Lecture 9
Cochrane Chapter 8 – Conditioning information

$$P_t = E_t\left[\frac{\beta u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] \quad or \quad P_t = E_t\left[m_{t+1}x_{t+1}\right] \quad or \quad P_t = E\left[m_{t+1}x_{t+1} \middle| I_t\right] \quad \uparrow$$

information at time t

If x_t and m_t are iid Vt, then unconditional expectations are the same as conditional expectations.

But this is not always the case.

We could make explicit assumptions about I_t (difficult). Instead of modeling conditional distributions, we would like to use unconditional moments. This chapter identifies what conditional distributions imply for unconditional moments.

<u>Conditioning down</u>

$$P_t = E_t\left[m_{t+1}x_{t+1}\right] \qquad P_t = E_t\left[m_{t+1}x_{t+1} \middle| \Omega\right]$$

$$E(P_t) = E\left[m_{t+1}x_{t+1}\right] \qquad E(P_t \mid I \, c \, \Omega) = E\left[m_{t+1}x_{t+1} \middle| I\right]$$

<u>Law of iterated expectations</u>: If you take an expected value (EV) using less information of an EV using more information, you get back the EV using less information.

$$E\left\{E_t(x)\right\} = E(x)$$

$$E_{t-1}\left[E_t(x_{t+1})\right] = E_{t-1}\left[x_{t+1}\right]$$

$$E\left[E(x|\Omega) \middle| I \, c \, \Omega\right] = E\left[x \middle| I\right]$$

Multiply the payoffs by an instrument z_t

$$z_t P_t = E_t \left[m_{t+1} \underbrace{x_{t+1} \, z_t}_{\text{New payoff}} \right]$$

$$\underbrace{E(z_t P_t)}_{\text{New price}} = E(m_{t+1} \, x_{t+1} \, z_t) \qquad\qquad E(z_t) = E(m_{t+1} \, R_{t+1} \, z_t)$$

Invest in asset according to $z_t \longrightarrow$ managed portfolio e.g. SMB,HML

Checking the (unconditional) expected price/return of <u>all</u> managed portfolios is sufficient to check <u>all</u> the implications of conditioning information.

So to deal with conditional distributions, add managed portfolios and use unconditional moments.

A conditional factor model does not imply an unconditional factor model.

$$m_{t+1} = b_t^{'} f_{t+1} \not\Rightarrow \exists \, b \text{ such that } \mathrm{m}_{t+1} = b^{'} f_{t+1}$$
$$E(R_{t+1}) = \beta_t^{'} \lambda_t \not\Rightarrow \exists \, \beta \text{ such that } E(R_{t+1}) = \beta^{'} \lambda$$

If sensitivities are time-varying, it is <u>not</u> OK to assume they are constant.

If a portfolio is MV efficient with respect to a conditional distribution, it does not imply that the portfolio is MV efficient unconditionally.

Cochrane Chapter 9 – Factor models

$$m_{t+1} = a + b' f_{t+1} \Leftrightarrow E(R_{t+1}) = \alpha + \beta' \lambda$$

What characteristics should factors have?

(1) based on economic foundation – e.g. related to consumption
(2) forecasting variables – predict returns or macro variables
(3) highly (maybe not completely) unpredictable

Classic derivations of CAPM

(1) Consumption CAPM, single period
(2) Quadratic utility, arbitrary returns, single period
(3) Negative exponential utility (general utility), normally distributed returns, single period
(4) Quadratic value function/Bellman equation, arbitrary returns, multiperiod (dynamic programming)
(5) Intertemporal CAPM

1. Consumption CAPM

We did this at the end of Penati/Pennacchi's notes entitled "State Preference Theory"

$Assume\ \ U'(\tilde{C}_{t+1}) = -\gamma \tilde{r}_m \leftarrow$ perfectly negatively correlated

2. Quadratic utility, arbitrary returns

$$U(c_t) = \frac{1}{2}\beta^t (c_t - c^*)^2$$
$$U(c_t, c_{t+1}) = \frac{-1}{2}(c_t - c^*)^2 - \frac{1}{2}\beta E\left[(c_{t+1} - c^*)^2\right]$$

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \beta \frac{(c_{t+1} - c^*)}{(c_t - c^*)}$$

$$c_{t+1} = W_{t+1}$$
$$W_{t+1} = R_t^w (W_t - c_t)$$
$$R_t^w = \sum_{i=1}^{N} w_i R_t^i \quad ; \quad \sum_{i=1}^{N} w_i = 1$$

$$So \quad m_{t+1} = \beta \frac{R_{t+1}^w (w_t - c_t) - c^*}{c_t - c^*} = \frac{-\beta c^*}{c_t - c^*} + \frac{\beta (w_t - c_t)}{c_t - c^*} R_{t+1}^w$$

This is a single factor model.

3. Negative exponential utility, normally distributed returns $u(c) = -e^{-\alpha c}$, $\alpha > 0$ coefficient of absolute risk aversion

We looked at some characteristics of portfolios chosen by an investor with constant ARA in Penati/Pennacchi's notes entitled "Risk Aversion and Portfolio Choice".

$$If \;\; c \sim \eta\left[E(c),\, \sigma^2(c)\right] \Rightarrow E\left[u(c_t)\right] = e^{-\alpha E(c) + \frac{\alpha^2}{2}\sigma^2(c)}$$

wealth W, R_f, risky assets return R, var-cov Σ

y = wealth ($) in each asset

$c = y_f R_f + y' R \quad \leftarrow$ end of period consumption

$W = y^f + y'1 \quad \leftarrow$ budget constraint

$$E\left[u(c)\right] = -e^{-\alpha\left[y_f R_f + y' E(R)\right] + \frac{\alpha^2}{2} y' \Sigma y}$$

$Max \;\; y_f, y \;\; E\left[u(c)\right] \;\; gives$

$$y = \Sigma^{-1} \frac{E(R) - R_f}{\alpha}$$

Amount invested in risky assets independent of wealth.

$$E(R) - R_f = \alpha \sum y = \alpha \operatorname{cov}(R, R_m)$$
$$where \;\; R_m = y_f R_f + y' R$$

$$\frac{E(R_m) - R_f}{\sigma^2 (R_m)} = \alpha$$

4. Quadratic Bellman equation, multiperiod

$$U = u(c_t) + \beta \, E_t \left[V(W_{t+1}) \right]$$

$$\uparrow$$

$$\frac{-\eta}{2} (W_{t+1} - W^*)^2$$

If you have a quadratic utility function $u(c_t) = \frac{-1}{2}(c_t - c^*)^2$,
you get a quadratic Bellman, so the problem looks like the usual single period quadratic utility case.

5. Intertemporal CAPM

We will do this later – after we go through continuous time mathematics.

<u>Concerns about these models</u>

(1) conditional or unconditional?
In general, these are conditional models unless the structure is constant through time (e.g. investment opportunity set is iid, utility function is the same, etc.).

(2) Does CAPM price options?
Generally no – applies to assets with normally distributed payoffs. Options on stocks do not have normally distributed returns.

Yes if quadratic utility.

Finance 400

A. Penati - G. Pennacchi

Arbitrage Pricing Theory

The notion of arbitrage is simple. It involves the possibility of getting something for nothing while having no possibility of loss. More specifically, suppose that an asset portfolio can be constructed without requiring any initial wealth.[1] If this zero-net-investment portfolio can sometimes produce a positive return, but can never produce a negative return, then it represents an arbitrage: starting from zero wealth, a profit can sometimes be made but a loss can never occur. A special case of arbitrage would be if this zero-net-investment portfolio produces a riskless return. If this certain return is positive (negative), an arbitrage is to buy (sell) the portfolio and reap a riskless profit or "free lunch." Only if the return was zero would there be no arbitrage.

An arbitrage opportunity can also be defined in a slightly different context. If a portfolio that requires a non-zero initial net investment is created such that it earns a certain rate of return, then this rate of return must equal the current (competitive maket) risk free interest rate. Otherwise, there would also be an arbitrage opportunity. For example, if the portfolio required a positive initial investment but earned less than the risk free rate, an arbitrage would be to (short) sell the portfolio and invest the proceeds at the risk free rate, thereby earning a riskless profit equal to the difference between the risk free rate and the portfolio's certain (lower) rate of return.[2]

In efficient, competitive, asset markets, it seems reasonable to think that easy profits deriving from arbitrage opportunities are rare and fleeting. Should an arbitrage opportunity temporarily exist, then trading by investors to earn this riskless profit will tend to move asset prices in a direction that eliminates the arbitrage. For example, if a zero-net-investment portfolio produces

[1] This would likely involve some borrowing or short selling of assets as well as long positions in assets.

[2] Arbitrage defined in this context is really equivalent to the previous definition of arbitrage. For example, if a portfolio requiring a positive initial investment produces a certain rate of return in excess of the riskless rate, then an investor should be able to borrow the initial funds needed to create this portfolio and pay an interest rate on this loan that equals the risk-free interest rate. That the investor should be able to borrow at the riskless interest rate can be seen from the fact that the portfolio produces a return that is always sufficient to repay the loan in full, making the borrowing risk-free. Hence, combining this initial borrowing with the non-zero portfolio investment results in an arbitrage opportunity that requires zero initial wealth.

a riskless positive return, as investors create (buy) this portfolio, the prices of the assets in the portfolio will be bid up. The cost of creating the portfolio will then exceed zero. The portfolio's cost will rise until it equals the present value of the portfolio's riskless return, thereby eliminating the arbitrage opportunity. Hence, in competitive asset markets, it may be reasonable to assume that equilibrium asset prices are such that no arbitrage opportunities exist. As will be shown, by assuming the absence of arbitrage, powerful asset pricing results can often be derived.

An early use of the arbitrage principle is the covered interest parity condition in foreign exchange markets. To illustrate, let F_{0t} be the current (date 0) t-period forward price of one unit of foreign exchange. What this forward price represents is the dollar price to be paid t periods in the future for delivery of one unit of foreign exchange t periods in the future. Let S_0 be the spot price of foreign exchange, that is, the current (date 0) dollar price of one unit of foreign currency to be delivered immediately. Also let r_{0t} be the risk free borrowing or lending rate for dollars over the period 0 to t, and denote as r_{0t}^* the risk free borrowing or lending rate for the foreign currency over the period 0 to t.[3]

Next consider setting up the following portfolio which requires zero net wealth. First, let us sell forward (take a short forward position in) one unit of foreign exchange at price F_{0t}.[4] Since we are now committed to deliver one unit of foreign exchange at date t, let us also purchase the present value of one unit of foreign currency, $1/(1+r_{0t}^*)^t$, and invest it at the foreign risk free rate, r_{0t}^*. In terms of the domestic currency, this purchase costs $S_0/(1+r_{0t}^*)^t$, which we finance by borrowing dollars at the rate r_{0t}.

At date t, our foreign currency investment yields $(1+r_{0t}^*)^t/(1+r_{0t}^*)^t = 1$ unit of the foreign currency which we then deliver to satisfy our short position in the forward foreign exchange contract. For delivering the currency, we receive F_{0t} dollars. But we also now owe from our dollar borrowing a sum of $(1+r_{0t})^t S_0/(1+r_{0t}^*)^t$. Thus, our net proceeds are

$$F_{0t} - (1+r_{0t})^t \, S_0/\,(1+r_{0t}^*)^t$$

[3]For example, if the foreign currency is the Japanese yen, r_{0t}^* would be the interest rate for a yen-denominated risk-free investment or loan.

[4]Taking a long or short position in a forward contract requires zero initial wealth, as payment and delivery all occur at the future date t.

Note that these net proceeds are a certain return, that is, this amount is known at date 0 since it depends only on prices and riskless rates quoted at date 0. If this amount was positive, then we should indeed create this portfolio as it represents an arbitrage. If, instead, this amount was negative, then an arbitrage would be for us to sell this portfolio, that is, we reverse each trade discussed above (take a long forward position, invest in the domestic currency financed by borrowing in foreign currency markets). Thus, the only instance in which arbitrage would not occur would be if the net proceeds were zero, that is,

$$F_{0t} = S_0 \left(1 + r_{0t}\right)^t / \left(1 + r_{0t}^*\right)^t$$

which is referred to as the *covered interest parity* condition.

Note that the forward exchange rate, F_{0t}, is determined without knowledge of the utility functions of individuals or their expectations regarding future values of foreign currency. For this reason, pricing (valuing) assets or contracts by using arguments that rule out the existence of arbitrage opportunities can be very appealing.

To motivate how arbitrage pricing might apply to a very simple version of the CAPM, suppose that there is a risk free asset that returns R_f and multiple risky assets. However, it is assumed that only a single source of (market) risk determines all risky asset returns and that these returns can be expressed by the linear relationship

$$\widetilde{R}_i = a_i + b_i \tilde{f} \tag{1}$$

where $\widetilde{R}_i$ is the return on the i^{th} asset, a_i is this asset's expected return, that is, $E[\widetilde{R}_i] = a_i$. Further, $\tilde{f}$ is the single risk factor generating all asset returns, where $E[\tilde{f}] = 0$, and b_i is the sensitivity of asset i to this risk factor. b_i can be viewed as asset i's beta coefficient. Note that this is a highly simplified example in that all risky assets are perfectly correlated with each other.

Now suppose that a portfolio of two assets is constructed, where a proportion of wealth of w is invested in asset i and the remaining proportion of $(1 - w)$ is invested in asset j. This portfolio's return is given by

$$\widetilde{R}_p = wa_i + (1-w)a_j + wb_i\tilde{f} + (1-w)b_j\tilde{f} \tag{2}$$

$$= w(a_i - a_j) + a_j + [w(b_i - b_j) + b_j]\,\tilde{f}$$

If the portfolio weights are chosen such that

$$w^* = \frac{b_j}{b_j - b_i} \tag{3}$$

then the uncertain (random) component of the portfolio's return is eliminated. The absence of arbitrage then requires that $R_p = R_f$, so that

$$R_p = w^*(a_i - a_j) + a_j = R_f$$

or

$$\frac{b_j(a_i - a_j)}{b_j - b_i} + a_j = R_f$$

which implies

$$\frac{a_i - R_f}{b_i} = \frac{a_j - R_f}{b_j} \equiv \lambda \tag{4}$$

This condition states that the expected return in excess of the risk free rate, per unit of risk, must be equal for all assets, and we define this ratio as λ. λ is the risk premium per unit of the factor risk. The denominator, b_i, can be interpreted as asset i's quantity of risk from the single risk factor, while $a_i - R_f$ can be thought of as asset i's compensation or premium in terms of excess expected return given to investors for holding asset i. Thus, this no-arbitrage condition is like a law of one price in that the "price of risk," λ, which is the premium divided by the quantity, must be the same for all assets.

Suppose that asset m has the same degree of risk as the factor, that is, $b_m = 1$. Thus, from the above equilibrium condition $\lambda = a_m - R_f$, or if we interpret asset m as the beta = 1 "market" portfolio, then $\lambda = \overline{R}_m - R_f$. In terms of asset i, the equilibrium condition can then be written as

$$a_i = R_f + b_i\lambda \tag{5}$$

$$= R_f + b_i\left(\overline{R}_m - R_f\right)$$

which is the CAPM relation. Thus, the CAPM can be derived by assuming there is only a single linear risk factor and that this risk factor has the same risk as the market portfolio.

Let us now generalize the arbitrage pricing principle to the case of multiple risk factors and allow individual asset returns to have idiosyncratic components. Thus, let there be k risk factors and N assets in the economy, where $k < N$. Let b_{iz} be the sensitivity of the i^{th} asset to the z^{th} risk factor, given by $\tilde{f}_z$. Also let $\widetilde{\varepsilon}_i$ be the idiosyncratic risk component specific to asset i, which by definition is independent of the k risk factors, $\tilde{f}_1,...,\tilde{f}_k$, and the specific risk component of any other asset j, $\widetilde{\varepsilon}_j$. $\widetilde{\varepsilon}_i$ must be independent of the risk factors or else it would affect all assets, thus not being truly a specific source of risk to just asset i. If a_i is the expected return on asset i, then the return generating process for asset i is given by the linear model

$$\widetilde{R}_i = a_i + \sum_{z=1}^{k} b_{iz}\tilde{f}_z + \widetilde{\varepsilon}_i \tag{6}$$

where $E[\widetilde{\varepsilon}_i] = E\left[\tilde{f}_z\right] = E[\widetilde{\varepsilon}_i\widetilde{\varepsilon}_j] = E\left[\widetilde{\varepsilon}_i\tilde{f}_z\right] = 0$. For simplicity, we will also assume that $E\left[\tilde{f}_z\tilde{f}_x\right] = 0$, that is, the risk factors are mutually independent. As it turns out, this last assumption is not important, as a linear transformation of correlated risk factors can always be found such that they can be redefined as independent risk factors.

Another assumption is that the idiosyncratic risk (variance) for each asset be finite, that is

$$E\left[\widetilde{\varepsilon}_i^2\right] \equiv s_i^2 < S^2 \tag{7}$$

where S^2 is some finite number. Finally, one can always normalize each risk factor to have variance equal to one, so that we will assume $E\left[\tilde{f}_z^2\right] = 1$. Under these assumptions, note that $cov\left(\widetilde{R}_i, \tilde{f}_z\right) = cov\left(b_{iz}\tilde{f}_z, \tilde{f}_z\right) = b_{iz}cov\left(\tilde{f}_z, \tilde{f}_z\right) = b_{iz}$. Thus, b_{iz} is the covariance between the return on asset i and factor z.

Let us now define an asymptotic arbitrage opportunity.

Definition: Let a portfolio containing n assets be described by the vector of investment amounts in each of the n assets, $w^n \equiv [w_1^n \ w_2^n \ldots w_n^n]'$. Consider a sequence of these portfolios where n is increasing, $n = 1, 2, \ldots$. Let σ_{ij} be the covariance between the returns on assets i and j. Then an asymptotic arbitrage exists if the following conditions hold:

(A) The portfolio requires zero net investment:

$$\sum_{i=1}^{n} w_i^n = 0$$

(B) The portfolio return becomes certain as n gets large:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i^n w_j^n \sigma_{ij} \to 0$$

(C) The portfolio return is always bounded above zero

$$\sum_{j=1}^{n} w_i^n a_i \geq \delta > 0$$

We can now state the Arbitrage Pricing Theorem (APT):

Theorem: If no asymptotic arbitrage opportunities exist, then the expected return of asset i, $i = 1, \ldots, n$, will be described by the following linear relation

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz} \lambda_z + \nu_i \qquad (*)$$

where λ_0 is a constant, λ_z can be interpreted as the risk premium for factor z, $z = 1, \ldots, k$, and the expected return deviations, ν_i, satisfy

$$\sum_{i=1}^{n} \nu_i = 0 \qquad (i)$$

$$\sum_{i=1}^{n} b_{iz} \nu_i = 0, \quad z = 1, \ldots, k \qquad (ii)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \nu_i^2 = 0 \tag{iii}$$

Note that condition (iii) says that the average squared error (deviation) from the pricing rule $(*)$ goes to zero as n becomes large. Thus, as the number of assets increase relative to the risk factors, expected returns will, on average, become closely approximated by the relation $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z$. Also note that if the economy contains a risk free asset (implying $b_{iz} = 0$, $\forall\ z$), the risk free return will be approximated by λ_0.

Proof: For a given number of n assets $> k$, "run a regression" of the a_i's on the b_{iz}'s. In other words, project the dependent variable vector $a = [a_1\ a_2 \ldots a_n]'$ on the k explanatory variable vectors $b_z = [b_{1z}\ b_{2z} \ldots b_{nz}]$, $z = 1, ..., k$. Define ν_i as the regression residual for observation i, $i = 1, ..., n$. Denote λ_0 as the regression intercept and λ_z, $z = 1, ..., k$, as the estimated coefficient on explanatory variable z. The regression estimates and residuals must then satisfy

$$a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i \tag{8}$$

where by the properties of an orthogonal projection (Ordinary Least Squares regression) the residuals sum to zero, $\sum_{i=1}^{n} \nu_i = 0$, and are orthogonal to the regressors, $\sum_{i=1}^{n} b_{iz}\nu_i = 0$, $z = 1, ..., k$. Thus, we have shown that $(*)$, (i), and (ii), can be satisfied. The last, but most important part of the proof, is to show that (iii) must hold in the absence of asymptotic arbitrage.

Thus, let us construct an arbitrage portfolio with the following weights

$$w_i = \frac{\nu_i}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \tag{9}$$

so that greater weights are given to assets having the greatest expected return deviation. The total arbitrage portfolio return is given by

$$\tilde{R}_p = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \tilde{R}_i \right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \sum_{z=1}^{k} b_{iz}\tilde{f}_z + \tilde{\varepsilon}_i \right) \right] \tag{10}$$

Since $\sum_{i=1}^{n} b_{iz}\nu_i = 0$, $z = 1, ..., k$, this equals

$$\tilde{R}_p = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \left(a_i + \tilde{\varepsilon}_i \right) \right] \tag{11}$$

Let us calculate this portfolio's mean and variance. Taking expectations, we obtain

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i a_i \right] \tag{12}$$

since $E[\tilde{\varepsilon}_i] = 0$. Substituting in for $a_i = \lambda_0 + \sum_{z=1}^{k} b_{iz}\lambda_z + \nu_i$, we have

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\lambda_0 \sum_{i=1}^{n} \nu_i + \sum_{z=1}^{k} \left(\lambda_z \sum_{i=1}^{n} \nu_i b_{iz} \right) + \sum_{i=1}^{n} \nu_i^2 \right] \tag{13}$$

and since $\sum_{i=1}^{n} \nu_i = 0$ and $\sum_{i=1}^{n} \nu_i b_{iz} = 0$, this simplifies to

$$E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \sum_{i=1}^{n} \nu_i^2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{14}$$

To calculate the portfolio's variance, start by subtracting (12) from (11)

$$\tilde{R}_p - E\left[\tilde{R}_p\right] = \frac{1}{\sqrt{\sum_{i=1}^{n} \nu_i^2 n}} \left[\sum_{i=1}^{n} \nu_i \tilde{\varepsilon}_i \right] \tag{15}$$

Then because $E[\tilde{\varepsilon}_i \tilde{\varepsilon}_j] = 0$ for $i \neq j$ and $E[\tilde{\varepsilon}_i^2] = s_i^2$, the portfolio variance is

$$E\left[\left(\tilde{R}_p - E\left[\tilde{R}_p\right] \right)^2 \right] = \frac{\sum_{i=1}^{n} \nu_i^2 s_i^2}{n \sum_{i=1}^{n} \nu_i^2} < \frac{\sum_{i=1}^{n} \nu_i^2 S^2}{n \sum_{i=1}^{n} \nu_i^2} = \frac{S^2}{n} \tag{16}$$

Thus, as n becomes large ($n \to \infty$), the variance of the portfolio goes to zero, that is, the expected return on the portfolio becomes *certain*. This implies that in the limit the actual return equals the expected return in (14)

$$\lim_{n \to \infty} \tilde{R}_p = E\left[\tilde{R}_p\right] = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \nu_i^2} \tag{17}$$

and so if there are no asymptotic arbitrage opportunities, this certain return on the portfolio must equal zero, that is,

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\nu_i^2} = 0 \tag{18}$$

which is condition (iii). *Q.E.D.*

Note that the APT can be viewed as a multi-beta generalization of CAPM. However, whereas CAPM says that its single beta should be the sensitivity of an asset's return to that of the market portfolio, APT gives no guidance as to what are the economy's multiple underlying risk-factors. Empirical researchers have tended to select risk factors based on those factors that provide the "best fit" to historical asset returns. We will see another multi-beta asset pricing model, namely Merton's Intertemporal CAPM, which is derived from an intertemporal consumer-investor optimization problem. However, that model predicts that the multiple betas are not likely to remain constant through time, which would cause significant difficulties when attempting to estimate betas from historical data.

APT – Cochrane Chapter 9

APT starts with a statistical characterization of outcomes/payoffs/returns. This effectively places restrictions on the structure of the covariance matrix.

e.g. $\text{cov}(r_i, r_j) = \beta_i \beta_j \sigma_m^2$

Alternatively, you can start with the investor's utility function and ask what variables/factors drive marginal utility.

The law of one price (no arbitrage condition) is very powerful when pricing redundant assets, but if you want to price a new non-redundant asset (i.e. one that has sensitivity to a "new" factor), the APT will not help you.

Exact factor model – no residuals
Approximate factor model – includes residuals

If the residuals are small or idiosyncratic, can the price of an individual asset be very different from the price predicted by APT?

In general, yes. It depends on $\text{cov}(m, \varepsilon_i)$. See figure 17.

How do you get from the approximate to the exact factor model?