Lecture 3
State Preference Theory

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states: s = 1,...., S | s = 1,...., k

assets: i = 1,...., N | i = 1,...., n

Payoff tableau/matrix

payoff in state s for asset i

$$\text{Y}=\begin{bmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{S1} & \cdots & Y_{Sn} \end{bmatrix}(S*N) \qquad R=\begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{k1} & \cdots & R_{kn} \end{bmatrix}(k*N)$$

$v_i =$ current value/price of asset i | $\text{S}_i =$ current value/price of asset i

$$\text{Z}_{\text{si}} = \frac{Y_{si}}{v_i} \leftarrow \text{total return} \qquad \text{r}_{\text{si}} = \frac{R_{si}}{S_i} - 1 \leftarrow \text{rate of return}$$

Returns tableau/matrix

$$\text{Z}=\begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{S1} & \cdots & Z_{Sn} \end{bmatrix}(S*N) \qquad r=\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{k1} & \cdots & r_{kn} \end{bmatrix}(k*N)$$

$n_i =$ number of shares of asset i held

$\text{w}_{\text{i}} = \frac{n_i v_i}{\text{W}_{\text{o}}};\ 1'w = 1 \leftarrow$ portfolio weight

$\eta_{\text{i}} = n_i v_i =$ amount of wealth in asset i

$\text{W}_{\text{o}} = n'v = 1'\eta =$ current wealth

$(\text{S*N})(\text{N*1}) \rightarrow (\text{S*1})$

$\text{Z}\eta =$ payoff (in wealth) for every state

Example: $\text{Z}=\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$\eta = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad Z\eta = \begin{pmatrix} 1+9 \\ 2+3 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 9 \end{pmatrix}$$

$$w = \begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix} \qquad Zw = \begin{pmatrix} 2.5 \\ 1.25 \\ 2.25 \end{pmatrix}$$

Define zero-investment (arbitrage)portfolio $1'w$ (not w) = 0, $w \neq 0$

riskless portfolio $Zw = R1$ where $1'w = 1$

redundant asset – contributes to a duplicable payoff where $Zw_1 = Zw_2$ and $w_1 \neq w_2$. (use augmented returns tableau)

Arrow-Debreu security – pays one (dollar) if sate s occurs, zero otherwise

$$\text{payoff} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} s^{th} \text{ state}$$

Also called elementary security, state security, pure security

insurable state – a state for which an Arrow-Debreu security exists

Thm: s is insurable $\Leftrightarrow$ $Z_{s\cdot}$ is linearly independent of all other asset returns in other states, $Z_1, \ldots, Z_{s-1}, \ldots, Z_{s+1}, \ldots, Z_S$

$$Z = \begin{pmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad \eta = \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{4} \end{pmatrix} \qquad Z\eta = \begin{pmatrix} \frac{3}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \\ 1 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Arbitrage opportunity

Type I : $1'\eta \le 0$ and $Z\eta \ge 0$ e.g. free lottery ticket

Type II : $1'\eta < 0$ and $Z\eta \geqq 0$ e.g. violation of law of one price

$$Z=\begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \eta=\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad Z\eta=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \textit{Type}\ \mathrm{I}$$

$$Z=\begin{pmatrix} 5 & 2 \\ -5 & -2 \end{pmatrix} \qquad \eta=\begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad Z\eta=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{Type}\ \mathrm{II}$$

if limited liability ($Z\eta \ge 0$), then Type II arb $\Rightarrow$ Type I arb

if arbitrage opportunity $\Rightarrow$ no equilibrium solution

$$p=\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_S \end{pmatrix} \quad \text{state pricing vector; } p_s = \text{ state prices}$$

P_s is the price/value of Arrow-Debreu security s.

Define: Z’p = 1   or   Y’p = v

p may not be unique. (p supports the economy)

When does a positive p exist?

Theorem : $p \geqq 0 \Leftrightarrow$ no Type II arbitrage opportunities
$p > 0 \Leftrightarrow$ no Type I or II arbitrage opportunities

Define:
Complete market – all states are insurable; any payoff pattern is achievable
Rank (Z) = S

With N assets, S states, and S > N,
no arbitrage $\longrightarrow$ N restrictions on p with N – S "degrees of freedom"

Here, incomplete market.

If there exists a riskless asset with total return R_f and no arbitrage opportunities, then

$$p'1 = \frac{1}{R_f}\left[\sum_s p_s = \frac{1}{R_f} = \frac{1}{1+r_f}\right].$$

If there is no riskless asset and no arbitrage opportunities, then

$$\frac{1}{\overline{R}} \le p'1 \le \frac{1}{\underline{R}}.$$

Now, introduce probabilities.

${}_k\pi$ – probability distribution (subjective) of individual k

Suppose we have an economy without any arbitrage opportunities supported by p > 0.
All individuals agree on this p. (might not be unique)

$$v_i = \frac{{}_kE(\widetilde{y_i})}{{}_kE(\widetilde{Z_i})}$$

${}_kE(\widetilde{y_i}) \longleftarrow$ Expected payoff

${}_kE(\widetilde{Z_i}) \longleftarrow$ (subjective) appropriate discount rate

Risk-neutral probabilities (martingale probabilities)
Define:

$$\hat{\pi}_s \equiv p_s \hat{R} \qquad \hat{R} = \text{ risk neutralized risk-free rate}$$

$$\hat{R} \equiv (1'p)^{-1} = \frac{1}{\sum_s p_s}$$

$$1'\pi = 1, \; \hat{\pi} > 0$$

$$v = Y'p \Rightarrow v = Y'\frac{\hat{\pi}}{\hat{R}}, \quad so \quad v_i = \frac{\hat{E}(\tilde{y}_i)}{\hat{R}} \left[\hat{E}(\tilde{Z}_i) = \hat{R} \right]$$

Value of an asset is its expected value under $\hat{\pi}$ discounted by the risk-free rate.

$V_i -$ like a certainty equivalent assigned by the market.

Now define a state price per unit probability (state price density) ← Cochrane

$$m_s \equiv \frac{P_s}{\pi_s} \qquad \text{(called } \Lambda_s \text{ in Ingersoll)}$$

True probability

$$v_i = \sum_s P_s Y_{si} = \sum_s \pi_s m_s Y_{si}$$

$$\boxed{v_i = E(\tilde{m}\tilde{Y}_i) \qquad or \qquad 1 = E(\tilde{m}\tilde{Z}_i)}$$

This is the CENTRAL ASSET PRICING EQUATION.

Cochrane: p = E(mx) or 1 = E(mR)

where $m(s) = \dfrac{p(s)}{\pi(s)}$

m is called the stochastic discount factor, pricing kernel, state-price density, ...

p(s) = m(s) π(s) - like a discounted probability

In a consumption-based asset pricing model, m(s) is the marginal rate of substitution.

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State Preference Theory

Consider a one period model. Assume that there are a finite number of end-of-period "states of the world," with state s having probability π_s. Let R_{sa} be the cashflow generated by one share (unit) of asset a in state s. Also assume that there are k states of the world and n assets. The following vector describes the payoffs to financial asset a:

$$\text{Asset } a \text{ cashflows} = \begin{bmatrix} R_{1a} \\ \vdots \\ R_{ka} \end{bmatrix}. \tag{1}$$

Thus, the per-share cashflows of the universe of all assets can be represented by the $k \times n$ matrix

$$R = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{k1} & \cdots & R_{kn} \end{bmatrix}. \tag{2}$$

We will assume that $n = k$ and that R is of full rank. A necessary condition for there to be a complete market (and unique state prices) is that $n \geq k$. If $n > k$, there will be some "redundant" assets (ones whose cashflows in the k states are linear combinations of others), so we could always reduce the number of assets down to k by combining them into k linearly independent (portfolios of) assets without loss of generality.

Suppose an individual's wealth is divided among the n assets, with the number of shares owned of asset i given by w_i for $i = 1, ..., n$. Let us then consider the problem of an individual who wants to provide for specific amounts of wealth in different future states of the world. Note that these amounts may not necessarily be equal across states of the world because the marginal utility of wealth may be state dependent. This could be because utility is state dependent, or the individual might receive different amounts of (non-tradeable) wage income in the different states.

Suppose an individual has target levels of wealth in the k states given by

$$W^* = \begin{bmatrix} W_1^* \\ \vdots \\ W_k^* \end{bmatrix}. \tag{3}$$

Then to obtain this given set of target wealth levels, the individual will need to initially purchase shares in the n assets, which we denote as $w = [w_1 \ldots w_n]'$, satisfying

$$Rw^* = W^* \tag{4}$$

or

$$w^* = R^{-1}W^*. \tag{5}$$

If $P = [P_1 \ldots P_n]'$ is the nx1 vector of beginning-of-period per share prices of the n assets, then the amount of initial wealth required to produce the target level of wealth given in (3) is simply $P'w^*$.

In this state preference framework, we can define a primitive (or elementary) security as a security that pays a cashflow of 1 unit in one of the states and zero in each of the other states. For example, primitive security "s" has the vector of cashflows

$$\text{Cashflows of elementary security } s = \begin{bmatrix} R_1 \\ \vdots \\ R_s \\ \vdots \\ R_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

Let p_s be the beginning of period price or value of this primitive security s. Note that the portfolio composed of the sum of all primitive securities will give a cashflow of 1 unit with certainty. The price of this portfolio will then define the risk free interest rate, r_f, by the relation:

$$\sum_{s=1}^{k} p_s = \frac{1}{1 + r_f}. \tag{7}$$

Given the prices of primitive securities, the price of every security can be obtained by arbitrage. The price, P_a, of a multi-cashflow security with a cashflow of R_{sa} in state s (see (1) above), must then have a beginning of period value equal to

$$P_a = \sum_{s=1}^{k} p_s \, R_{sa}. \tag{8}$$

Our goal is now to derive the equilibrium prices of primitive securities, and hence the equilibrium prices of all securities, based on a model of expected utility maximization. Define C_s as the number of primitive securities of type s that an individual holds in equilibrium. Thus, the value of wealth in state s will also be equal to C_s, since each primitive security pays 1 unit. Since the individual has no further utility following the end of the period, it is optimal to consume all end of period wealth. Thus, C_s will also be the individual's end of period consumption in state s. An individual's expected end-of-period utility of consumption will then be

$$\sum_{s=1}^{k} \pi_s \, U(C_s). \tag{9}$$

(Note that $U(\,\cdot\,)$ may be state dependent, that is, a function of s, $U(C_s,\, s)$.) The optimal portfolio chosen by an investor with utility of consumption at both the beginning and end of the period is the solution to the problem

$$\max_{C_0, \{C_s\}} U(C_0) + \delta \sum_{s=1}^{k} \pi_s \, U(C_s) \tag{10}$$

where $\delta = \frac{1}{1+\rho}$ and ρ is the rate of time preference, subject to the budget constraint

$$W_0 = C_0 + \sum_{s=1}^{k} p_s \, C_s \tag{11}$$

which says that initial wealth must equal current consumption, C_0, plus savings. The Lagrangian is

$$\max_{C_0, \{C_s\}} L = \max_{C_0, \{C_s\}} U(C_0) + \delta \sum_{s=1}^{k} \pi_s \, U(C_s) + \lambda (W_0 - C_0 - \sum_{s=1}^{k} p_s \, C_s) \tag{12}$$

which leads to the first order conditions

$$\frac{\partial L}{\partial C_0} = U'(C_0) - \lambda = 0 \tag{13}$$

$$\frac{\partial L}{\partial C_s} = \delta\,\pi_s\,U'(C_s) - \lambda\,p_s = 0, \qquad s = 1,\ldots,k. \tag{14}$$

Eliminating λ and solving for p_s gives

$$p_s = \frac{\delta\,\pi_s\,U'(C_s)}{U'(C_0)}. \tag{15}$$

Interpreted at the individual investor level, equation (15) says that the investor takes the primitive security price, p_s, as given, and then allocates C_0 and C_s so that the ratio of marginal utilities, times the probability of state s, equals this given primitive security price.

But (15) can be interpreted in another way as well. Suppose all investors in the economy have identical utility functions and initial wealths. Thus, we can think of there being a single "representative" consumer-investor. C_0 and C_s are now per-capita aggregate consumption levels. Then (15) can be viewed as an equilibrium pricing relationship that deterines p_s. It says that the equilibrium primitive security price equals the probability of state s times the ratio of the marginal utilities of per-capital aggregate consumption in state s and the current period.

Note that (15) can be written as

$$p_s\,U'(C_0) = \delta\,\pi_s\,U'(C_s) \tag{16}$$

which has the interpretation that if one reduces consumption today to invest in state s, there is a current utility loss of $p_s\,U'(C_0)$. Thus, when acting optimally, the discounted expected value of future consumption in state s, which is $\delta\,\pi_s\,U'(C_s)$, must equal this current utility loss.

Summing (16) over all s, we obtain

$$U'(C_0)\sum_{s=1}^{k} p_s = \delta\sum_{s=1}^{k}\pi_s\,U'(C_s) \tag{17}$$

and simplifying

$$U'(C_0)\,\frac{1}{1+r_f} \;=\; \delta\, E[U'(\tilde{C})] \tag{18}$$

or

$$U'(C_0) = (1+r_f)\,\delta\, E[U'(\tilde{C})] \tag{19}$$

Equation (19) is sometimes referred to as the Euler equation of consumption theory. We will see such a condition in our future coverage of intertemporal consumption-portfolio choice problems.

Note that the condition (16) is satisfied for every primitive security, say primitive security s and primitive security z. Rewriting (16) for security z and then substituting out for $U'(C_0)$ gives

$$\frac{\pi_z\, U'(C_z)}{p_z} \;=\; \frac{\pi_s\, U'(C_s)}{p_s} \tag{20}$$

with the implication that the individual is optimally allocating wealth among the k primitive securities when the ratio of the expected marginal utility of consumption to the current cost of obtaining it is equal across each of the states of the world.

We can also derive general expressions for the risk premiums of each financial asset. Let P_0 be the price of a riskless security that has a cashflow of 1 unit at the end of the period. From (18), we have shown that this price must satisfy

$$P_0 \;=\; \sum_{s=1}^{k} p_s \;=\; \frac{1}{1+r_f} \;=\; \frac{\delta\, E[U'(\tilde{C})]}{U'(C_0)} \tag{21}$$

Also define P_a as the price of security a which generates the cash flow R_{sa} in state s. Its return in state s is

$$r_{sa} \;=\; \frac{R_{sa}}{P_a} - 1 \tag{22}$$

and its price is given by

$$P_a \;=\; \sum_{s=1}^{k} p_s\, R_{sa} \;=\; \sum_{s=1}^{k} \frac{\pi_s\, \delta\, U'(C_s)\, R_{sa}}{U'(C_0)} \;=\; \frac{\delta\, E[U'(\tilde{C})\, \tilde{R}_a)]}{U'(C_0)} \tag{23}$$

Divide (23) by P_a and obtain

$$1 \;=\; \frac{\delta\, E[U'(\tilde{C})\,(1+\tilde{r}_a)]}{U'(C_0)}. \tag{24}$$

Then use the definition of covariance to re-write (24) as

$$\frac{\delta\, E[U'(\tilde{C})]\, E[1+\tilde{r}_a]}{U'(C_0)} + \frac{\delta\, Cov[U'(\tilde{C}),\, \tilde{r}_a]}{U'(C_0)} - 1 \,=\, 0 \tag{25}$$

or

$$\left(1 - \frac{\delta\, Cov[U'(\tilde{C}),\, \tilde{r}_a]}{U'(C_0)}\right) \frac{U'(C_0)}{\delta\, E[U'(\tilde{C})]} \,=\, 1 + E[\tilde{r}_a]. \tag{26}$$

Using (19) to substitute $1 + r_f \,=\, \frac{U'(C_0)}{\delta\, E[U'(\tilde{C})]}$, gives

$$1 + r_f - \frac{Cov[U'(\tilde{C}),\, \tilde{r}_a]}{E[U'(\tilde{C})]} \,=\, 1 + E[\tilde{r}_a] \tag{27}$$

or

$$E[\tilde{r}_a] \,=\, r_f - \frac{Cov[U'(\tilde{C}),\, \tilde{r}_a]}{E[U'(\tilde{C})]}. \tag{28}$$

Thus, the risk premium is minus the covariance between the marginal utility of end-of-period consumption and the asset return, divided by the expected end-of-period marginal utility of consumption. If an asset pays a higher return when consumption is high, its return will have a negative covariance with the marginal utility of consumption, and therefore the investor will demand a positive risk premium over the risk free rate.

Conversely, if an asset pays a higher return when consumption is low, so that its return will positively covary with the marginal utility of consumption, then it will have an expected return less than the risk-free rate. Investors will be satisfied with this lower return because the asset is providing insurance against low consumption states of the world, that is, it is helping to smooth consumption across states.

Now suppose there exists a portfolio with return, $\tilde{r}_m$, that is perfectly negatively correlated with the marginal utility of consumption, $U'(\tilde{C})$:

$$U'(\tilde{C}) \,=\, -\gamma\, \tilde{r}_m, \qquad \gamma > 0. \tag{29}$$

Then this implies

$$Cov[U'(\tilde{C}),\, \tilde{r}_m] \,=\, -\gamma\, Cov[\tilde{r}_m,\, \tilde{r}_m] \,=\, -\,\gamma\, Var[\tilde{r}_m] \tag{30}$$

and

$$Cov[U'(\tilde{C}),\, \tilde{r}_a] \;=\; -\gamma\, Cov[\tilde{r}_m,\, \tilde{r}_a]. \tag{31}$$

For the portfolio having return $\tilde{r}_m$, the risk premium relation (28) is

$$E[\tilde{r}_m] \;=\; r_f - \frac{Cov[U'(\tilde{C}),\, \tilde{r}_m]}{E[U'(\tilde{C})]} \;=\; r_f + \frac{\gamma\, Var[\tilde{r}_m]}{E[U'(\tilde{C})]} \tag{32}$$

Using (28) and (32) to substitute for $E[U'(\tilde{C})]$, and using (31), we obtain

$$\frac{E[\tilde{r}_m] - r_f}{E[\tilde{r}_a] - r_f} \;=\; \frac{\gamma\, Var[\tilde{r}_m]}{\gamma\, Cov[\tilde{r}_m,\, \tilde{r}_a]} \tag{33}$$

and re-arranging

$$E[\tilde{r}_a] - r_f \;=\; \frac{Cov[\tilde{r}_m,\, \tilde{r}_a]}{Var[\tilde{r}_m]}\, \left(E[\tilde{r}_m] - r_f\right) \tag{34}$$

or

$$E[\tilde{r}_a] \;=\; r_f + \beta_a\, \left(E[\tilde{r}_m] - r_f\right). \tag{35}$$

So we obtain the CAPM if the market portfolio is negatively correlated with the marginal utility of end-of-period consumption (or wealth). Note that for an arbitrary distribution of asset returns, this will always be the case if utility is quadratic (so that marginal utility is linear in consumption).

State Preference Theory can easily be generalized to an infinite number of states and primitive securities. It basically amounts to defining probability densities of states and replacing the summations in expressions like (6) and (7) with integrals. For example, let states be indexed as all possible points on the real line between 0 and 1, that is, the state $s \in (0, 1)$. Also let $p(s)$ be the price (density) of a primitive security that pays 1 unit in state s. Further, define $R_a(s)$ as the cashflow paid by security a in state s. Then, analogous to (6) we can write

$$\int_0^1 p(s)\, ds \;=\; \frac{1}{1 + r_f} \tag{36}$$

and instead of (7) we can write the price of security a as

$$P_a \;=\; \int_0^1 p(s)\, R_a(s)\, ds. \tag{37}$$

In some cases, namely where markets are complete intertemporally, State Preference Theory can be extended to the case where assets have cashflows occurring at different points in time in the future. This generalization is sometimes referred to as Time State Preference Theory. See Stewart C. Myers (1968) "A Time-State Preference Model of Security Valuation," Journal of Financial and Quantitative Analysis 3, p.1-34. For example, suppose that assets can pay cashflows at both one and two periods in the future. Let s_1 be a state at the end of the first period and let s_2 be a state at the end of the second period. States at the end of the second period can depend on which states were reached at the end of the first period.

For example, suppose there are two events each period, economic recession (r) or expansion (e). Then we could define $s_1 \in \{r_1, e_1\}$ and $s_2 \in \{r_1r_2, r_1e_2, e_1r_2, e_1e_2\}$. By assigning suitable probabilities and primitive security state prices for assets that pay cashflows of 1 unit in each of these six states, we can sum (or integrate) over both time and states at a given point in time to obtain prices of complex securities. Thus, when primitive security prices exist at all states for all future dates, we are essentially back to a single period complete markets framework, and the analysis is essentially the same as what we derived above.