Lecture 4
Asset Pricing Kernels

From last lecture:

$$1 = E_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)} R^i\right] \qquad 1 = E(mR) \Rightarrow 0 = E(mR^e); \quad R^e = R^i - R_f$$

$$p_t = E_t\left[\frac{\beta u'(C_{t+1})}{u'(C_t)} x_{t+1}\right] \qquad p = E(mx)$$

$$m = \frac{\beta u'(C_{t+1})}{u'(C_t)}$$

m is a measure of aggregate discomfort
bad states ⟶ high m ; good states ⟶ low m

If x is high when m is high ⟶ high P ⟶ low E(r)

If x is low when m is high ⟶ low P ⟶ high E(r) (risky asset)

If asset is riskless, its price should be $\frac{x_{t+1}}{R_f} = \frac{x_{t+1}}{1+r_f}$

$p = E(mx) = x_{t+1}E(m)$ *so*

$$\boxed{E(m) = \frac{1}{R_f}} \leftarrow \text{ first restriction on m}$$

For any risky asset, $p_t^i = \frac{E_t(x_{t+1}^i)}{R^i}$ ⟵ Appropriate risk-adjusted rate

$p_t = E_t\left[m_{t+1}x_{t+1}\right]$ means all assets can be priced by defining a single m_{t+1}

Each asset-pricing model corresponds to different choices of m.

- We have not assumed complete markets.
- We have not assumed representative investor.
- We have not restricted the payoff distributions or the form of the utility function.
- We are not restricted to a one-period model $E_0[\sum_{t=0}^{\infty}\beta^t u(c_t)]$
- Investors can have non-marketable assets (e.g. human capital).

$$R_f = \frac{1}{E(m)}$$

What affects how R_f is set in equilibrium?

Assume: u(c) = $\frac{1}{1\text{-}\gamma} c^{1-\gamma}$, $u'(c) = c^{-\gamma}$ where $\gamma > 0$ $R_f(c) = \gamma$

consumption growth $\left(\frac{c_{t+1}}{c_t}\right)$ is log normally distributed

$$\Delta \ln c_{t+1} = \ln\left(\frac{c_{t+1}}{c_t}\right) = \ln c_{t+1} - \ln c_t$$

$$\Delta \ln c_{t+1} \sim n\left[E_t\, \Delta \ln c_{t+1},\ \ \sigma_t^2\, \Delta \ln c_{t+1}\right]$$

$$R_{f,t} = \frac{1}{E_t(m)} = \frac{1}{E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}$$

define: $r_t^f = \ln R_{f,t}$, $\beta = e^{-\delta}$

$$R_{f,t} = \frac{1}{e^{-\delta} e^{-\gamma E_t \Delta \ln C_{t+1} + \frac{\gamma^2}{2}\sigma_t^2 \Delta \ln C_{t+1}}}$$

$$\Rightarrow r_f = \delta + \gamma E_t\ \Delta \ln c_{t+1} - \frac{\gamma^2}{2}\sigma_t^2\ \Delta \ln c_{t+1}$$

more impatient (δ increases) $\Rightarrow$ r_f increases

E_t $\Delta \ln c_{t+1}$ increases $\Rightarrow$ r_f increases

(high r_f induces savings $\Rightarrow$ higher consumption growth)

$\sigma_t^2 \Delta \ln c_{t+1}$ increases $\Rightarrow$ r_f decreases

If γ is high, then investor prefers smooth consumption more. When consumption growth is volatile, investor wants to smooth consumption so he is more willing to save (precautionary savings).

Risk correction to price

$$p = E(mx)$$
$$E(mx) = \text{cov}(mx) + E(m)E(x)$$

$$p = \frac{E(x)}{R_f} + \text{cov}(mx)$$

$\frac{E(x)}{R_f}$: Risk-neutral valuation

$\text{cov}(mx)$: Risk adjustment

If $\text{cov}(mx) > 0 \Rightarrow \text{high } p \quad (\text{low } E(r) - \text{like insurance})$

$$\text{cov}(mx) = \frac{\text{cov}\left[\beta\, u'(c_{t+1}), x_{t+1}\right]}{u'(c_t)}$$

$u'(c)$ is inversely related to c, so m_{t+1} is inversely related to c_{t+1}.

If an asset's payoff covaries positively with consumption $\Rightarrow$ low p.
This asset won't help you smooth consumption, so it looks risky.

Risk correction to expected returns

$$1 = E(mR)$$
$$1 = E(m)\,E(R) + \text{cov}(m, R)$$
$$E(R) = \frac{1}{E(m)} - \frac{\text{cov}(m, R)}{E(m)}$$
$$E(R) = R_f - R_f\,\text{cov}(m, R)$$

If an asset's return covaries positively with consumption, it covaries negatively with m, so E(R) > R_f.

Idiosyncratic risk

Let R be very volatile but still have cov(m,R) = 0.
Then E(R) = R_f ⟵ no risk adjustment.
Only systematic risk is priced.
What matters is the projection of R onto m.

$$R_s^i = \underbrace{\hat{\alpha}^i + \hat{\beta}^i m_s} + e_s^i$$

CAPM

$$E(R) = R_f - \frac{\text{cov}(m,R)}{E(m)}$$

$$E(R) = R_f + \underbrace{\frac{\text{cov}(m,R)}{\text{var}(m)}}_{\beta_i} \underbrace{\left[\frac{-\text{var}(m)}{E(m)}\right]}_{\lambda_m} \quad \leftarrow SML$$

the factor is the pricing kernel
λ_m is the price of β risk
Is λ_m + or - ? What about $\beta_i\, s$?

Mean-variance (efficient) frontier

$$1 = E(mR^i)$$

$$1 = E(m)E(R^i) + \rho_{m,R^i}\, \sigma(R^i)\, \sigma(m)$$

$$E(R^i) = R_f - \rho_{m,R^i} \frac{\sigma(m)}{E(m)} \sigma(R^i)$$

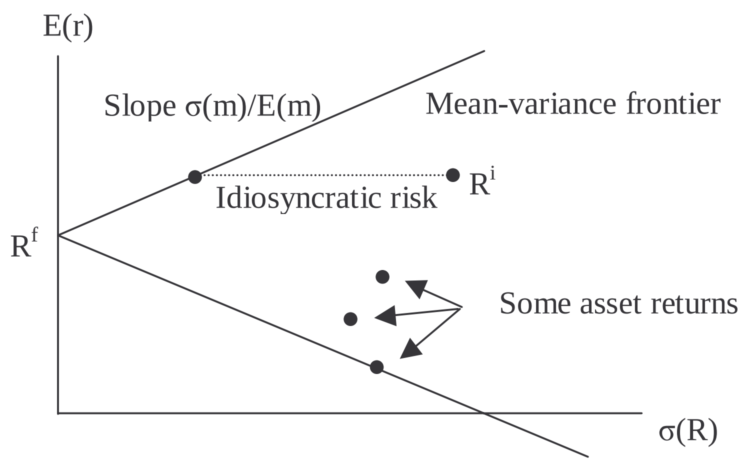
* $$\left|E(R^i) - R_f\right| \le \frac{\sigma(m)}{E(m)} \sigma(R^i)$$

$$\left|E(R_i^e)\right| \le \frac{\sigma(m)}{E(m)} \sigma(R_i^e)$$

$$\boxed{\sigma(m) \ge \frac{1}{R_f}\left[\frac{E(R_i^e)}{\sigma(R_i^e)}\right]} \leftarrow \text{second restriction on m}$$

$\frac{E(R_i^e)}{\sigma(R_i^e)}$ is asset i's Sharpe ratio. This relation is used to construct the Hansen-Jaganathan bound, a region of permissible values for the moments of m.

- gives a region where stocks can be



ρ_{m,R^i} captures the degree of systematic risk.

If $\rho_{m,R^i} = -1$, asset is perfectly negatively correlated with m and perfectly positively correlated with consumption (no idiosyncratic risk). These assets receive the highest expected return.

$$E(R^i) = R_f + \frac{\sigma(m)}{E(m)}\sigma(R^i) \quad \leftarrow CML$$

You can reach any position on the efficient frontier if you have two assets on it.

Any efficient portfolio carries all pricing information!

$E(R^i) = R^f + \beta_{i,mv}\left[E(R^{mv}) - R_f\right] \leftarrow SML$ with respect to any efficient portfolio

$\frac{\left|E(R^i) - R_f\right|}{\sigma(R^i)} = \frac{\sigma(m)}{E(m)} = R_f\sigma(m) \leftarrow$ slope of CML

Again, let $u'(c) = c^{-\gamma}$, lognormal consumption growth.

$$\frac{\sigma(m)}{E(m)} = \frac{\sigma\left[e^{-\delta}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}{E\left[e^{-\delta}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]}$$

← σ big if $\frac{c_{t+1}}{c_t}$ volatile, γ big

$$\frac{\sigma(m)}{E(m)} = \sqrt{e^{\gamma^2\sigma^2(\Delta \ln c_{t+1})} - 1} \quad \approx \gamma \sigma \Delta \ln c_{t+1}$$

Steep slope if economy is risky (consumption growth is volatile) and if consumers are very risk averse.

<u>Time-varying expected returns</u>

$$E_t(R_{t+1}) = R_{f,t} - \frac{\text{cov}_t\left[m_{t+1}, R_{t+1}\right]}{E(m_{t+1})}$$

Can have predictable returns as long as it is explained by changing expected consumption growth, changing covariance of return with consumption growth, or changing risk aversion.

<u>Present Value</u>

$$P_t = E_t \sum_{j=0}^{\infty} \underbrace{\beta^j \frac{u'(c_{t+j})}{u'(c_t)}}_{m_{t,t+j}} d_{t+j}$$