Non-expected Utility.

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1. Standard utilities.

Preferences are additive over time and across states of nature.

Time is discrete, with dates $t = 0, 1, 2, ...$

At each $t > 0$, an event z_t is drawn from a finite set Z, following an initial event z_0.

The t-period history of events is denoted by $z_t = (z_0, z_1, ..., z_t)$ and the set of possible t-histories by Z_t.

A typical agent in such a setting has preferences over payoffs $c(z_t)$ for each possible history.

More common is to impose the additive expected utility structure:

$$U(c(z_t)) = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^t} p\left(z^t\right) u\left[c\left(z^t\right)\right] = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{1}$$

where $0 < \beta < 1$, $p(z^t)$ is the probability of history z^t, and u is a period/state utility function.

These preferences are remarkably parsimonious: behavior over time and across states depends solely on the discount factor β, the probabilities p, and the function u.

Some alternatives have developed to account for the anomalous predictions of expected utility in experimental work.

Others arose from advances in the pure theory of intertemporal. choice.

They offer greater flexibility along several dimensions.

Preferences over time, preferences across states or histories, and (especially) combinations of the two.

Why bother?

more flexible functional forms for approximating features of data the equity premium, for example.

the ability to ask questions that have no counterpart in the additive model.

- How should we make decisions if we do not know the probability model that generates the data?

- Can preferences be dynamically inconsistent? If they are, how do we make decisions?

Assume that agent's period utility function is:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma > 0$$

The single parameter γ captures

- the agent's sensitivity to risk
- the agent's sensitivity to consumption variation across time periods, i.e., his willingness to substitute consumption in one period for consumption in another.

Consider an agent solving the following two-period consumption-saving problem:

$$\max_s E\left\{U(W_0 - s) + \beta U(s\widetilde{R})\right\}$$
$$s.t. s > 0$$

Now consider a deterministic version of this equation:

$$\max_s E\left\{U(W_0 - s) + \beta U(s)\right\}$$
$$s.t. s \leq W_0$$

with

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \gamma > 0$$

The necessary and sufficient first-order conditions:

$$-(W_0 - s)^{-\gamma} + \beta s^{-\gamma} = 0$$

or

$$\left(\frac{1}{\beta}\right)^{\frac{1}{\gamma}} = \left(\frac{W_0 - s}{s}\right)$$

with $\beta < 1$, as $\gamma \to \infty$, $\left(\frac{W_0 - s}{s}\right) \to 1$, i.e. $c_0 \approx c_1$.

For this preference structure, a highly risk-averse agent will also seek an intertemporal consumption profile which is very smooth.

1. Epstein and Zin Utility.

Epstein and Zin (1989, 1991) propose a class of utility functions that allows each dimension to be parameterized separately.

In Epstein and Zin (1991), utility is defined recursively thus relaxing the assumption of separability across states in the standard CRRA utility. In particular, utility is now recursive and becomes

$$U_t = U(c_t, c_{t+1}, c_{t+2}, ...) = W\left(c_t, \mu\left(\widetilde{U}_{t+1}|I_t\right)\right) \tag{2}$$

where W is an 'aggregator' function that maps current consumption c_t and the certainty equivalent μ of future utility $\widetilde{U}_{t+1}$ into current utility. μ is the certainty equivalent in terms of period t consumption of the uncertain utility in all future periods.

Intertemporal substitution is incoded in W while the certainty equivalent function μ reflects the degree of risk aversion.

Specify

$$W_t = \left\{(1-\kappa)\, c_t^{\rho} + \kappa \mu_{t+1}^{\rho}\right\}^{\frac{1}{\rho}}, 0 \neq \rho < 1$$

or

$$W_t = (1-\kappa)\log c_t + \kappa \log \mu_{t+1}, \rho = 0 \qquad (3)$$

$\kappa = \frac{1}{1+\beta}, \beta > 0$. When future consumption in (3) is deterministic function results in intertemporal CRRA function with elasticity of substitution $\delta = \frac{1}{1-\rho}$ and rate of time preference β.

Consider the following specification for CE

$$\mu_{t+1} = E_t(\widetilde{U}_{t+1}^{1-\gamma})^{\frac{1}{1-\gamma}}, 1 \neq \gamma > 0 \tag{4}$$

$$\log \mu_{t+1} = E_t(\log \widetilde{U}_{t+1}), \gamma = 1 \tag{5}$$

Combining parametric forms for μ and W we can rewrite the utility function as

$$U_t = \left\{ (1-\kappa)\, c_t^{\rho} + \kappa E_t(\widetilde{U}_{t+1}^{1-\gamma})^{\frac{\rho}{1-\gamma}} \right\}^{\frac{1}{\rho}} \tag{6}$$

Epstein and Zin (1989) show that γ can be interpreted as a measure of risk aversion in the sense that as γ declines the agent indeed exhibits a more risk averse behavior. Similarly, when the time preference parameter ρ becomes smaller, the agent becomes less willing to substitute consumption intertemporally.

If $\gamma = \rho$, we are back to CRRA utility.

Epstein-Zin preferences do not resolve the equity premium puzzle.

Weil (1989) finds that if γ is fixed at implausibly high levels can a properly calibrated model replicate the premium.

If ρ is calibrated according to empirical studies than the model predict a risk-free rate which is too high.

CCAPM with Epstein-Zin Preferences

The incorporation of separate time and risk preferences may enhance the ability of that class of models to explain the general pattern of security returns.

N assets with returns $\widetilde{r}_{j,t+1}$, $j = 1, ..., N$

Market portfolio with return $\widetilde{r}_{M,t+1}$

Asset pricing equation:

$$E_t\left\{\left[\beta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{\rho-1}\right]^{\frac{\gamma}{\rho}}\left[\frac{1}{1+\widetilde{r}_{M,t+1}}\right]^{1-\frac{\gamma}{\rho}}\left(1+\widetilde{r}_{j,t+1}\right)\right\}=1 \qquad (7)$$

If $\gamma = \rho$ then (7) reduces to the pricing equation of the standard time-separable CCAPM case.

The pricing kernel is:

$$\left[\beta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{\rho-1}\right]^{\frac{\gamma}{\rho}}\left[\frac{1}{1+\widetilde{r}_{M,t+1}}\right]^{1-\frac{\gamma}{\rho}} \tag{8}$$

is a geometric average (with weights γ/ρ) and $1-(\gamma/\rho)$ respectively) of the pricing kernel of the standard CCAPM, $\left[\beta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{\rho-1}\right]$ and the pricing kernel for the log $\rho = 0$ case $\frac{1}{1+\widetilde{r}_{M,t+1}}$

Epstein and Zin next consider a linear approximation to the geometrical average in (8)

$$\frac{\gamma}{\rho}\left[\beta\left(\frac{\widetilde{c}_{t+1}}{c_t}\right)^{\rho-1}\right]+\left(1-\frac{\gamma}{\rho}\right)\left[\frac{1}{1+\widetilde{r}_{M,t+1}}\right] \tag{9}$$

Substituting (9) into (7) gives

$$E_t \left\{ \left[\frac{\gamma}{\rho} \left[\beta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{\rho-1} \right] + \left(1 - \frac{\gamma}{\rho} \right) \left[\frac{1}{1 + \widetilde{r}_{M,t+1}} \right] \right] \left(1 + \widetilde{r}_{j,t+1} \right) \right\} \approx 1 \tag{10}$$

or

$$\frac{\gamma}{\rho} E_t \left\{ \beta \left(\frac{\widetilde{c}_{t+1}}{c_t} \right)^{\rho-1} \left(1 + \widetilde{r}_{j,t+1} \right) \right\} + \left(1 - \frac{\gamma}{\rho} \right) E_t \left\{ \left(\frac{1}{1 + \widetilde{r}_{M,t+1}} \right) \left(1 + \widetilde{r}_{j,t+1} \right) \right\} \approx 1 \tag{11}$$

Equation (11) is revealing.

CAPM: relates the non-diversifiable risk of an asset to the covariance of its returns with market portfolio.

CCAPM: relates the riskiness of an asset to the covariance of its returns with the growth rate of consumption.

Equation (11) suggests that both covariances matter for an asset return.

The covariance of an asset with market portfolio captures its attemporal, non-diversifiable risk.

The covariance of an asset return with the growth rate of consumption fundamentally captures its risk across successive time periods.

Campbell et al (1997) assumed lognormality and heteroscedasticity in consumption and asset returns. The equation (11), expressing the risk premium on asset i satisfies:

$$E_t\left(\widetilde{r}_{i,t+1}\right) - r_{f,t+1} + \frac{\sigma_i^2}{2} = \frac{1-\gamma}{\rho}\left(1-\rho\right)\sigma_{ic} + \left(1 - \frac{1-\gamma}{\rho}\right)\sigma_{iM} \quad (12)$$

Both sources of risk are clearly present.