used, and are generally not significant. This is true both for the whole sample period and for the two subperiods. Moreover, for the three-factor model the estimated risk premia for the *SMB* (*HML*) factors of -0.30% (-0.38%) per month are quite different from the sample mean returns on the corresponding portfolios of 0.10% (0.36%) per month. As a result, a one-pass GRS F-test, which includes in the null hypothesis the equality of the factor risk premia and the corresponding portfolio risk premia (sample mean return), strongly rejects the three-factor model both for the whole sample period and for the two halves of the period, with p -values well under 1%. The CAPM is also rejected under the one-pass GRS test with a p -value of about 4%. The ICAPM does not constrain the factor risk premia, so the GRS test is not directly applicable to this model. Furthermore, if factor-mimicking portfolios²⁰ are constructed for the innovations in r and η , then in contrast to the CAPM and three-factor models, the ICAPM is not rejected by the GRS F-test using these portfolios as factors (p -

value = 8.3%). However, we do not place much weight on these mimicking portfolio results, because the portfolio weights of the mimicking portfolios are estimated with considerable error and in any case, as we shall see, the ICAPM is rejected by the two-stage cross-sectional tests reported below when the test assets include both the industrial portfolios and the size and book-to-market sorted portfolios. It is interesting to note that for all three models it is the tobacco industry, “Smoke,” that is the most mispriced - its return is from 3.6% to 6.2% *underpredicted* by the models.

Given the contrasting point estimates of the factor premia from the two sets of data, we examine how the models perform when the 55 portfolio returns are combined in a single test. The results are reported in Table VII. First, the estimated market risk premium is quite similar across the three models and the two subperiods at around 0.6% to 0.7% per month, which is close to the sample mean excess return on the market portfolio. For the three-factor model, the estimated risk premia associated with *SMB* and *HML* are now close to zero and insignificant: For *SMB*, the point estimate is around 25 basis points per year, and for *HML*, it is around 144 basis points, in both cases well below the sample mean returns on the corresponding portfolios. On the other hand, both ICAPM factor risk premia are highly significant in the full sample and though not significant, have the same sign in both subperiods. However, when the test assets include all 55 portfolios, all three models are rejected at better than the 1% level for each of the three sample periods. The GMM/DF tests yield similar rejections except that the ICAPM is not rejected in the first subperiod.

Insert Table VII about here

The fact that the simple ICAPM is rejected is not entirely surprising in view of the evidence of model mis-specification that we reported in Section III.A: Since r and η do not follow a joint Markov process, there must be additional state variables in the stochastic process for these two variables that describe the instantaneous investment opportunity set, and in general we should expect them to be priced. Moreover, in specifying the empirical version of the simple ICAPM, we have treated the stock market index as the market portfolio and the resulting mismeasurement of the return on aggregate wealth could also account for the model rejection.

IV. Conclusion

In this paper we have developed, estimated, and tested a simple model of asset valuation for a setting in which real interest rates and risk premia vary stochastically. The model implies that zero-coupon nominal bond yields are linearly related to the state variables r and η , the real interest rate and the maximal Sharpe ratio, as well as to the expected rate of inflation, π . Data on bond yields and expected inflation were used to provide estimates of the state variables and of the parameters of their joint stochastic process. The estimated real interest rate and Sharpe ratio both show strong business cycle-related variation, with the Sharpe ratio rising and the real interest rate falling, during recessions. The Sharpe ratio estimate was shown to be related to the excess return on the equity market portfolio, consistent with the model predictions. However, there was evidence of model mis-specification in that the filtered series for r and η did not possess the hypothesized Markov property.

The model was tested first on the 25 size and book-to-market portfolios. The estimated risk premia for both r and η were significant, and the model pricing restrictions were not rejected for the whole sample period, although they were rejected for one-half of the period. In contrast, both the CAPM and the Fama-French three-factor model were rejected for the whole sample period and the two subperiods.

Lo and Mackinlay (1990) have warned against testing asset pricing models on the returns of portfolios that have been formed on the basis of some characteristic that is known to be associated with returns. Since the size and book-to-market portfolios seem to meet this criterion, we also tested the model using the returns on 30 industry portfolios. None of the three models was rejected using these returns. However, the point estimates of the risk premia associated with r and η and with SMB and HML were quite different from those obtained using the 25 portfolios. The estimates of the risk premia associated with SMB and HML were also quite different from their corresponding sample means, and the one-pass GRS F-test strongly rejects the Fama-French three-factor model.

Finally, the models were estimated and tested using the combined sample of 55 portfolios. With these combined test assets the market risk premium was significant for all three models; and, while neither of the premia associated with the Fama-French arbitrage portfolios were significant, the premia associated with both r and η were highly significant. Despite this finding, the pricing restrictions of all three models were rejected. It is hypothesized that the rejection of the simple ICAPM could be due either to mismeasurement of the return on aggregate wealth or to the failure of the assumption that the two investment opportunity set state variables follow a Markov process. Nevertheless, these results with a highly simplified

ICAPM are sufficiently encouraging to warrant further empirical investigation of the ICAPM. We stress the need in empirical implementation of the ICAPM to pay careful attention to the selection of state variables: The ICAPM is not just another “factor model;” the state variables of the model must be limited to those that predict future investment opportunities.

Appendix A. Proof of Theorems 1 and 2

The real part of the economy is described by the processes for the real pricing kernel, the real interest rate, and the maximum Sharpe ratio, (5.1) to (5.3), while the nominal part of the economy is described by the processes for the price level and the expected inflation rate, (10) to (11). Under the risk neutral probability measure Q , we can write these processes as:

$$dr = \kappa_r(\bar{r} - r)dt - \sigma_r \rho_{mr} \eta dt + \sigma_r dz_r^Q \quad (\text{A1})$$

$$d\pi = \kappa_\pi(\bar{\pi} - \pi)dt - \sigma_\pi \rho_{m\pi} \eta dt + \sigma_\pi dz_\pi^Q \quad (\text{A2})$$

$$d\eta = \kappa_\eta^*(\bar{\eta}^* - \eta)dt + \sigma_\eta dz_\eta^Q, \quad (\text{A3})$$

where $\kappa_\eta^* = \kappa_\eta + \sigma_\eta \rho_{m\eta}$ and $\bar{\eta}^* = \frac{\kappa_\eta \bar{\eta}}{\kappa_\eta^*}$.

Let y , whose stochastic process is given by (6), denote the expectation of a nominal cash flow at a future date T , X_T . The process for $\xi \equiv y/P$, the deflated expectation of the nominal cash flow, under the risk neutral probability measure can be written as:

$$\frac{d\xi}{\xi} = [-\pi - \sigma_y \sigma_P \rho_{yP} + \sigma_P^2 - \eta(\sigma_y \rho_{ym} - \sigma_P \rho_{Pm})] dt + \sigma_y dz_y^Q - \sigma_P dz_P^Q. \quad (\text{A4})$$

The real value at time t of the claim to the nominal cash flow at time T , X_T , is given by the

expected discounted value of the real cash flow under Q :

$$\begin{aligned}
V(\xi, r, \pi, \eta, T - t) &= \mathbb{E}_t^Q \left[\frac{X_T}{P_T} \exp^{-\int_t^T r(s) ds} \right] = \mathbb{E}_t^Q \left[\frac{y_T}{P_T} \exp^{-\int_t^T r(s) ds} \right] \\
&= \mathbb{E}_t^Q \left[\xi_T \exp^{-\int_t^T r(s) ds} \right].
\end{aligned} \tag{A5}$$

Using equation (A4), we have

$$\begin{aligned}
\xi_T &= \xi_t \exp \left\{ \left(-\frac{1}{2} \sigma_y^2 + \frac{1}{2} \sigma_P^2 \right) (T - t) - (\sigma_y \rho_{ym} - \sigma_P \rho_{Pm}) \int_t^T \eta(s) ds \right. \\
&\quad \left. - \int_t^T \pi(s) ds + \sigma_y \int_t^T dz_y^Q - \sigma_P \int_t^T dz_P^Q \right\}.
\end{aligned} \tag{A6}$$

Tedious calculations from equations (A1), (A2), and (A3) easily lead to results for $\int_t^T \eta(s) ds$, $\int_t^T \pi(s) ds$, and $\int_t^T r(s) ds$. Substituting the expressions for $\int_t^T \eta(s) ds$, $\int_t^T \pi(s) ds$, and $\int_t^T r(s) ds$ into equation (A5) yields

$$V(\xi, r, \pi, \eta, T - t) = \xi_t G \mathbb{E}_t^Q [\exp^\varphi], \tag{A7}$$

where G is given by

$$G = \exp \{ E(\tau) - B(\tau) r_t - C(\tau) \pi_t - D(\tau) \eta_t \} \tag{A8}$$

and

$$B(\tau) = \frac{1 - e^{-\kappa_r(T-t)}}{\kappa_r} \quad (\text{A9})$$

$$C(\tau) = \frac{1 - e^{-\kappa_\pi(T-t)}}{\kappa_\pi} \quad (\text{A10})$$

$$D(\tau) = d_1 + d_2 e^{-\kappa_\eta^* \tau} + d_3 e^{-\kappa_r \tau} + d_4 e^{-\kappa_\pi \tau} \quad (\text{A11})$$

$$\begin{aligned} E(\tau) &= \left(-\frac{1}{2} \sigma_y^2 + \frac{1}{2} \sigma_P^2 - \bar{r} - \bar{\pi} - d_1 \kappa_\eta^* \bar{\eta}^* \right) \tau + (\bar{r} - d_3 \kappa_\eta^* \bar{\eta}^*) B(\tau) \\ &+ (\bar{\pi} - d_4 \kappa_\eta^* \bar{\eta}^*) C(\tau) - d_2 \kappa_\eta^* \bar{\eta}^* d(\tau) \end{aligned} \quad (\text{A12})$$

with

$$d_1 = -\frac{\sigma_P \rho_{mP} - \sigma_y \rho_{my}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{\kappa_r \kappa_\eta^*} - \frac{\sigma_\pi \rho_{m\pi}}{\kappa_\pi \kappa_\eta^*} \quad (\text{A13})$$

$$\begin{aligned} d_2 &= \frac{\sigma_P \rho_{mP} - \sigma_y \rho_{my}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r) \kappa_\eta^*} - \frac{\sigma_\pi \rho_{m\pi}}{(\kappa_\eta^* - \kappa_\pi) \kappa_\eta^*} \\ &= -d_1 - d_3 - d_4 \end{aligned} \quad (\text{A14})$$

$$d_3 = \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r) \kappa_r} \quad (\text{A15})$$

$$d_4 = \frac{\sigma_\pi \rho_{m\pi}}{(\kappa_\eta^* - \kappa_\pi) \kappa_\pi}. \quad (\text{A16})$$

The stochastic variable φ is a linear function of the Brownian motions:

$$\begin{aligned} \varphi &= \sigma_\eta \int_t^T [d_2 (1 - e^{-\kappa_\eta^*(T-s)}) + d_3 (1 - e^{-\kappa_r(T-s)}) + d_4 (1 - e^{-\kappa_\pi(T-s)})] dz_\eta^*(s) \\ &- \frac{\sigma_r}{\kappa_r} \int_t^T (1 - e^{-\kappa_r(T-s)}) dz_r^*(s) - \frac{\sigma_\pi}{\kappa_\pi} \int_t^T (1 - e^{-\kappa_\pi(T-s)}) dz_\pi^*(s) \\ &+ \sigma_y \int_t^T dz_y^*(s) - \sigma_P \int_t^T dz_P^*(s). \end{aligned} \quad (\text{A17})$$

Since φ is normally distributed with mean zero, V is given by

$$V(\xi, r, \pi, \eta, T - t) = \xi_t G \exp \left\{ \frac{1}{2} \text{Var}_t(\varphi) \right\}. \quad (\text{A18})$$

Calculating $\text{Var}_t(\varphi)$ and collecting terms, we get that

$$V(\xi, r, \pi, \eta, T - t) = \xi_t \exp \{ A(\tau) - B(\tau)r_t - C(\tau)\pi_t - D(\tau)\eta_t \}, \quad (\text{A19})$$

where

$$\begin{aligned} A(\tau) = & a_1\tau + a_2 \frac{1 - e^{-\kappa_r\tau}}{\kappa_r} + a_3 \frac{1 - e^{-\kappa_\pi\tau}}{\kappa_\pi} + a_4 \frac{1 - e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*} \\ & + a_5 \frac{1 - e^{-2\kappa_r\tau}}{2\kappa_r} + a_6 \frac{1 - e^{-2\kappa_\pi\tau}}{2\kappa_\pi} + a_7 \frac{1 - e^{-2\kappa_\eta^*\tau}}{2\kappa_\eta^*} \\ & + a_8 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r} + a_9 \frac{1 - e^{-(\kappa_\eta^* + \kappa_\pi)\tau}}{\kappa_\eta^* + \kappa_\pi} + a_{10} \frac{1 - e^{-(\kappa_r + \kappa_\pi)\tau}}{\kappa_r + \kappa_\pi}. \end{aligned} \quad (\text{A20})$$

Define $a_0 \equiv \frac{\sigma_{r\eta}}{\kappa_r} + \frac{\sigma_{\pi\eta}}{\kappa_\pi} + \sigma_{P\eta} - \sigma_{y\eta} - \kappa_\eta^* \bar{\eta}^*$, $\bar{r}^* \equiv \bar{r} - \frac{\sigma_{Pr} - \sigma_{yr}}{\kappa_r}$, and $\bar{\pi}^* \equiv \bar{\pi} - \frac{\sigma_{P\pi} - \sigma_{y\pi}}{\kappa_\pi}$, then

a_1, \dots, a_{10} are expressed as

$$a_1 = \sigma_P^2 - \sigma_{yP} + \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\pi^2}{2\kappa_\pi^2} + \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} + \frac{\sigma_\eta^2}{2}d_1^2 - \bar{r}^* - \bar{\pi}^* + a_0d_1 \quad (\text{A21})$$

$$a_2 = \bar{r}^* - \frac{\sigma_r^2}{\kappa_r^2} - \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{r\eta}}{\kappa_r}d_1 + a_0d_3 + \sigma_\eta^2d_1d_3 \quad (\text{A22})$$

$$a_3 = \bar{\pi}^* - \frac{\sigma_\pi^2}{\kappa_\pi^2} - \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_1 + a_0d_4 + \sigma_\eta^2d_1d_4 \quad (\text{A23})$$

$$a_4 = a_0d_2 + \sigma_\eta^2d_1d_2 \quad (\text{A24})$$

$$a_5 = \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\eta^2}{2}d_3^2 - \frac{\sigma_{r\eta}}{\kappa_r}d_3 \quad (\text{A25})$$

$$a_6 = \frac{\sigma_\pi^2}{2\kappa_\pi^2} + \frac{\sigma_\eta^2}{2}d_4^2 - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_4 \quad (\text{A26})$$

$$a_7 = \frac{\sigma_\eta^2}{2}d_2^2 \quad (\text{A27})$$

$$a_8 = -\frac{\sigma_{r\eta}}{\kappa_r}d_2 + \sigma_\eta^2d_2d_3 \quad (\text{A28})$$

$$a_9 = -\frac{\sigma_{\pi\eta}}{\kappa_\pi}d_2 + \sigma_\eta^2d_2d_4 \quad (\text{A29})$$

$$a_{10} = \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_3 - \frac{\sigma_{r\eta}}{\kappa_r}d_4 + \sigma_\eta^2d_3d_4. \quad (\text{A30})$$

Theorems 1 and 2 follow as special cases of equation (A19). Theorem 1 is obtained by setting σ_P and the parameters in the expected inflation process (A2) to zero. Theorem 2 is obtained by setting σ_y to zero.

Appendix B. Details of Kalman Filter

The yield-based estimates of the state variable dynamics are derived by applying a Kalman filter to data on bond yields and inflation using equation (15). The transition equations for the state variables, r , π , and η , are derived by discretizing equations (5.2), (5.3), and (11):

$$\begin{pmatrix} r_t \\ \pi_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} e^{-\kappa_r \Delta t} & 0 & 0 \\ 0 & e^{-\kappa_\pi \Delta t} & 0 \\ 0 & 0 & e^{-\kappa_\eta \Delta t} \end{pmatrix} \begin{pmatrix} r_{t-\Delta t} \\ \pi_{t-\Delta t} \\ \eta_{t-\Delta t} \end{pmatrix} + \begin{pmatrix} \bar{r} [1 - e^{-\kappa_r \Delta t}] \\ \bar{\pi} [1 - e^{-\kappa_\pi \Delta t}] \\ \bar{\eta} [1 - e^{-\kappa_\eta \Delta t}] \end{pmatrix} + \begin{pmatrix} \epsilon_r(t) \\ \epsilon_\pi(t) \\ \epsilon_\eta(t) \end{pmatrix} \quad (\text{B1})$$

where the vector of innovations is related to the standard Brownian motions, dz_r , dz_π and dz_η , by

$$\begin{pmatrix} \epsilon_r(t) \\ \epsilon_\pi(t) \\ \epsilon_\eta(t) \end{pmatrix} = \begin{pmatrix} \sigma_r e^{-\kappa_r t} \int_{t-\Delta t}^t e^{\kappa_r \tau} dz_r(\tau) \\ \sigma_\pi e^{-\kappa_\pi t} \int_{t-\Delta t}^t e^{\kappa_\pi \tau} dz_\pi(\tau) \\ \sigma_\eta e^{-\kappa_\eta t} \int_{t-\Delta t}^t e^{\kappa_\eta \tau} dz_\eta(\tau) \end{pmatrix}, \quad (\text{B2})$$

and the variance-covariance matrix of the innovations is

$$Q = \begin{pmatrix} \frac{\sigma_r^2}{2\kappa_r} [1 - e^{-2\kappa_r \Delta t}] & \frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\kappa_r + \kappa_\pi} [1 - e^{-(\kappa_r + \kappa_\pi) \Delta t}] & \frac{\sigma_r \sigma_\eta \rho_{r\eta}}{\kappa_r + \kappa_\eta} [1 - e^{-(\kappa_r + \kappa_\eta) \Delta t}] \\ \frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\kappa_r + \kappa_\pi} [1 - e^{-(\kappa_r + \kappa_\pi) \Delta t}] & \frac{\sigma_\pi^2}{2\kappa_\pi} [1 - e^{-2\kappa_\pi \Delta t}] & \frac{\sigma_\pi \sigma_\eta \rho_{\pi\eta}}{\kappa_\pi + \kappa_\eta} [1 - e^{-(\kappa_\pi + \kappa_\eta) \Delta t}] \\ \frac{\sigma_r \sigma_\eta \rho_{r\eta}}{\kappa_r + \kappa_\eta} [1 - e^{-(\kappa_r + \kappa_\eta) \Delta t}] & \frac{\sigma_\pi \sigma_\eta \rho_{\pi\eta}}{\kappa_\pi + \kappa_\eta} [1 - e^{-(\kappa_\pi + \kappa_\eta) \Delta t}] & \frac{\sigma_\eta^2}{2\kappa_\eta} [1 - e^{-2\kappa_\eta \Delta t}] \end{pmatrix}. \quad (\text{B3})$$

The first n observation equations assume that the observed yields at time t , $y_{\tau_j, t}$, on bonds with maturities τ_j , $j = 1, \dots, n$, are given by equation (15) plus measurement error terms, ϵ_{τ_j} :

$$y_{\tau_j, t} \equiv -\frac{\ln V(t, t + \tau_j)}{\tau_j} = -\frac{A(t, \tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j} r_t + \frac{C(\tau_j)}{\tau_j} \pi_t + \frac{D(\tau_j)}{\tau_j} \eta_t + \epsilon_{\tau_j}(t). \quad (\text{B4})$$

The measurement errors, $\epsilon_{\tau_j}(t)$, are assumed to be serially and cross-sectionally uncorrelated

and are uncorrelated with the innovations in the transition equations. The $n + 1$ observation equation uses the Livingston estimate of the rate of inflation:

$$\pi_{\text{liv}} = \pi + \epsilon_{\text{liv}}, \tag{B5}$$

where ϵ_{liv} is assumed to be uncorrelated with the yield measurement errors and the innovations in the transition equation.

Table I
Summary Statistics on Bond Yields and Model Parameter Estimates

Panel A of the table reports summary statistics for the bond yield data. The bond data are monthly observations on constant maturity zero-coupon U.S. Treasury yields for the period from January 1952 to December 2000. Panel B reports the summary statistics for the inflation data. CPI inflation is calculated using CPI data for the same sample period while the Livingston inflation is calculated using the Livingston survey data. Panel C reports estimates of the parameters of the stochastic process of the investment opportunity set, equations (5.2) to (5.3), obtained from a Kalman filter applied to the inflation and bond yield data with $\bar{r} = 1.62\%$, $\bar{\pi} = 3.85\%$ and $\bar{\eta} = 0.7$, where \bar{r} and $\bar{\pi}$ are the sample means, and $\bar{\eta}$ is 20% higher than the CRSP value-weighted market index Sharpe ratio. The notation η is the Sharpe ratio; r is the real interest rate; m is the pricing kernel; π is the expected rate of inflation, and P is the price level. Asymptotic t -ratios are in parentheses.

A. Bond Yields (% per year)								
Bond maturity (years)	0.25	0.5	1	2	3	4	5	10
Mean	5.47	5.70	5.90	6.12	6.26	6.36	6.44	6.64
Std. dev.	2.86	2.90	2.88	2.83	2.79	2.77	2.76	2.72
B. Inflation (% per year)								
	CPI Inflation	Livingston Inflation						
Mean	3.85%	3.15%						
Std. dev.	1.16%	0.68%						
C. Parameter Estimates								
	σ_b	σ_r	σ_π	σ_η	κ_r	κ_π	κ_η	
Estimate	0.16%	1.11%	0.75%	30.79%	0.074	0.000	0.047	
t -ratio	(86.75)	(11.76)	(12.58)	(8.52)	(14.15)	(0.75)	(5.44)	
	$\rho_{r\pi}$	$\rho_{r\eta}$	$\rho_{r\pi\eta}$	$\rho_{\pi\eta}$	$\rho_{\pi m}$	$\rho_{\eta m}$	ρ_{Pm}	
Estimate	0.114	-0.528	-0.852	-0.283	-0.219	0.962	0.383	
t -ratio	(1.30)	(6.27)	(11.21)	(3.52)	(1.05)	(4.98)	(1.89)	
	\bar{r}	$\bar{\pi}$	$\bar{\eta}$	σ_P	σ_{liv}	ML		
Pre-set value	1.62%	3.85%	0.700	1.16%	0.35%	30,962.63		

Table II
Joint Test of ICAPM, FF Three-Factor Model, and CAPM for 25 Size and
Book-to-Market Sorted Portfolios: $\overline{\mathbf{R}} - \overline{R}_f = \hat{\beta}\lambda + \epsilon$

This table reports the two-stage cross-sectional regression results under the ICAPM, the FF three-factor model, and the CAPM. In the first stage, the excess returns on 25 size and book-to-market sorted portfolios are regressed on the model factors to obtain estimates of the factor loadings, $\hat{\beta}$. Under the ICAPM, the factors are the market risk premium, $R_M - R_f$, and the innovations in the state variables, Δr and $\Delta\eta$. Under the FF three-factor model, the factors are $R_M - R_f$, SMB , and HML . Under the CAPM, the factor is simply $R_M - R_f$. In the second stage, the sample means of the monthly portfolio excess returns are regressed on the betas without the intercept: $\overline{\mathbf{R}} - \overline{R}_f = \hat{\beta}\lambda + \epsilon$, and then the mispricing is calculated as $\hat{\alpha} = \overline{\mathbf{R}} - \overline{R}_f - \hat{\beta}\lambda$. A χ^2 test is then performed on $\hat{\alpha}$ to test the joint significance of the mispricing. The individual t -ratios and the variance covariance matrix, Σ , are calculated using Shanken's adjustment. The sample period is from January 1952 to December 2000.

Panel A: January 1952 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.56%	-0.43%	13.59%	0.58%	0.08%	0.40%	0.70%	0.62%	0.10%	0.36%
t -ratios	3.11	2.39	3.44	3.28	0.62	3.48	3.75	3.55	0.82	3.26
$\alpha' \Sigma^{-1} \alpha$		27.31			70.04		94.43			
p -value		19.97%			< 1%		< 1%			
Panel B: January 1952 to June 1976										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.43%	-0.15%	7.53%	0.53%	0.03%	0.44%	0.62%	0.55%	0.06%	0.39%
t -ratios	1.75	1.43	2.67	2.23	0.21	3.23	2.41	2.32	0.72	4.68
$\alpha' \Sigma^{-1} \alpha$		28.83			46.16		57.39			
p -value		14.98%			< 1%		< 1%			
Panel C: June 1976 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.69%	-0.43%	15.93%	0.64%	0.13%	0.34%	0.78%	0.68%	0.14%	0.34%
t -ratios	2.60	2.17	2.57	2.47	0.68	1.84	2.87	2.65	0.74	1.90
$\alpha' \Sigma^{-1} \alpha$		41.94			89.10		108.95			
p -value		< 1%			< 1%		< 1%			

Table III
Beta Estimates under the ICAPM for 25 Size and Book-to-Market Sorted
Portfolios: January 1952 to December 2000

This table reports the beta estimates under the ICAPM, $\hat{\beta}_k$ ($k = 1, \dots, 3$), in the first stage of the two-stage cross-sectional regression for 25 size and book-to-market sorted portfolios, where the portfolio excess returns are regressed in the market excess return, and the innovations to the two state variables, Δr and $\Delta \eta$. The sample period is from January 1952 to December 2000.

Size	Book-to-Market					Book-to-Market				
	Low	2	3	4	High	Low	2	3	4	High
	$\hat{\beta}_{mkt}$					$t(\hat{\beta}_{mkt})$				
Small	1.36	1.21	1.08	1.02	1.03	23.27	26.12	24.67	24.16	22.76
2	1.38	1.16	1.02	0.98	1.06	31.45	29.59	26.31	26.97	23.23
3	1.32	1.09	0.99	0.93	0.98	37.47	35.02	27.72	26.50	21.48
4	1.22	1.06	0.98	0.92	1.01	44.08	33.77	29.92	26.80	21.78
Big	1.03	0.96	0.86	0.84	0.85	48.83	45.06	31.19	23.45	18.19
	$\hat{\beta}_\eta \times 100$					$t(\hat{\beta}_\eta)$				
Small	2.83	4.24	4.40	4.50	5.94	1.38	2.61	3.47	3.28	3.54
2	1.17	3.04	2.88	3.53	4.20	0.81	2.62	2.81	3.26	3.29
3	1.52	2.30	2.77	2.92	2.58	1.45	2.37	2.89	2.76	1.96
4	1.26	0.36	1.48	2.20	3.12	1.74	0.43	1.43	2.27	2.36
Big	-1.36	-1.19	-1.84	0.10	1.21	2.08	1.77	2.10	0.11	1.01
	$\hat{\beta}_r$					$t(\hat{\beta}_r)$				
Small	1.19	1.27	0.94	1.03	1.18	2.03	2.99	2.51	2.88	2.99
2	0.91	0.69	0.22	0.22	0.48	2.25	2.07	0.73	0.67	1.29
3	0.79	0.51	0.23	-0.19	-0.16	2.67	1.94	0.78	0.54	0.43
4	0.49	-0.19	-0.26	-0.45	-0.04	1.92	0.73	0.94	1.43	0.10
Big	0.06	-0.21	-0.59	-0.51	-0.12	0.26	1.04	2.39	1.62	0.28

Table IV
Mispricing under ICAPM, FF Three-Factor Model, and CAPM for 25 Size and Book-to-Market Sorted Portfolios: January 1952 to December 2000

This table reports the model mispricing, $\hat{\alpha}_i$, in the two-stage cross-sectional regression for 25 size and book-to-market sorted portfolios. For the ICAPM, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M + \hat{\beta}_{ir}\hat{\lambda}_r + \hat{\beta}_{i\eta}\hat{\lambda}_\eta)$. For the FF three-factor model, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M + \hat{\beta}_{iSMB}\hat{\lambda}_{SMB} + \hat{\beta}_{iHML}\hat{\lambda}_{HML})$. For the CAPM, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M)$. The sample period is from January 1952 to December 2000.

Size	Book-to-Market					Book-to-Market				
	Low	2	3	4	High	Low	2	3	4	High
	$\alpha(\%)$					$t(\alpha)$				
ICAPM										
Small	-0.35	0.03	0.00	0.26	0.21	2.31	0.33	0.04	3.45	2.20
2	-0.11	-0.08	0.03	0.01	0.08	1.20	0.92	0.33	0.19	0.96
3	-0.04	0.07	-0.06	-0.06	-0.01	0.45	0.75	0.75	0.80	0.12
4	-0.02	-0.14	-0.04	-0.14	-0.07	0.20	1.68	0.50	1.43	0.60
Big	0.24	0.13	0.20	0.00	0.07	2.07	1.40	1.95	0.10	0.42
Fama-French Three-Factor Model										
Small	-0.36	0.02	0.05	0.20	0.15	3.99	0.23	0.93	4.30	3.01
2	-0.15	-0.04	0.12	0.10	0.05	2.54	0.65	2.37	1.90	0.99
3	0.04	0.06	-0.04	0.06	-0.04	0.64	1.05	0.69	1.26	0.70
4	0.15	-0.13	0.02	0.02	-0.09	3.10	2.28	0.39	0.34	1.20
Big	0.19	-0.02	0.05	-0.15	-0.19	2.86	0.31	0.66	2.32	2.27
CAPM										
Small	-0.66	-0.09	0.05	0.30	0.37	4.16	0.79	0.60	3.76	4.00
2	-0.53	-0.12	0.18	0.26	0.30	4.95	1.80	3.28	4.29	3.90
3	-0.35	0.02	0.08	0.28	0.26	3.60	0.27	1.21	3.97	3.09
4	-0.23	-0.16	0.13	0.22	0.23	2.36	2.23	1.70	2.75	2.16
Big	-0.12	-0.08	0.08	0.10	0.16	1.00	0.78	0.71	0.86	1.26

Table V
Joint Test of ICAPM, FF Three-Factor Model, and CAPM for 30 Industrial
Portfolios: $\overline{\mathbf{R}} - \overline{R}_f = \hat{\beta}\lambda + \epsilon$

This table reports the two-stage cross-sectional regression results under the ICAPM, the FF three-factor model, and the CAPM. In the first stage, the excess returns on 30 industrial portfolios are regressed on the model factors to obtain estimates of factor loadings, $\hat{\beta}$. Under the ICAPM, the factors are the market risk premium, $R_M - R_f$, and the innovations in the state variables, Δr and $\Delta\eta$. Under the FF three-factor model, the factors are $R_M - R_f$, SMB , and HML . Under the CAPM, the factor is simply $R_M - R_f$. In the second stage, the sample means of the monthly portfolio excess returns are regressed on the betas without the intercept: $\overline{\mathbf{R}} - \overline{R}_f = \beta\lambda + \epsilon$, and then the mispricing is calculated as $\hat{\alpha} = \overline{\mathbf{R}} - \overline{R}_f - \hat{\beta}\lambda$. A χ^2 test is then performed on $\hat{\alpha}$ to test the joint significance of the mispricing. The t -ratios and the variance covariance matrix, Σ , are calculated using Shanken's adjustment. Results are reported for the whole sample period of from January 1952 to December 2000 and two equal subsamples.

Panel A: January 1952 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.69%	-0.08%	-4.01%	0.75%	-0.30%	-0.38%	0.64%	0.62%	0.10%	0.36%
t -ratios	3.81	1.09	1.45	4.18	1.82	2.51	3.52	3.55	0.82	3.26
$\alpha' \Sigma^{-1} \alpha$		18.73			18.80		25.59			
p -value		87.95%			87.71%		64.72%			
Panel B: January 1952 to June 1976										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.65%	0.04%	-2.24%	0.69%	-0.08%	-0.25%	0.62%	0.55%	0.06%	0.39%
t -ratios	2.69	0.61	1.12	2.88	0.44	1.35	2.52	2.32	0.72	4.68
$\alpha' \Sigma^{-1} \alpha$		20.91			18.80		22.54			
p -value		79.04%			87.71%		79.73%			
Panel C: July 1976 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_M	λ_r	λ_η	λ_M	λ_{SMB}	λ_{HML}	λ_M	$R_M - R_f$	SMB	HML
Estimate	0.71%	-0.09%	-7.82%	0.79%	-0.86%	-0.49%	0.66%	0.68%	0.14%	0.34%
t -ratios	2.60	0.76	1.75	3.01	2.85	2.08	2.46	2.65	0.74	1.90
$\alpha' \Sigma^{-1} \alpha$		19.18			25.15		32.99			
p -value		86.33%			56.59%		27.82%			

Table VI
Mispricing under ICAPM, FF Three-Factor Model and CAPM for 30
Industrial Portfolios: January 1952 to December 2000

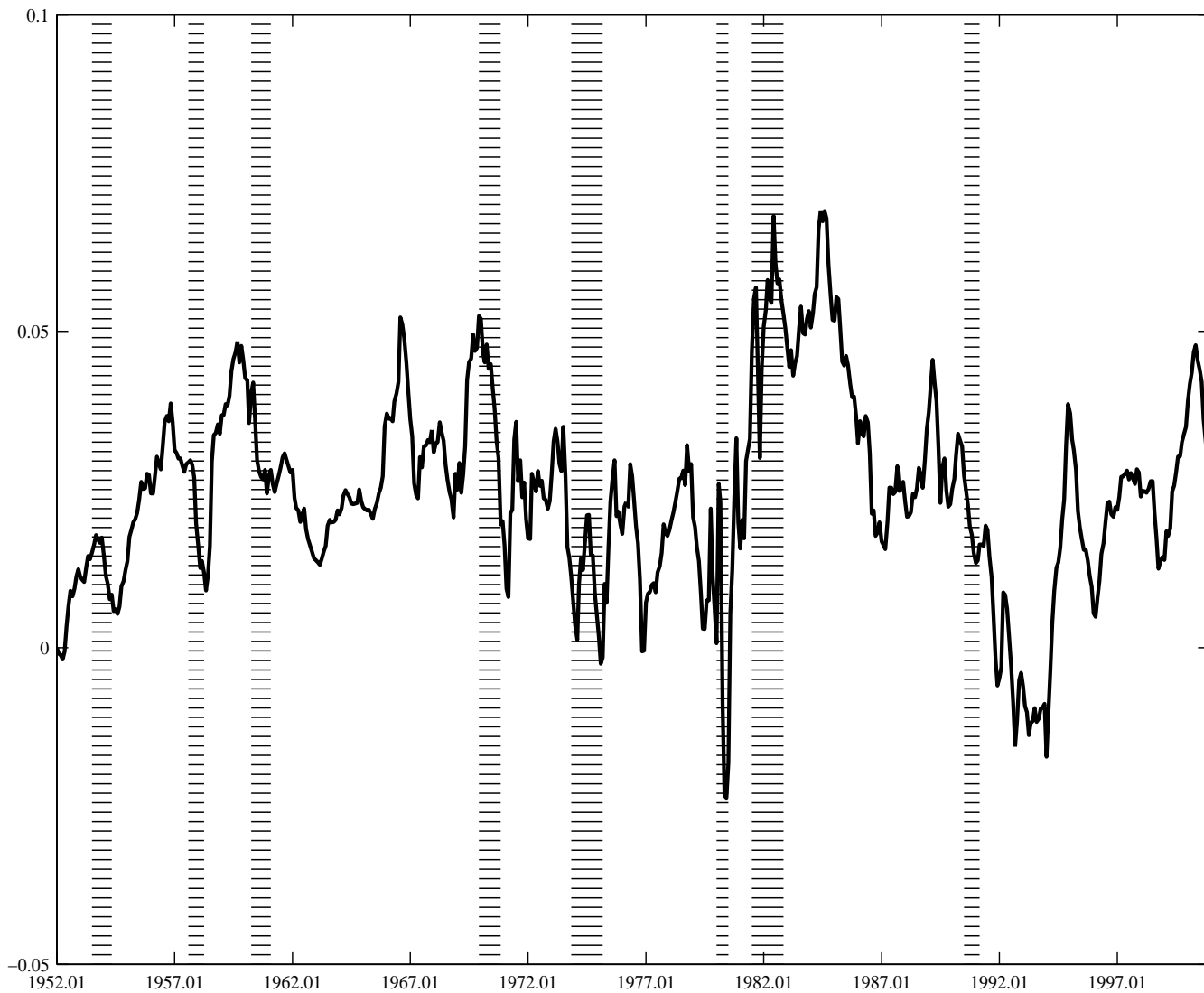
This table reports the model mispricing, $\hat{\alpha}_i$, in the two-stage cross-sectional regression for 30 industrial portfolios. For the ICAPM, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M + \hat{\beta}_{ir}\hat{\lambda}_r + \hat{\beta}_{i\eta}\hat{\lambda}_\eta)$. For the FF three-factor model, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M + \hat{\beta}_{iSMB}\hat{\lambda}_{SMB} + \hat{\beta}_{iHML}\hat{\lambda}_{HML})$. For the CAPM, $\hat{\alpha}_i \equiv \overline{R_i - R_f} - (\hat{\beta}_{iM}\hat{\lambda}_M)$. The sample period is from January 1952 to December 2000.

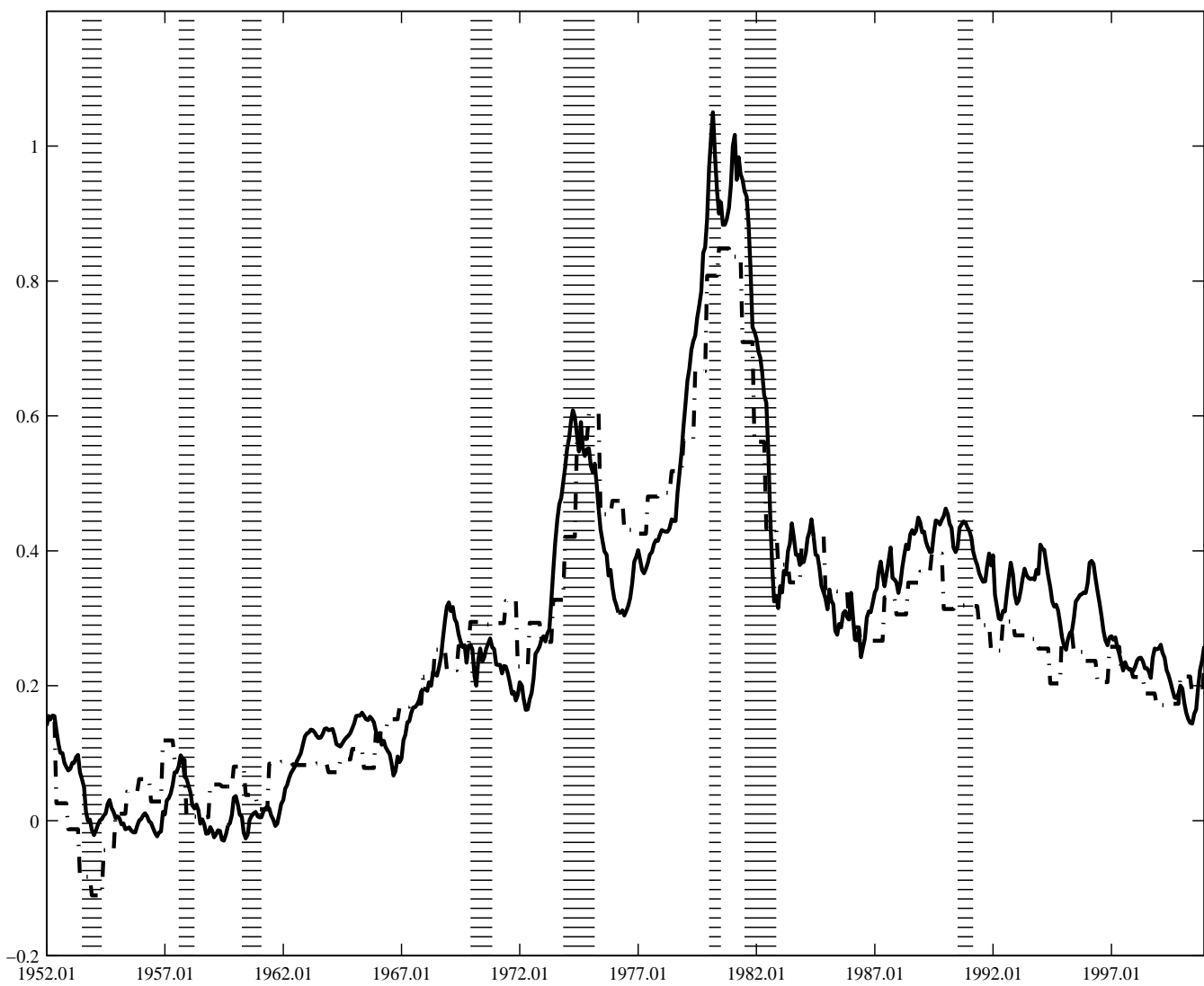
Industry	ICAPM		FF		CAPM	
	$\alpha(\%)$	$t(\alpha)$	$\alpha(\%)$	$t(\alpha)$	$\alpha(\%)$	$t(\alpha)$
Food	0.18	1.74	0.14	1.37	0.26	2.23
Beer	0.20	1.26	0.09	0.63	0.12	0.83
Smoke	0.30	1.55	0.42	2.25	0.52	2.63
Games	0.07	0.46	0.07	0.52	-0.01	0.03
Books	0.10	0.78	0.12	1.08	0.10	0.94
Hshld	0.11	0.97	-0.07	0.80	0.15	1.41
Clths	-0.19	1.15	-0.01	0.04	-0.20	1.43
Hlth	0.12	0.83	0.02	0.14	0.28	2.01
Chems	-0.11	0.86	-0.22	2.17	-0.11	1.05
Txtls	0.04	0.29	0.10	0.81	-0.19	1.36
Cnstr	-0.21	2.20	-0.11	1.39	-0.15	1.83
Steel	-0.10	0.77	-0.21	1.51	-0.31	2.17
FabPr	-0.17	1.68	-0.17	1.84	-0.18	1.97
ElcEq	0.09	0.56	-0.03	0.24	0.02	0.15
Autos	0.02	0.13	0.00	0.01	-0.01	0.07
Carry	0.01	0.09	0.16	1.09	0.11	0.75
Mines	-0.01	0.07	-0.11	0.57	-0.25	1.25
Coal	0.03	0.14	0.01	0.06	-0.08	0.35
Oil	0.07	0.39	-0.01	0.09	0.15	0.91
Util	-0.04	0.41	0.15	1.37	0.19	1.50
Telcm	0.03	0.20	0.03	0.25	0.15	1.09
Servs	0.06	0.37	0.05	0.50	0.05	0.34
BusEq	0.21	1.44	-0.04	0.34	0.12	0.81
Paper	-0.11	0.96	-0.15	1.55	-0.04	0.43
Trans	-0.17	1.29	-0.08	0.66	-0.17	1.44
Whlsl	-0.09	0.81	0.05	0.55	-0.07	0.70
Rtail	0.10	0.72	0.01	0.11	0.07	0.63
Meals	0.05	0.31	0.15	1.07	0.05	0.37
Fin	-0.16	1.72	0.00	0.01	0.06	0.64
Other	-0.25	2.05	-0.14	1.21	-0.19	1.68

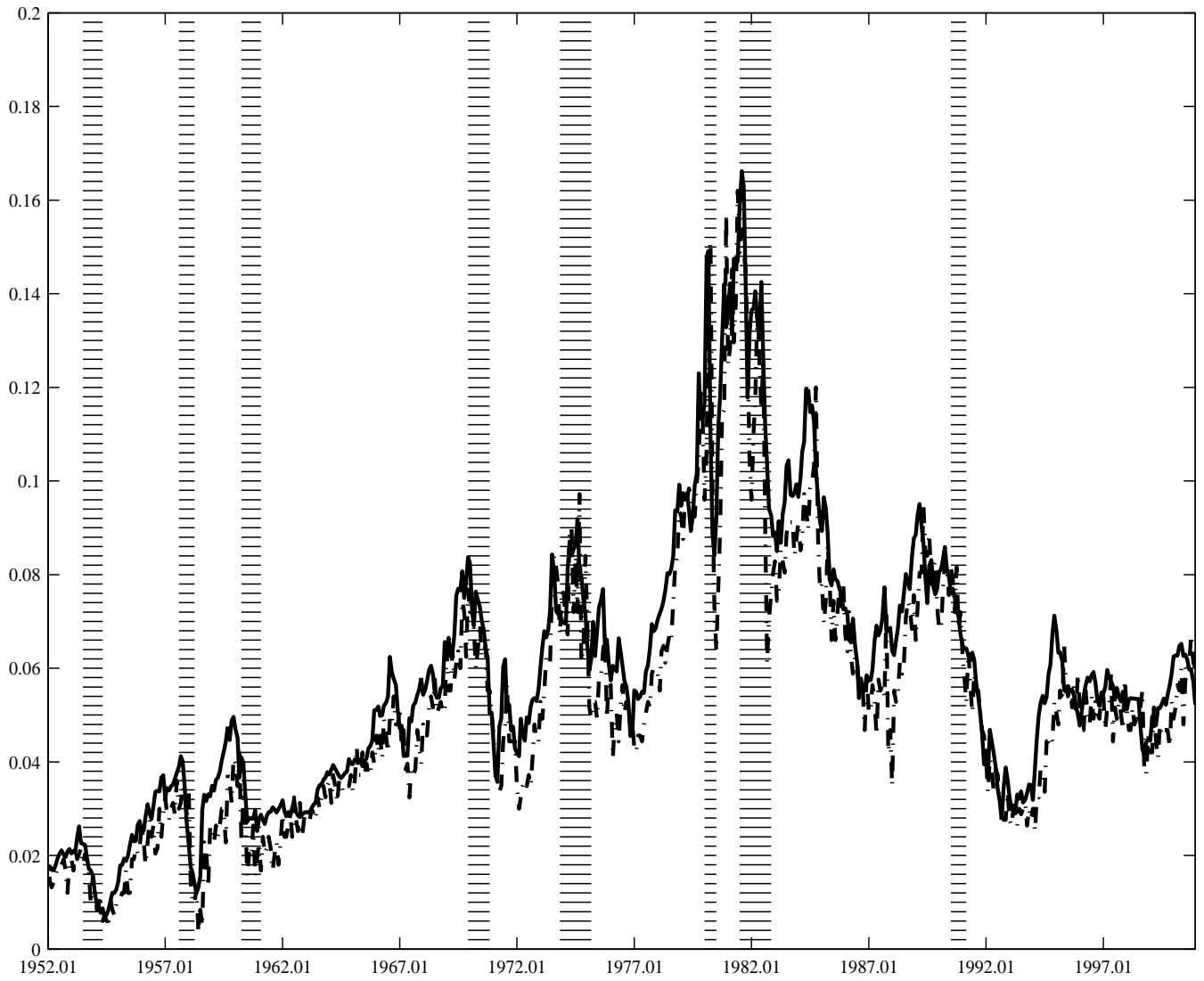
Table VII
Joint Test of ICAPM, FF Three-Factor Model, and CAPM using 25 size and B/M sorted and 30 Industrial Portfolios: $\overline{\mathbf{R}} - \overline{R}_f = \hat{\beta}\lambda + \epsilon$

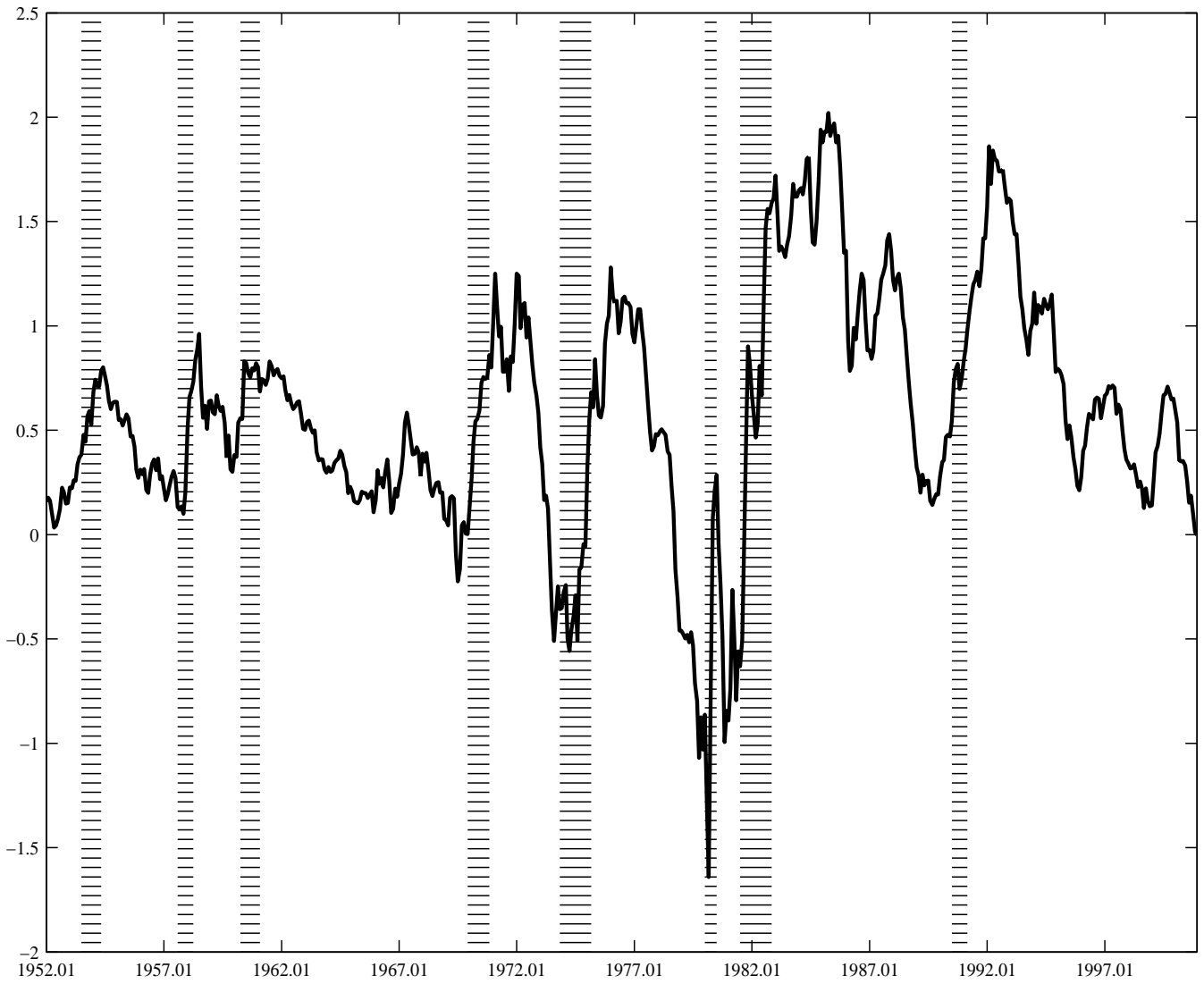
This table reports the two-stage cross-sectional regression results under the ICAPM, the FF three-factor model, and the CAPM. In the first stage, the excess returns on 55 portfolios are regressed on the model factors to obtain estimates of the factor loadings, $\hat{\beta}$. Under the ICAPM, the factors are the market risk premium, $R_M - R_f$, and the innovations in the state variables, Δr and $\Delta\eta$. Under the FF three-factor model, the factors are $R_M - R_f$, SMB , and HML . Under the CAPM, the factor is simply $R_M - R_f$. In the second stage, the sample mean of the excess returns is regressed on the beta without the intercept: $\overline{\mathbf{R}} - \overline{R}_f = \hat{\beta}\lambda + \epsilon$, and then the mispricing is calculated as $\hat{\alpha} = \overline{\mathbf{R}} - \overline{R}_f - \hat{\beta}\lambda$. A χ^2 test is then performed on $\hat{\alpha}$ to test the joint significance of the mispricing. The t -ratios and the variance covariance matrix, Σ , are calculated using Shanken's adjustment. Results are reported for the whole sample period of from January 1952 to December 2000 and two equal subsamples.

Panel A: January 1952 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_{mkt}	λ_r	λ_η	λ_{mkt}	λ_{SMB}	λ_{HML}	λ_{mkt}	$R_M - R_f$	SMB	HML
Estimate	0.64%	-0.25%	5.81%	0.65%	0.02%	0.12%	0.67%	0.62%	0.10%	0.36%
t -ratios	3.54	3.23	2.31	3.63	0.17	1.03	3.67	3.55	0.82	3.26
$\alpha' \Sigma^{-1} \alpha$		116.39			161.95		186.09			
p -value		< 1%			< 1%		< 1%			
Panel B: January 1952 to June 1976										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_{mkt}	λ_r	λ_η	λ_{mkt}	λ_{SMB}	λ_{HML}	λ_{mkt}	$R_M - R_f$	SMB	HML
Estimate	0.59%	-0.02%	1.36%	0.65%	-0.01%	0.15%	0.61%	0.55%	0.06%	0.39%
t -ratios	2.46	0.33	0.66	2.71	0.03	1.00	2.47	2.32	0.72	4.68
$\alpha' \Sigma^{-1} \alpha$		110.37			109.74		120.91			
p -value		< 1%			< 1%		< 1%			
Panel C: July 1976 to December 2000										
	ICAPM			FF Three-Factor Model			CAPM	Sample Mean		
	λ_{mkt}	λ_r	λ_η	λ_{mkt}	λ_{SMB}	λ_{HML}	λ_{mkt}	$R_M - R_f$	SMB	HML
Estimate	0.74%	-0.22%	1.55%	0.69%	0.01%	0.02%	0.71%	0.68%	0.14%	0.34%
t -ratios	2.79	2.10	0.46	2.63	0.07	0.10	2.69	2.65	0.74	1.90
$\alpha' \Sigma^{-1} \alpha$		180.21			209.66		229.59			
p -value		< 1%			< 1%		< 1%			









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Footnotes

¹“The multi-factor models of Merton (1973) and Ross (1976) ... can involve multiple factors and the cross-section of expected returns is constrained by the cross-section of factor loadings The multi-factor models are an empiricist’s dream ... that can accommodate ... any set of factors that are correlated with returns” (Fama (1991, p. 1594)).

²It is surprising that papers testing conditional versions of the CAPM that allow for time variation in expected returns typically do not allow for the pricing of the state variables they use to describe the investment opportunity set. Jagannathan and Wang (1996) explicitly assume that “the hedging motives are not sufficiently important [to warrant consideration of the ICAPM].” (p.7)

³Papers that are related to our general valuation framework in allowing for time-variation in interest rates and risk premia include Ang and Liu (2001) and Bekaert and Grenadier (2000). The valuation model in this paper differs from the models presented in these papers chiefly in its parsimonious specification of the relevant state variables.

⁴The ICAPM has been suggested by Fama and French (FF) themselves as one possible reason for the premia that they find to be associated with loadings on the *SMB* and *HML* hedge portfolios that are formed on the basis of firm size and book-to-market ratio. In FF (1995) they argue that the premia “are consistent with a multi-factor version of Merton’s (1973) intertemporal asset pricing model in which size and BE/ME proxy for sensitivity to risk factors in returns” (p.154).

⁵See Cochrane (2001) for a complete treatment.

⁶Kim and Omberg (1996) also assume an O-U process for the Sharpe ratio.

⁷For a structural model of time variation in investment opportunities that relies on habit formation, see Campbell and Cochrane (1999).

⁸The assumption of constant volatility is for convenience only. For example, as Samuelson (1965) has shown, the volatility of the expectation of a future cash flow will decrease monotonically with the time to maturity if the cash flow has a mean-reverting component.

⁹We thank Luis Viceira and Robert Bliss for providing the bond yield data. Our data start in January 1952 because the Federal-Treasury Accord that re-asserted the independence of the Fed from the Treasury was adopted in March 1952. Equity market data and portfolio returns are taken from the Web site of Ken French, which contains the description of portfolio formation procedure.

¹⁰In estimating a version of the model that has a constant value of η , Brennan and Xia (2002) find that the standard errors of the estimated bond yields decline with maturity out to five years.

¹¹The Livingston survey is carried out twice a year in June and December. The CPI forecast contains the economists' forecast of the CPI level in six and 12 months. We use the mean of the six-month forecast to construct the expected rate of inflation, and then use linear interpolation for the other months. The data are available from the Federal Reserve Bank of Philadelphia: <http://www.phil.frb.org/econ/liv>.

¹²The CPI data are obtained from the Federal Reserve Bank of St. Louis.

¹³The excess return of the CRSP value-weighted market portfolio during the sample period has a mean of about 0.62% and a standard deviation of 4.23% per month, implying a Sharpe ratio of 0.5, if volatility is assumed to be constant. Mackinlay (1995) reports an average Sharpe ratio of around 0.40 for the *S&P500* for the period 1981 to 1992.

¹⁴Babbs and Nowman (1999) report standard deviations of the measurement errors of 10 basis points for a one-year maturity and 6 basis points for an eight-year maturity for a three-factor generalized Vasicek model estimated over a much shorter sample period.

¹⁵Campbell and Viceira (2001) also find that the expected rate of inflation is close to a random walk in a similar setting, using a model with constant risk premia.

¹⁶The period of recession is measured from peak to trough as determined by the National Bureau of Economic Research.

¹⁷Boudoukh, Richardson, and Smith (1993) find evidence that the ex-ante *equity* market risk premium is negative in periods in which Treasury Bill rates are high. In March 1980 the Treasury Bill rate was over 14%.

¹⁸We thank Monika Piazzesi for making their series available to us.

¹⁹Note that the estimated correlation depends on the assumed value of $\bar{\eta}$. Shanken (1986) presents evidence that one can reject the hypothesis that the correlation between a linear combination of the CRSP *equal*-weighted index and a bond portfolio and the pricing kernel exceeds 0.7. Kandel and Stambaugh (1987) report similar evidence for the *value*-weighted index.

²⁰See Breeden (1979) and Breeden, Gibbons, and Litzenberger (1989) for details.

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Figure 1. Time Series of Real Interest Rate Estimates. The figure plots the estimated real interest rate series from January 1952 to December 2000, which is filtered out from the bond yield and inflation data. Shaded area indicates periods of U.S. recessions.

Figure 2. Time Series of Expected Inflation Rate Estimates and the Livingston Expected Inflation Rates. The figure plots the estimated expected inflation rates, π , against the expected inflation rate from the Livingston Survey, π_{liv} , from January 1952 to December 2000. The expected inflation rate is filtered out from the bond yield and Livingston inflation data. The Livingston Survey expected inflation rate, π_{liv} , is constructed from the economists' six-month ahead forecast of CPI level. The monthly data is constructed using linear interpolation. The data in the figure is in percent per month. Shaded area indicates periods of U.S. recessions. Legend: Solid line - π ; dash-dot line - π_{liv} .

Figure 3. Time Series of Nominal Interest Rate Estimates and the Realized One-month T-bill Rate. The figure plots the estimated nominal interest rate series, $R \equiv r + \pi$, against the realized one-month T-bill rate, R_f , from January 1952 to December 2000. The real interest rate r and the expected inflation π are filtered out from the bond yield and inflation data. Shaded area indicates periods of U.S. recessions. Legend: Solid line - $r + \pi$; dash-dot line - R_f .

Figure 4. Time Series of Sharpe Ratio Estimates. The figure plots the estimated Sharpe ratio series from January 1952 to December 2000, which are filtered out from the bond yield and inflation data. Shaded area indicates periods of U.S. recessions.