Lecture 4
Asset Pricing Kernels

From last lecture:

\[
1 = E_t \left[ \frac{\beta u'(C_{t+1})}{u'(C_t)} R^i \right] \quad 1 = E(mR) \Rightarrow 0 = E(mR^c); \quad R^i = R - R_f
\]

\[
p_t = E_t \left[ \frac{\beta u'(C_{t+1})}{u'(C_t)} x_{t+1} \right] \quad p = E(mx)
\]

\[m = \frac{\beta u'(C_{t+1})}{u'(C_t)}\]

m is a measure of aggregate discomfort
bad states \(\rightarrow\) high m \quad ; \quad good states \(\rightarrow\) low m

If x is high when m is high \(\rightarrow\) high P \(\rightarrow\) low E(r)
If x is low when m is high \(\rightarrow\) low P \(\rightarrow\) high E(r) (risky asset)

If asset is riskless, its price should be \(x_{t+1} = \frac{x_{t+1}}{R_f} \frac{1}{1+r_f}\)

\[p = E(mx) = x_{t+1} E(m) \quad \text{so} \quad \boxed{E(m) = \frac{1}{R_f}} \leftarrow \text{first restriction on } m\]

For any risky asset, \(p_t^* = \frac{E_t(x_{t+1}^i)}{R^i}\) \(\quad \text{Appropriate risk-adjusted rate}\)

\[p_t = E_t \left[ m_{t+1} x_{t+1} \right] \text{ means all assets can be priced by defining a single } m_{t+1}\]

Each asset-pricing model corresponds to different choices of m.
- We have not assumed complete markets.
- We have not assumed representative investor.
- We have not restricted the payoff distributions or the form of the utility function.
- We are not restricted to a one-period model \(E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)]\)
- Investors can have non-marketable assets (e.g. human capital).
What affects how $R_f$ is set in equilibrium?

Assume: $u(c) = \frac{1}{1-\gamma} e^{\gamma c}, \quad u'(c) = e^{\gamma c}$ where $\gamma > 0 \quad R_f(c) = \gamma$

consumption growth $\left(\frac{c_{t+1}}{c_t}\right)$ is log normally distributed

$\Delta \ln c_{t+1} = \ln \left(\frac{c_{t+1}}{c_t}\right) = \ln c_{t+1} - \ln c_t$

$\Delta \ln c_{t+1} \sim n\left[E_i \Delta \ln c_{t+1}, \sigma_i^2 \Delta \ln c_{t+1}\right]$

$R_{f,i} = \frac{1}{E_i(m)} = \frac{1}{E_i \left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}$

define: $r_i^f = \ln R_{f,i}, \quad \beta = e^{-\delta}$

$R_{f,i} = \frac{1}{e^{-\delta} e^{-\gamma E_i \Delta \ln c_{t+1} + \frac{\gamma^2}{2} \sigma_i^2 \Delta \ln c_{t+1}}}$

$\Rightarrow r_f = \delta + \gamma E_i \Delta \ln c_{t+1} - \frac{\gamma^2}{2} \sigma_i^2 \Delta \ln c_{t+1}$

more impatient ($\delta$ increases) $\Rightarrow r_f$ increases

$E_i \Delta \ln c_{t+1}$ increases $\Rightarrow r_f$ increases

(high $r_f$ induces savings $\Rightarrow$ higher consumption growth)

$\sigma_i^2 \Delta \ln c_{t+1}$ increases $\Rightarrow r_f$ decreases

If $\gamma$ is high, then investor prefers smooth consumption more. When consumption growth is volatile, investor wants to smooth consumption so he is more willing to save (precautionary savings).
Risk correction to price

\[ p = E(mx) \]
\[ E(mx) = \text{cov}(mx) + E(m)E(x) \]

\[ p = \frac{E(x)}{R_f} + \text{cov}(mx) \rightarrow \text{Risk adjustment} \]

Risk-neutral valuation

If \( \text{cov}(mx) > 0 \Rightarrow \text{high } p \) \ (low \( E(r) \) – like insurance)

\[ \text{cov}(mx) = \frac{\text{cov}[\beta u'(c_{t+1}), x_{t+1}]}{u'(c_t)} \]

\( u'(c) \) is inversely related to \( c \), so \( m_{t+1} \) is inversely related to \( c_{t+1} \).

If an asset's payoff covaries positively with consumption \( \Rightarrow \) low \( p \).
This asset won't help you smooth consumption, so it looks risky.

Risk correction to expected returns

\[ 1 = E(mR) \]
\[ 1 = E(m) E(R) + \text{cov}(m, R) \]
\[ E(R) = \frac{1}{E(m)} - \frac{\text{cov}(m, R)}{E(m)} \]
\[ E(R) = R_f - R_f \text{ cov}(m, R) \]

If an asset's return covaries positively with consumption, it
covaries negatively with \( m \), so \( E(R) > R_f \).
Idiosyncratic risk

Let $R$ be very volatile but still have $\text{cov}(m,R) = 0$. Then $E(R) = R_f$—no risk adjustment. Only systematic risk is priced. What matters is the projection of $R$ onto $m$.

$$R'_i = \hat{\alpha} + \hat{\beta}'m_i + e'_i$$

**CAPM**

$$E(R) = R_f - \frac{\text{cov}(m,R)}{E(m)}$$

$$E(R) = R_f + \frac{\text{cov}(m,R)}{\text{var}(m)} \left[ -\frac{\text{var}(m)}{E(m)} \right] \left[ \beta_i \lambda_m \right]$$

the factor is the pricing kernel

$\lambda_m$ is the price of $\beta$ risk

Is $\lambda_m$ + or - ? What about $\beta_i$ s?

Mean-variance (efficient) frontier

1 = $E(mR')$

$$1 = E(m)E(R') + \rho_{m,R'} \sigma(R') \sigma(m)$$

$$E(R') = R_f - \rho_{m,R'} \frac{\sigma(m)}{E(m)} \sigma(R')$$

$$\left| E(R') - R_f \right| \leq \frac{\sigma(m)}{E(m)} \sigma(R')$$

$$\left| E(R'_i) \right| \leq \frac{\sigma(m)}{E(m)} \sigma(R'_i)$$

$$\sigma(m) \geq \frac{1}{R_f} \left[ \frac{E(R'_i)}{\sigma(R'_i)} \right]$$

← second restriction on $m$
\( \frac{E(R'_i)}{\sigma(R'_i)} \) is asset i’s Sharpe ratio. This relation is used to construct the Hansen-Jaganathan bound, a region of permissible values for the moments of m.

- gives a region where stocks can be

\[
\begin{align*}
E(r) & \quad \text{Slope } \frac{\sigma(m)}{E(m)} \\
\sigma(R) & \quad \text{Mean-variance frontier}
\end{align*}
\]

\( \rho_{m,R}^i \) captures the degree of systematic risk.

If \( \rho_{m,R}^i = -1 \), asset is perfectly negatively correlated with m and perfectly positively correlated with consumption (no idiosyncratic risk). These assets receive the highest expected return.

\[
E(R') = R_f + \frac{\sigma(m)}{E(m)} \sigma(R') \quad \leftarrow \text{CML}
\]

You can reach any position on the efficient frontier if you have two assets on it.

Any efficient portfolio carries all pricing information!

\[
E(R') = R_f + \beta_{i,m} \left[ E(R'^{mv}) - R_f \right] \quad \leftarrow \text{SML with respect to any efficient portfolio}
\]

\[
\frac{|E(R') - R_f|}{\sigma(R')} = \frac{\sigma(m)}{E(m)} = R_f \sigma(m) \quad \leftarrow \text{slope of CML}
\]
Again, let \( u'(c) = c^{-\gamma} \), lognormal consumption growth.

\[
\frac{\sigma(m)}{E(m)} = \frac{\sigma}{E} \left[ e^{-\delta (\frac{c_{t+1}}{c_t})^{-\gamma}} \right] \Rightarrow \sigma \text{ big if } \frac{c_{t+1}}{c_t} \text{ volatile, } \gamma \text{ big}
\]

\[
\frac{\sigma(m)}{E(m)} = \sqrt{e^{2\sigma^2(\Delta \ln c_{t+1})}-1} \approx \gamma \sigma \Delta \ln c_{t+1}
\]

Steep slope if economy is risky (consumption growth is volatile) and if consumers are very risk averse.

**Time-varying expected returns**

\[
E_t(R_{t+1}) = R_{f,t} - \frac{\text{cov}_t[m_{t+1}, R_{t+1}]}{E(m_{t+1})}
\]

Can have predictable returns as long as it is explained by changing expected consumption growth, changing covariance of return with consumption growth, or changing risk aversion.

**Present Value**

\[
P_t = E_t \sum_{j=0}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} d_{t+j} \left( \frac{m_{t,t+j}}{m_{t,t}} \right)
\]