Asset Pricing in Specific Economies.
Asset Pricing

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1 Asset Pricing. Reminder.

Let $P(S^t)$ denote the time 0 price of one unit of consumption good delivered in history realization S^t using the price at time 0 as the numeraire. The representative consumer's problem can be stated as

$$\max_{c(S^t)} E_0 U\left[c\left(S^t\right)\right] \tag{1}$$

$$s.t. \sum_{S^t} P\left(S^t\right) c\left(S^t\right) = \sum_{S^t} P\left(S^t\right) e\left(S^t\right) \tag{2}$$

The first order condition is given by

$$\frac{\partial U[c\left(S^t\right)]}{\partial [c\left(S^t\right)]} = \lambda P\left(S^t\right) \tag{3}$$

where λ is the Lagrange multiplier associated with the budget constraint (2).

Using the conventional separable expected utility:

$$U\left[\{c\left(S^t\right)\}\right] = E_0\left[\sum_{t=0}^{\infty} \beta^t U\left[c\left(S^t\right)\right]\right] = \sum_{S^t} \beta^t \pi\left(S^t\right) U\left[c\left(S^t\right)\right] \tag{4}$$

where $\pi\left(S^t\right)$ is the probability of arriving to state S^t. We can rewrite the first order condition (3) as

$$\beta^t \pi \left(S^t\right) u' \left[c\left(S^t\right)\right] = \lambda P \left(S^t\right) \tag{5}$$

If we further normalize $P(S^0) \equiv P(0) = 1$ then we can solve for λ from (5) as $\lambda = u'[c(0)]$. Plugging λ back into (5) we obtain the Arrow-Debreu price as

$$P\left(S^t\right) = \beta^t \pi \left(S^t\right) \frac{u'\left[c\left(S^t\right)\right]}{u'\left[c\left(0\right)\right]} \tag{6}$$

Given the Arrow-Debreu price in (6) and the complete market assumption, we can price all kinds of assets. In particular,

The one-period risk-free bond price at time 0

$$P^b(S^0) = \sum P(S^1) = P(H) + P(L) \tag{7}$$

and the associated one-period risk-free rate

$$R^b(S^0) = \frac{1}{P^b(S^0)} \tag{8}$$

The one-period-ahead contingent claim price at time t is given by

$$p\left(S^{t+1}|S^t\right) = \frac{P\left(S^{t+1}\right)}{P\left(S^t\right)} = \beta^t \pi \left(S^{t+1}|S^t\right) \frac{u'\left[c\left(S^{t+1}\right)\right]}{u'\left[c\left(S^t\right)\right]} \tag{9}$$

where $\pi\left(S^{t+1}|S^t\right) \equiv \frac{\pi\left(S^{t+1}\right)}{\pi\left(S^t\right)}$ is the conditional probability of arriving to state S^{t+1} following state S^t, and the second equality follows from (6).

The risk-free one-period ahead bond price at time t is hence

$$P^b\left(S^t\right) = \frac{P(S^tH) + P(S^tL)}{P(S^t)} = \sum_{S^{t+1}\supset S^t} p\left(S^{t+1}|S^t\right) \tag{10}$$

while the associated risk-free rate is simply $R^b(S^t) = 1/P^b(S^t)$.

Stock defined as a claim to a stream of dividends $d(S^t)$. Its price at time 0 is

$$P^A(0) = \frac{P(S^tH) + P(S^tL)}{P(S^t)} = \sum_{S^t} P(S^t)d(S^t) \tag{11}$$

and the price at time t is

$$P^A\left(S^t\right) = \sum_{S^\tau \supset S^t} \frac{P(S^\tau)}{P(S^t)} d(S^\tau) \tag{12}$$

$$= \sum_{S^{t+1} \supset S^t} p\left(S^{t+1}|S^t\right)\left[d(S^{t+1}) + P^A(S^{t+1})\right] \tag{13}$$

Note since return is defined as

$$R^A\left(S^{t+1}|S^t\right) \equiv \frac{P^A(S^{t+1}) + d(S^{t+1})}{P^A(S^t)} \tag{14}$$

we can use return to rewrite (13) as

$$1 = \sum_{S^{t+1} \supset S^t} p\left(S^{t+1}|S^t\right) R^A\left(S^{t+1}|S^t\right) \tag{15}$$

which we can rewrite using (9) and the definition of conditional expectation as

$$1 = E_t\left[\beta \frac{u'\left[c\left(S^{t+1}\right)\right]}{u'\left[c\left(S^t\right)\right]} R^A\left(S^{t+1}|S^t\right)\right] \tag{16}$$

Moreover, we can use (6) to rewrite (12) as

$$P^A\left(S^t\right) = E_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'\left(c_{t+j}\right)}{u'\left(c_t\right)} d_{t+j}\right] \tag{17}$$

In equilibrium we have

$$c_t = e_t \tag{18}$$

Therefore, asset prices must adjust until consumers just want to consume the endowment process in equilibrium.

2 Quadratic Utility.

Assume the representative agent has a quadratic utility, i.e., $u(c_t) = -\frac{1}{2}\left(c_t - c^*\right)^2$, where c^* is the saturation point. Then $u'(c_t) = c^* - c_t$. By (9) and (10) the one-period ahead bond price at time t is given by

$$P_t^b = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right] = E_t\left[\beta \frac{c^* - c_{t+1}}{c^* - c_t}\right] = \beta \frac{c^* - E_t\left[c_{t+1}\right]}{c^* - c_t} \tag{19}$$

and the interest rate is then $R_t^b = 1/P_t^b$.

Note that the interest rate depends on consumer's subjective discount rate β and consumption in the current period relative to expected future consumption. Interest rate is low when consumption is high relative to future consumption. The intuition is that when endowment is high today relative that of tomorrow, the consumer would like to save to achieve smoothing. However, in aggregate they cannot do so because the endowment is fixed and condition (18). To clear the market, the interest rate has to go down to offset the consumer's saving incentive until he or she is just happy holding e_t again.

Upon specifying the exogenous endowment process, we can characterize asset prices further. For example, suppose the endowment follows an AR(1) process

$$e_{t+1} = \rho e_t + \varepsilon_{t+1} \tag{20}$$

Then (18) and (19) imply that

$$R_t^b = \frac{1}{P_t^b} = \frac{1}{\beta}\frac{c^* - c_t}{c^* - \rho c_t} \tag{21}$$

Suppose $c_t = E_t[c_{t+1}]$ as in the case of quadratic preferences and permanent income hypothesis then (19) implies that $P_t^b = \beta$ and $R_t^b = 1/\beta$.

3 CES Utility.

Assume that the representative agent has CES utility, i.e., $u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$, and hence $u^{'}(C_t) = C_t^{-\gamma}$. Then

$$1 = E_t\left[(1 + R_{i,t+1})\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \tag{22}$$

We assume that asset returns and aggregate consumption are jointly homeskedastic and lognormal.

When a rv X is conditionally lognormally distributed

$$\log E_t[X] = E_t[\log X] + \frac{1}{2}Var_t[\log X] \tag{23}$$

If in addition, X is conditionally homoskedastic than $Var_t[\log X] = E[(\log X - E_t[\log X])^2] = Var[\log X - E_t[\log X]]$.

Thus, with joint conditional lognormality and homoscedasticity of asset returns and consumption, we take logs of (22):

$$0 = E_t[r_{i,t+1}] + \log \beta - \gamma E_t[\Delta c_{t+1}] + \frac{1}{2}[\sigma_i + \gamma^2 \sigma_c^2 - 2\gamma\sigma_{ic}] \tag{24}$$

Equation (25) has both time-series and cross-sectional implications:
The riskless rate obeys:

$$r_{t+1}^f = -\log(\beta) + \gamma E_t(\Delta c_{t+1}) - \frac{1}{2}\gamma^2\sigma_c^2 \tag{25}$$

Therefore, interest rate is high when consumption growth is expected to be high as expected from certainty equivalence model in quadratic utility case. However, there is another effect that is absent in the quadratic utility model in that interest rate is also low when consumption is volatile. Specifically, if you add $+1/-1$ randomness with probability .5, then agents would save more. The reason is that marginal utility rises more when consumption goes down by one unit then it goes down when consumption goes up by one unit. This is the case since $u''(c)$ is increasing or $u'''(c) > 0$. (Recall that in quadratic utility case, $u''(c)$ is constant.) The effect is called precautionary saving. That is, when endowment stream is more volatile, agents save more thus pushing down interest rate.

The assumption of homoskedasticity makes the log risk risk premium on any asset over the riskless real rate constant, so expected real returns on other assets are also linear in expected consumption growth:

$$E_t\left[r_{t+1}^i - r_{t+1}^f\right] + \frac{1}{2}\sigma_i^2 = \gamma\sigma_{ic} \tag{26}$$

The variance term on the let hand side of (26) is a Jensen's inequality adjustment. We can write

$$\log E_t\left[(1 + R_{t+1}^i)/(1 + R_{t+1}^f)\right] = \gamma\sigma_{ic} \tag{27}$$

4 Mehra and Prescott (1985)

Mehra and Prescott assume a slightly more general exogenous process. Consider an economy with risky trees ("equity"), denoted s, and sure claims ("bonds"), denoted B. We could price contingent claims and then back out the implications for bonds, but in this exercise we will go straight to the latter. Unlike our earlier asset pricing problems, this model will have a non-stationary environment. In particular, dividends (which in equilibrium will also equal consumption) will have a stochastic trend and so exhibit unit-root-like dynamics. To model this, let equity pay random dividends y each period.

Specifically, denote consumption growth by $\lambda_t = c_t/c_{t-1}$ and dividend growth by $\xi_t = d_t/d_{t-1}$. Assume that

1. λ_t and ξ_t take values in a finite set, i.e., $\lambda_t \in \{\lambda_1, \lambda_2, ..., \lambda_N\}$ and $\chi_t \in \{\chi_1, \chi_2, ..., \chi_N\}$ and that $S_t = (\lambda_t, \chi_t)$ follows a N-state Markov chain with transition matrix $\Pi_{N\times N}$ such that

$$\Pi_{ij} = prob\{S_{t+1} = (\lambda_j, \chi_j)\,|S_t = (\lambda_i, \chi_i)\} \tag{28}$$

   and that the Markov chain is ergodic.

2. $\lambda_t, \xi_t > 0$ and $c_0, d_0 > 0$.

3. Define matrix $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}_{ij} = \beta\Pi_{ji}\lambda_j^{1-\gamma}\mathbf{B}_{ij} = \beta\Pi_{ji}\lambda_j^{-\gamma}\xi_j \tag{29}$$

   and assume that both $\mathbf{A}$ and $\mathbf{B}$ are stable, i.e., $\lim_{m\to\infty}\mathbf{A}^m = \lim_{m\to\infty}\mathbf{B}^m = 0$. Equivalently, it is assumed that all eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ lie inside the unit circle.

Define P_c^t to be the price of a stock that is a claim on the consumption stream. Continuing to assume CES utility and by (17), we have

$$\begin{aligned} P_t^c(c_t, \lambda_t) &= E\left[\sum_{j=1}^{\infty}\beta^j\left(\frac{c_t}{c_{t+j}}\right)^{\gamma}c_{t+j}|c_t, \lambda_t\right] \\ &= E\left[\beta\left(\frac{c_t}{c_{t+j}}\right)^{\gamma}\left[P_{t+1}^c(c_{t+1}, \lambda_{t+1}) + c_{t+j}\right]|c_t, \lambda_t\right] \end{aligned}$$

Note that since $c_{t+j} = c_t\lambda_t\lambda_{t+1}...\lambda_{t+j}$ the price of equity is homogenous of degree one in c_t. To see this, let θ be any constant and note that

$$\begin{aligned} P_t^c(\theta c_t, \lambda_t) &= E_t\left[\sum_{j=1}^{\infty}\beta^j\left(\frac{\theta c_t}{\theta c_{t+j}}\right)^{\gamma}\theta c_{t+j}\right] \\ &= \theta E_t\left[\sum_{j=1}^{\infty}\beta^j\left(\frac{c_t}{c_{t+j}}\right)^{\gamma}c_{t+j}\right] = \theta P_t^c(c_t, \lambda_t) \end{aligned}$$

We want to solve for stationary or time-invariant prices. By homogeneity of degree one in consumption, we assume

$$P^c(c,i) = \omega_i c \tag{30}$$

Now using the definition of conditional expectation, we can rewrite (30) as

$$P^c(c,i) = \beta \sum_{j=1}^{\infty} \Pi_{ij} \left(\frac{\lambda_j c}{c}\right)^{-\gamma} \left[P^c(\lambda_j c, i) + c\lambda_j\right] \tag{31}$$

which is equivalent to, by (30),

$$\omega_i = \beta \sum_{j=1}^{\infty} \Pi_{ij} \lambda_j^{1-\gamma} (\omega_j + 1) \, \forall i = 1, ..., N \tag{32}$$

This is a system of N linear equations in N unknowns. Using the matrix notation defined in (29), we thus have

$$P^c_{N\times 1} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{A}\underline{\mathbf{1}} \tag{33}$$

where $\underline{1}$ is an $N \times 1$ vector of $1s$ and $\mathbf{I}$ is the identity matrix.

Similarly, if we define P^e_t to be the price of a stock that is a claim on the dividend, as opposed to consumption, stream, then

$$P^e_t(c_t, \lambda_t) = E\left[\sum_{j=1}^{\infty} \beta^j \left(\frac{c_t}{c_{t+j}}\right)^{\gamma} d_{t+j} | \lambda_t, \xi_t\right] \tag{34}$$

An argument similar to (33) leads to

$$P^e_{N\times 1} = (\mathbf{I} - \mathbf{B})^{-1} \mathbf{B}\underline{\mathbf{1}} \tag{35}$$

Now, the realized returns from state i to state j are given by

$$R^c(i,j) = \frac{\lambda_j \left(1 + P^c_j\right)}{P^c_j} \tag{36}$$

$$R^e(i,j) = \frac{\xi_j \left(1 + P^e_j\right)}{P^e_j} \tag{37}$$

and the conditional expected equity return given current period state i is

$$R^e(i) = \sum_{j=1}^{N} \Pi_{ij} R^e(i,j) \tag{38}$$

The risk-free one-period-ahead bond is

$$P^f(i) = \beta \sum_{j=1}^{N} \Pi_{ij} \lambda_j^{-\gamma} \tag{39}$$

and the risk-free rate is $R^f(i) = 1/P^f(i)$.

Furthermore, if we let $\phi \in N$ be the vector of unconditional probabilities. By ergodicity, we know that ϕ exists and can be obtained by

$$\phi = \Pi' \phi \tag{40}$$

Using ϕ the unconditional expected returns are thus given by

$$R^e = \sum_{i=1}^{N} \phi R^e(i) \tag{41}$$

$$R^f = \sum_{i=1}^{N} \phi R^f(i) \tag{42}$$

5 Mehra and Prescott. Calibration.

To calibrate this model, Mehra and Prescott (1985) set $n = 2$ and $S_1 = 1 + \lambda - \delta$ (a "bad" state) and $S_2 = 1 + \lambda + \delta$ (a "good" state). They further set λ to be the long-run average annual growth rate of per capita consumption and δ to the standard deviation of per capita consumption over the years 1889-1978. The first order autocorrelation coefficient of per capita consumption is governed by a single parameter ϕ such that if we define

$$\Pi_{ij} = prob\{S_{t+1} = (\lambda_j, \xi_j) \,|S_t = (\lambda_i, \xi_i)\} \tag{43}$$

then the transition matrix is symmetric with

$$\begin{pmatrix} \Pi_{11} & 1-\Pi_{11} \\ 1-\Pi_{22} & \Pi_{22} \end{pmatrix} = \begin{pmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{pmatrix}$$

This calibration leads to $\lambda = 0.018$, annual average consumption growth of just less than 2%, $\delta = 0.036$, annual standard deviation of consumption growth of 3.6%, and $\phi = 0.43$, average first order autocorrelation of consumption growth of

$2\,\phi - 1 = -0.14$

Mehra and Prescott then experiment with different numbers for relative risk aversion $\gamma \in [0, 10]$ and $\beta \in [0.95, 1]$ the model only generates equity premium, $R^e - R^f$, of approximately at most .1% per annum, as opposed to 6% per annum observed in the data.

6 Habit Formation

- Allow for nonseparability in utility over time.
- Positive effect of today's consumption on tomorrow's marginal utility of consumption.
- period utility function $U_t(C_t, X_t)$, where X_t-habit, subsistence level

Modeling issues:

- power function of the ratio C_t/X_t (Abel (1990))
- power function of the difference $C_t - X_t$ (Constantinides (1990), Campbell and Cochrane (1995))
- external habit models (Abel (1990), Campbell and Cochrane (1995)) habit depends on aggregate consumption
- internal habit models (Constantinides (1990)), habit depends on agent's own consumption

6.1 Habit Formation. Catching Up with the Jones. Abel (1990).

Another form of habit formation is given by the Catching Up with the Jones proposed by Abel (1990). The utility function is given by

$$U(C_t, X_t) = E_t \sum_{j=0}^{\infty} \beta^j \frac{(C_{t+j}/X_{t+j})^{1-\gamma} - 1}{1-\gamma} \tag{44}$$

where X_t summarizes the influence of past consumption. X_t can be either external or internal consumption. Abel (1990) specifies $X_t = \overline{C}_{t-1}^{\kappa}$ where aggregate past consumption matters. Parameter κ governs the degree of time separability.

The derivation of Euler equation is complicated by the fact that time t consumption affect the summation in (44) through the term dated $t+1$ and term dated t.

$$\frac{\partial U_t}{\partial C_t} = \left[1 - \beta\kappa \left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} \left(\frac{X_{t+1}}{X_t}\right)^{\gamma-1}\right] \left(\frac{C_t}{X_t}\right)^{1-\gamma} \left(\frac{1}{C_t}\right) \tag{45}$$

Expression in (45) is random at time t because it depends on consumption at time $t+1$. Substituting for X_t and imposing the condition that the agent's own consumption equals aggregate consumption, this becomes:

$$\frac{\partial U_t}{\partial C_t} = C_{t-1}^{\kappa(\gamma-1)} C_t^{-\gamma} - \beta\kappa C_t^{\kappa(\gamma-1)} C_{t+1}^{-\gamma} \left(\frac{C_{t-1}}{C_t}\right) \tag{46}$$

To capture the idea of habit formation, we need $\kappa(\gamma-1) \geq 0$ to ensure that an increase in yesterday's consumption increases the marginal utility of consumption today. The Euler equation can now be written as:

$$E_t \left[\frac{\partial U_t}{\partial C_t}\right] = \beta E_t \left[(1 + R_{t+1}) \frac{\partial U_{t+1}}{\partial C_{t+1}}\right] \tag{47}$$

where the expectation operator on LHS is needed because of the randomness of $\frac{\partial U_t}{\partial C_t}$

(46) and (48) can be combined to give

$$1 = \beta E_t \left[(1 + R_{t+1}) \left(\frac{C_t}{C_{t-1}}\right)^{\kappa(\gamma-1)} \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \tag{48}$$

If we assume homoscedasticity and joint lognormality of asset returns and consumption growth, this implies the following restriction on risk premia and the riskless interest rate:

$$E_t \left[r_{t+1}^i - r_{t+1}^f\right] + \frac{1}{2}\sigma_i^2 = \gamma\sigma_{ic} \tag{49}$$

$$r_{t+1}^f = -\log(\beta) + \gamma E_t(\Delta c_{t+1}) - \frac{1}{2}\gamma^2 - \kappa(\gamma-1)\Delta c_t \tag{50}$$

Equation (50) says that the riskless rate of return equals its value under power utility, less $\kappa(\gamma-1)\mu_t$. Holding consumption today and expected consumption tomorrow constant, an increase in consumption yesterday increases the marginal utility of consumption today. This makes the representative agent to borrow from the future, driving up the real interest rate. Equation (49) describing the risk premium is exactly the same as the risk premium formula for the power utility. The external habit simply adds a term to the Euler equation which is known at time t, and this does not affect the risk premium.

Abel (1990) nevertheless argues that catching up with the Joneses can help to explain the risk-premium puzzle.

Two considerations:

- The average level of the risk free rate in (50) is $-\log(\beta) - \frac{1}{2}\gamma^2\sigma_t^2 + (\gamma - \kappa(\gamma-1))g$, where g is the average consumption growth rate. When risk aversion γ is very large, a positive κ reduces the average riskless rate. Thus, it enables to increase risk aversion to solve the equity premium puzzle without encountering the risk-free puzzle.

- A positive κ is likely to make the riskless real interest rate more variable because of the term $-\kappa(\gamma - 1)\mu_t$ in (50). If one solves for the stock returns implied by the assumption that stock dividends equal consumption, a more variable real interest rate increases the covariance of stock returns and consumption σ_{ic} and drives up the equity premium.

The second of these points can be regarded as a weakness rather than a strength of the model. The equity premium puzzle is that the ratio of the measured equity premium to the measured covariance σ_{ic} is large; increasing the value σ_{ic} implied by a model that equates stock dividends with consumption does not improve the matters.

Also, the real interest rate does not vary greatly in the short run. Since the standard deviation of the consumption is also small, large values of κ and γ tend to produce counterfactual volatility in the expected real interest rate.

The problem with the riskless real interest rate is a fundamental problem for habit formation models. Time-nonseparable preferences make marginal utility volatile even when consumption is smooth, because consumers derive utility from consumption relative to its recent history rather than from the absolute level of consumption. Time non-separability also creates large swings in expected marginal utility at successive dates, and this implies large movements in real interest rate.

6.2 External Habit (Campbell and Cochrane (1999))

Still another variant of habit formation is the external habit proposed by Campbell and Cochrane (1999).

The utility function they specify is given by

$$u(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma} \tag{51}$$

where X_t denotes levels of habit. Campbell and Cochrane (1999) judiciously choose the latent stochastic process X_t to have properties such that pricing kernel would yield many of the asset prices that we observe.

This model differs from the Abel's ratio model in two ways:

- in this model the agent's risk aversion varies with the level of consumption relative to habit, whereas risk aversion is constant in Abel's ratio model
- in this model consumption must always be above habit for utility to be well defined, whereas it is nor required in the ratio model

To understand the first point, it is convenient to work with the surplus consumption ratio:

$$S_t \equiv \frac{C_t - X_t}{C_t} \tag{52}$$

If habit X_t is held fixed as consumption C_t varies, the normalized curvature of the utility function is

$$\frac{-Cu_{cc}}{u_c} = \frac{\gamma}{S_t} \tag{53}$$

The measure of risk aversion rises as the surplus consumption ratio S_t declines, i.e., consumption declines towards habit.

To ensure that consumption is always above habit specify a non-linear process by which habit adjusts to consumption, remaining below consumption at all times.

Write a process for the log surplus consumption ratio $s_t = \log(S_t)$

Consumption growth is $i.i.d$, that is $\Delta c_{t+1} = g + v_{t+1}$ where $v_{t+1} \sim i.i.dN(0, \sigma^2)$.

AR(1) model for s_t:

$$s_t = (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)v_{t+1} \tag{54}$$

$\overline{s}$- steady state surplus consumption ratio

ϕ persistence parameter

$\lambda(s_t)$ sensitivity function to innovations in consumption growth v_{t+1}

Take a linear approximation around a steady state. (54) can be shown to be a traditional habit formation model in which log habit responds slowly to log consumption:

$$x_{t+1} \approx [(1-\phi)h + g] + \phi x_t + (1-\phi)c_t \tag{55}$$

$$= \left[h + \frac{g}{1-\phi}\right] + (1-\phi)\sum_{j=0}^{\infty}\phi^j c_{t-j} \tag{56}$$

where $h = ln(1-\overline{S})$ is the steady state value of $x - c$.

Since habit is external the marginal utility of consumption is $u'(c_t) = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma}$. The stochastic discount factor is then:

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma} \tag{57}$$

In habit formation models one can volatile sdf not only through high γ but also through a volatile surplus consumption ratio S_t.

The riskless rate:

$$(1 + R_{t+1}^f) = \frac{1}{E_t(M_{t+1})} \tag{58}$$

Taking logs and using (57) and (54):

$$r_{t+1}^f = -\log(\beta) + \gamma g - \gamma(1-\phi)(s_t - \overline{s}) - \frac{1}{2}\gamma^2 \sigma_v^2 [\lambda(s_t) + 1]^2 \tag{59}$$

The first two terms are familiar from the power utility.

The third term reflects intertemporal substitution, or mean-reversion in marginal utility. If the surplus consumption ratio is low, the marginal utility of consumption is high. However, the surplus consumption ratio is expected to revert to its mean, so marginal utility is expected to fall in the future. Therefore, the consumer would like to borrow and this drives up the equilibrium risk free rate.

The fourth term reflects precautionary savings. As uncertainty increases, consumers become more willing to save and this drives down the equilibrium risk free rate.

To generate stable real interest rates, the serial correlation parameter ϕ must be near one. Also, the sensitivity function $\lambda(s_t)$ must decline with s_t so that uncertainty is high when s_t is low and the precautionary savings term offsets the intertemporal substitution term.

External-habit model can produce a large equity premium, volatile stock prices. The basic mechanism is is time variation in risk aversion. When consumption falls relative to habit, the resulting increase in risk aversion drives up the risk premium on risky assets such as stocks.

6.3 5.3. Constantinides (1990)

Suppose utility is derived over services you get from consumption $u(s_t) = \frac{s_t^{1-\gamma}}{1-\gamma}$ where the service flow is defined as

$$s_t = c_t + \theta c_{t-1} \tag{60}$$

for some constant θ.

The representative agent's problem now becomes

$$\max_{\{c_t\},\{x_t\}} E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{(c_{t+j} + \theta c_{t+j-1})^{1-\gamma}}{1-\gamma} \right] \tag{61}$$

$$s.t. c_t + p_t x_{t+1} \leq (y_t + p_t) x_t \tag{62}$$

The first order conditions are

$$[c_t] : \beta^t (c_t + \theta c_{t-1})^{-\gamma} + \beta^t \psi_t + \beta^{t+1} E_t [c_{t+1} + \theta c_t]^{-\gamma} \theta = 0 \tag{63}$$

$$[x_t] : -\beta^t \psi_t p_t \beta^{t+1} E_t [\psi_{t+1} (p_{t+1} + d_{t+1})] = 0 \tag{64}$$

By (64),

$$p_t = E_t \left[\beta \frac{\psi_{t+1}}{\psi_t} (p_{t+1} + d_{t+1}) \right] \tag{65}$$

Therefore, in the habit formation model the pricing kernel becomes $m_{t+1} = \beta \frac{\psi_{t+1}}{\psi_t}$ instead of $m_{t+1} = \beta \left(\frac{c_t}{c_{t+1}} \right)^{\gamma}$ as in CES utility case.

Next by (63) we solve for ψ_t

$$\psi_t = (c_t + \theta c_{t-1})^{-\gamma} + \beta \theta E_t \left[(c_{t+1} + \theta c_t)^{-\gamma} \right] \tag{66}$$

It follows that

$$m_{t+1} = \beta \frac{\psi_{t+1}}{\psi_t} = \beta \frac{(c_{t+1} + \theta c_t)^{-\gamma} + \beta\theta E_{t+1}\left[(c_{t+2} + \theta c_{t+1})^{-\gamma}\right]}{(c_t + \theta c_{t-1})^{-\gamma} + \beta\theta E_t\left[(c_{t+1} + \theta c_t)^{-\gamma}\right]} \tag{67}$$

Since we want to work with stationary variables, dividing and multiplying (67) by c_{t-1}^{γ} and c_t^{γ} to obtain

$$m_{t+1} = \beta \left[\lambda_t^{-\gamma} \frac{(\lambda_{t+1} + \theta)^{-\gamma} + \beta\theta\lambda_{t+1}^{-\gamma} E_{t+1}\left[(\lambda_{t+2} + \theta)^{-\gamma}\right]}{(\lambda_t + \theta)^{-\gamma} + \beta\theta\lambda_t^{-\gamma} E_t\left[(\lambda_{t+1} + \theta)^{-\gamma}\right]}\right] \tag{68}$$

Therefore, when $\theta < 0$ in the case of habit formation, the pricing kernel (68) can indeed be quite volatile.

Finally, we can also represent stock prices and returns in a matrix form. Continue to assume the exogenous processes λ and χ as in Mehra and Prescott (1985) and let $\Phi_{N\times 1} = (\lambda + \theta)^{-\gamma}$ where $\Phi_i = (\lambda_i + \theta)^{-\gamma}$ for $i = 1, ..., N$. Furthermore, we define $\Delta_{N\times 1} = \Pi_{N\times N}\Phi_{N\times 1}$ which gives the conditional expectation $E_t\left[(\lambda_{t+1} + \theta)^{-\gamma}\right]$. In particular, element i in vector Δ is $\Delta_i = E_t\left[(\lambda_{t+1} + \theta)^{-\gamma} | state = i\right] = \sum_{j=1}^{N} \prod_{ij} \Phi_j$. Using λ and Δ we can rewrite the pricing kernel (68) as

$$m(i,j) = \beta\lambda_i^{-\gamma} \left[\frac{(\lambda_j + \theta)^{-\gamma} + \beta\theta\lambda_j^{-\gamma}\Delta_j}{(\lambda_i + \theta)^{-\gamma} + \beta\theta\lambda_i^{-\gamma}\Delta_i}\right] \tag{69}$$

Define P^c to be the price of a stock that is a claim on future consumption stream and P^e that of future dividend stream. Defining matrix $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A}_{ij} = \prod_{ij} m_{ij}\lambda_j$ and $\mathbf{B}_{ij} = \prod_{ij} m_{ij}\xi_j$, respectively, and using argument similar to (33) we can write that $P^c = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{A}\underline{\mathbf{1}}$ and $P^e = [\mathbf{I} - \mathbf{B}]^{-1}\mathbf{B}\underline{\mathbf{1}}$ and that $R^c(i,j) = \lambda_j(1 + P_j^c)/P_i^c$ and $R^e(i,j) = \xi_j(1 + P_j^e)/P_i^e$.