Lecture 5
More on Stochastic Discount Factors

Cochrane (Ch 3)

m in continuous time

Consumption-based models don't describe the real world very well

$$m = \beta \frac{u'(c_{t+1})}{u'(c_t)} \quad ; \quad p = E(mx)$$

With power utility where CARA = γ, $p = E\left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} x_{t+1} \right]$

$$E(R^i) = R_f - R_f \operatorname{cov}(m, R^i)$$

$$\text{Bounds}: E(m) = \frac{1}{R_f}; \quad \frac{\sigma(m)}{E(m)} \geq \frac{E(R^e)}{\sigma(R^e)}$$

In the real world, the geometric excess return on the stock market has been 6% per year. Its standard deviation has been 17% per year.

The risk free rate has been fairly stable and close to zero, and the average annual consumption growth has been 2%.

With log utility ($\gamma = 1$),

$$\sigma(m) \geq (1.01)\left(\frac{.06}{.17} \right) = .356$$

But the annual standard deviation of consumption growth has been only 1%.

So $\sigma(m)$ is not nearly big enough. In other words, the volatility of consumption growth is much too small to be reconciled with such a high expected market risk premium.

Shiller (1981) was the first to suggest that stock returns are too volatile.

Mehra and Prescott (1985) coined this the "equity premium puzzle" – why do stocks have such a high-risk premium?

Only cov(m,R) should matter, but the variation of consumption growth is very small compared to the variation of stock returns. Since σ(m) is small, cov(m,R) should be small, so $E(R^i) - R_f$ should be small.

cov(m,R) is small – changes in aggregate consumption are not very sensitive to stock returns.

Explanations

1. People are unbelievably risk averse $(\gamma \approx 25)$
2. Utility functional form is misspecified
    - habit formation
    - prospect theory and path dependency
3. Consumption growth is measured incorrectly

Bottom Line

Consumption-based models do a very poor job. m includes other things besides consumption growth.

We need to look for alternative specifications for m ~ f(data)

CAPM $m_{t+1} = a + b\, r_{m,t+1}$

Cochrane (Ch 4)

Ingersoll

Arrow-Debreu security – contingent claim, state price = pc(s) ⟵ p_s
price of asset = p ⟵ v_i

$$m(s) = \frac{pc(s)}{\pi(s)}$$

$$p = \sum_s pc(s)\ x(s) = \sum_s \pi(s)\ m(s)\ x(s) = E(mx)$$

"happy meal logic" ← no combination effect in pricing (linear)

risk neutral probabilities: $\pi^*(s) = R_f\ m(s)\,\pi(s) = R_f\ pc(s)$; $\quad p = \dfrac{E^*(x)}{R_f}$

π^* – higher weight to states that are very unpleasant ← effect of risk aversion
Pay lots of attention to states that are highly probable or resultin disaster.

$\pi^* = \dfrac{m(s)}{E(m)}\pi_s$ ← m tells how to transform probabilities

Back to the consumption model

$$p = E\left[\underbrace{\beta\frac{u'(c_{t+1})}{u'(c_t)}}_{m}\ x_{t+1}\right]$$

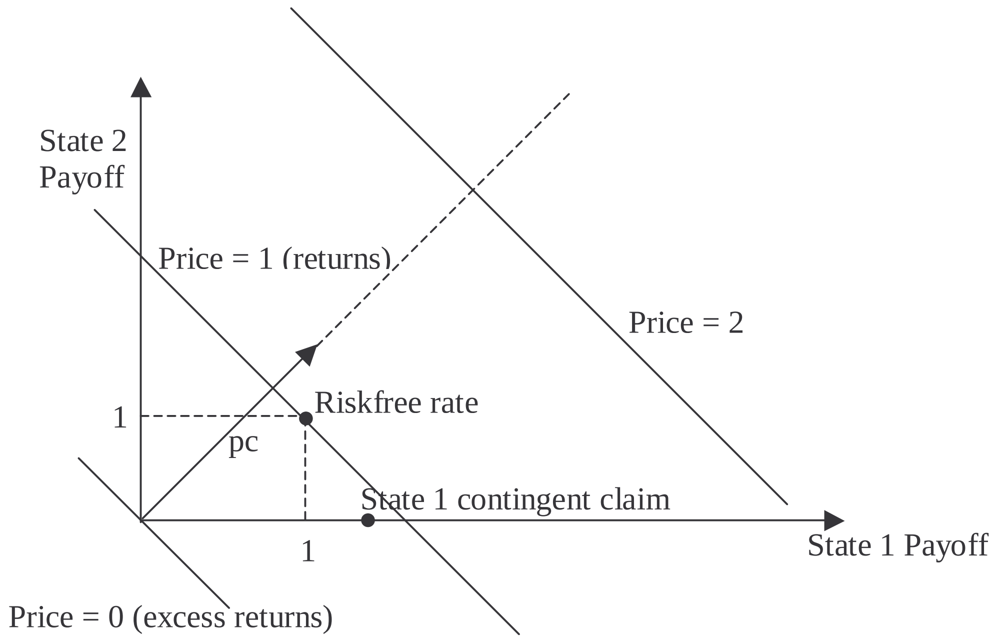
$\dfrac{m(s_1)}{m(s_2)} = \dfrac{u'(c_1)}{u'(c_2)}$ ← relative prices (state 1 at time t+1)

Rate you can give up consumption in state 2 for consumption in state 1 equals MRS between states.

In equilibrium, marginal utility growth is the same for all consumers.

$$\beta^i\,\frac{u'(c^i_{t+1})}{u'(c^i_t)} = \beta^j\,\frac{u'(c^2_{t+1})}{u'(c^2_t)}$$

All consumers share all risks. New securities allow for improved risk sharing.



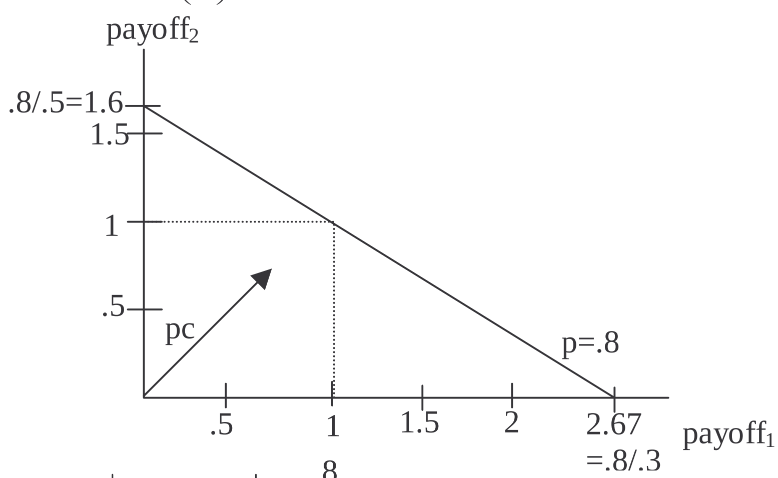
State pricing vector points to the first quadrant.
Marginal utility is always positive, so m > 0 and pc > 0.
Payoff vectors with the same price are on a line orthogonal to pc.

$$p = \sum_s pc(s)\, x(s) = pc \cdot x = |pc|\,|proj(x\,|pc)|$$

\# of units of pc to line

p = 1 is “returns” line; p = 0 is “excess returns” line
R^f at (1,1)

$$pc = \begin{pmatrix} .3 \\ .5 \end{pmatrix} \qquad R_f = \frac{1}{.3+.5} = 1.25 \qquad |pc| = \sqrt{(.3)^2 + (.5)^2} = .5831$$



$$|proj(x|pc)| = \frac{.8}{.5831} = 1.3720; \quad proj(x|pc) = (pc \cdot pc)^{-1}(pc \cdot x)\,pc = \frac{E(pc \cdot x)}{E(pc^2)}\,pc$$

regress x on pc

$$pc \cdot pc = .34, \;\; pc \cdot x = .8 \;\; ; \quad proj(x|pc) = \frac{.8}{.34}\,pc = 2.353 \quad pc = \begin{pmatrix} .706 \\ 1.177 \end{pmatrix}; \sqrt{.706^2 + 1.177^2} = 1.372$$

Link between no arbitrage and a positive m.

X – space of all available payoffs

LOOP $\Rightarrow$ there exists a unique x^* (a discount factor) where $x^* \in \underline{X}$ and $p = E(x^* x)$ for all $x \in \underline{X}$

Any discount factor m can be expressed as $m = x^* + \varepsilon$ where $E(\varepsilon x) = 0$.

If market is complete, x^* the only possible discount factor.

x^* is the projection of any valid m on X.

Cochrane's "no arbitrage" is Ingersoll's "no Type I or II arbitrage"

m might not be unique, and if this is the case, it may be that some of the valid m's might have negative or zero elements. But at least one m must be positive.