# Information Ambiguity Explains Asset Pricing Anomalies: An Expectile CAPM Approach

January 14, 2014

## Abstract

The concept of 'information ambiguity' is utilized to develop an 'expectile-based' asset pricing framework to simultaneously resolve two categories of asset pricing anomalies: the equity premium puzzle, from the perspective of the stochastic discount factor (SDF) mean and variance, and momentum, from the perspective of the SDF factor structure. We show that information ambiguity directly introduces two categories of information premium that can explain observed asset pricing anomalies: compensation for taking information ambiguity risk, and compensation for providing additional information. Ambiguity risk will also activate the 'latent' nature of market risk, indirectly affecting how this is compensated. (*JEL* G02, G11, G12, C51)

**Key words:** Ambiguity asset pricing, Equity premium puzzle, Momentum, Risk-reward measure, Expectile

Asset pricing theory states that the asset price of any contingent claim is the expectation of its terminal payoff discounted by a stochastic discount factor (SDF) (Cochrane, 2001). However, empirical evidence has consistently been at odds with the predictions of the established theoretical models. This is evident from the perspective of SDF mean-variance anomalies, e.g., the equity premium puzzle (Mehra and Prescott, 1985), risk-free rate puzzle (Weil, 1989), correlation puzzle (Friend and Blume, 1975), etc., and from the perspective of SDF's factor structure anomalies, e.g., size effect (Banz, 1981), value effect (Fama and French, 1993), and momentum (Jegadeesh and Titman, 1993). The purpose of this paper is to explain simultaneously two of the aforementioned anomalies: the equity premium puzzle and the momentum issue, by developing an 'information ambiguity' enhanced asset pricing model.

Information ambiguity is defined as a state of having limited knowledge in which it is impossible to describe the probability of future outcomes exactly. This paper, therefore, relates to the literature on the ambiguity asset pricing approach. See Gilboa and Schmeidler (1989), Ghirardato et al. (2004), Klibanoff et al. (2005), Bossaerts et al. (2010), Ui (2011), Araujo et al. (2012). It adds to the existing ambiguity literature by establishing a coherent expectile (Newy and Powell, 1987) risk-reward measurement framework to reflect investors' view bias towards ambiguity. In this instance, 'view bias' is defined as asymmetric weighting which amplifies the probability of the good state and reduces the probability of the bad state for optimistic investors, or vice-versa, for pessimistic investors.

The focus, therefore, is on the axiomatization of investor's decision-making, thus extending the standard marginal condition for a consumption-based optimum. The asset price of any contingent claim becomes the expectation of the terminal payoff discounted by the product of the SDF and a view bias adjustment, or equivalently the expectile of the terminal payoff discounted by SDF.

While building on the existing ambiguity literature, the work presented in this paper provides added dimensions to both the market sentiment, see Yu and Yuan (2011) and Stambaugh et al. (2012), and the consumption-based approach, including: 1) Loss aversion by Benartzi and Thaler (1995), Fielding and Stracca (2007), and Giorgi and Post (2011). 2) Disappointment ex post by Athanasoulis and Sussmann (2007) and Gollier and Muermann (2010). 3) Habit formation by Meyer and Meyer (2005), Du (2011), and Otrok et al. (2002). 4) The relationship between time preference and risk aversion coefficients by Kang and Kim (2012). In contrast to the former, view bias is implied by the equity premium puzzle rather than being extracted through the use of news and also takes into account the existing empirical evidence on the existence of (on aggregate) a pessimistic view bias in the post-war United States (US). In contrast with the utility consumption-based approach, we replace the expectation operator with an 'expectile' which has the advantage of conceptually reflecting investors' tendency to amplify the probability of bad states and to reduce the probability of good states if the investor is pessimistic, or vice versa, for optimistic investors. It is the introduction of these concepts that enables the model to account for arbitrage pricing puzzles.

The key theoretical result contradicts the standard multifactor theory. Firstly, we hypothesize that when pessimistic (optimistic) investors with low confidence in their beliefs on probabilities are required to choose a deterministic value to represent the future uncertainty, they will choose a value lower (higher) than the expectation, and that will introduce a negative (positive) information adjustment, which is proportional to the volatility of the portfolio rate of return. Therefore, they require a higher (lower) compensation for accepting this ambiguity risk. We name this premium the 'first category of information premium'. Secondly, the amount of the information contained in the distribution of the portfolio rate of return is less than the amount in the joint distribution of individual securities within the portfolio. This paper extends the law of one price under imperfect information in such way that if pessimism exists, investors should assemble the securities into a portfolio; if optimism exists, investors should break down the portfolio into securities. The positive excess rate of return they earn by doing so is termed the 'second category of information adjustment', which rewards them for providing additional information to the market price. Thirdly, information ambiguity activates the latent nature of the market risk, which is orthogonal to the systematic risk. Although the extended CAPM still admits a single beta expression, the beta becomes the weighted average of the systematic risk and the latent risk. The weights are uniquely determined by the information ambiguity. The price of risk becomes the 'view bias- adjusted market risk premium'. Not surprisingly, the nonlinear structure above has an inherently stronger explanatory power.

Applying the information ambiguity concept to explain the equity premium puzzle in the US stock market, we find that, under imperfect information, investors' view bias will significantly reduce the required risk premium in the equity market. We also utilize the concept of ambiguity to solve the momentum puzzle by answering the following two questions.

Firstly, why do recent winners (stocks with strong past performance) continue to outperform the losers over the next period on a 3-12 month horizon (Jegadeesh and Titman, 1993) with betas for the winners being even lower than those of the losers (Wang, 2005), and why does the effect disappear after 12 months? As view bias appears to feature a cyclical dynamic in processing past information and accommodating itself to new information, view bias has a periodic effect on how the extended CAPM works, with a significant period of 19 months. Hence the pessimistic part of the view bias dynamic is not a phase-lagged mirror reflection of the optimistic part, and the pessimistic part occupies the majority of the dynamic, roughly corresponding to the description of the horizon in Jegadeesh and Titman (1993). Whenever view bias returns to neutral or optimistic, the effect disappears; otherwise, real data shows that the pessimistic view bias in the post-war era adjusted the market price of risk to a negative value. Hence, a low (high) level of risk implies a winner (loser) position, and it holds as the pessimistic view bias continues. Another contributor is the first category of information premium. Since it is proportional to the volatility of the portfolio rate of return, a higher volatility implies a winner position.

Secondly, why is this reversed in the long term, namely where losers over the past 3-5 years tend to substantially outperform prior-period winners during the subsequent 3-5 years (De Bondt and Thaler, 1985)? The view bias has another significant period of approximately 5 years. Although the optimistic view bias is only the minority of the dynamic, the optimistic part occurs over a sufficiently long period to manifest a reversal. Under the optimistic view bias, the market price of risk is always positive; hence a low (high) amount of risk implies a loser (winner) position, and this holds as long as the optimistic view bias continues.

The remainder of the paper is organized as follows. Section 1 establishes the axiomatic expectile based reward-risk measurement framework. Section 2 replaces the expected utility maximization axiom with the expectile utility maximization axiom, reformulates the Merton problem, and theoretically resolves the anomalies. Section 3 reports empirical results and quantitative implications. Section 4 concludes. Proofs are presented in the Appendix.

# 1 Expectile Risk-Reward Measurement Framework

In this section, we establish an expectile-based risk-reward measurement framework, check the coherency of both reward and risk measures, summarize the advantages of using the expectile as a reward measure to reflect an investor's view bias towards information ambiguity. We term the methodology as 'view bias approach'. Finally, we compare the 'view bias approach' and the 'risk preference approach' in Table 2.

## 1.1 Expectile definition

We assume the following probabilistic setting holds throughout the paper: the Brownian motion *W* will be defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$, and we shall denote by $\mathcal{F}_t$ the $\mathcal{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \leqslant s \leqslant t)$. For simplicity, at this stage, we drop time $t$, and denote $W(t)$ as $W$, and obviously $W$ is absolute-integrable, i.e., $\int |W| f_W(w)dw < \infty$.

**Definition 1** The *unconditional expectile* (Newy and Powell, 1987) of $W$ is

$$\mathbb{E}_\theta(W) \triangleq argmin_q \left[ (1-\theta) \int_{W<q} (W-q)^2 f_W(w)dw + \theta \int_{W \geqslant q} (W-q)^2 f_W(w)dw \right], \qquad (1)$$

where $f_W(\cdot)$ is the probability density function of $W$, and $\theta \in (0,1)$.

We interpret $\theta$ as view bias coefficient, and term $\mathbb{E}_\theta(W) - \mathbb{E}(W)$ as the 'first category of information premium'. Under perfect information, where the probability distribution is clearly known to the investors, $\theta$ is 50%, which implies neutral view bias. Under imperfect information, where the homogeneous investors have strong doubt about the probability distribution, if they are holding pessimistic view bias, they will overweight the probability of the bad state (i.e., $(1-\theta) \geqslant 50\%$) and underweight the probability of the good state (i.e., $\theta < 50\%$); if they are holding optimistic view bias, they will do the opposite. Thus, $\theta$ captures how the investors tend to respond to the information ambiguity. Taking the first order derivative to optimize Equation (1) with respect to (w.r.t.) $q$, we equivalently define the unconditional expectile as follows,

$$\mathbb{E}_\theta(W) \triangleq q^* = \int \pi_W(\theta) w f_W(w) dw, \tag{2}$$

where $\pi_W(\theta) = \Theta[\int \Theta f_W(w)dw]^{-1}$ and $\Theta = (1-\theta)\mathbf{1}_{\{W<q^*\}} + \theta\mathbf{1}_{\{W\geqslant q^*\}}$. The unconditional expectile $\mathbb{E}_\theta(W)$ (for simplicity, denoted as $q^*$) appears on both sides of Equation (2). Hence Equation (2) is an implicit function w.r.t. $\mathbb{E}_\theta(W)$.

## 1.2 Necessity of introducing view bias

Since the anomalies contradict the standard marginal condition for basic consumption-based optimum, the optimum should be the starting point for this research. By choosing the optimal consumption sequence and the weight of each security in the portfolio, the investor is able to maximize the expected total utility generated by lifetime consumption and the utility generated by lump sum terminal wealth, within their budget constraints. However in certain periods, e.g., in the post-war era, investors only have information on the possible outcomes of an uncertain event, and do not know the exact probabilities of each state; instead they simply have a vague assessment[1]. Therefore, investors are unable to form coherent expectations. As a result, a pessimistic (an optimistic) investor will adopt a maxmin (maxmax) strategy, assuming that the worst (best) state will happen with certainty, and will accordingly choose the contingent claim providing the highest payoff in the worst (best) state. Table 1 provides an example.

[1] Knight (1921) distinguishes uncertainty, knowing the outcomes without being aware of the corresponding probabilities, from risk.

**Table 1**
**Decision making under perfect and imperfect information**

| | Contingent claim X | | | | Contingent claim Y | | | |
|---|---|---|---|---|---|---|---|---|
| State | S1 | S2 | S3 | S4 | S1 | S2 | S3 | S4 |
| Rate of return | 1% | 3% | 50% | 100% | -1000% | 3% | 50% | 100% |
| Probability | 5% | 15% | 70% | 10% | 0% | 10% | 80% | 10% |
| Summary | Mean=45.5%, VAR=0.068 | | | | Mean=50.29%, VAR=0.048 | | | |

Assuming that risk-averse investors face risk with perfect information on the probability of each state, they will choose contingent claim Y, because Y generates a higher expected rate of return (50.29%) than X (45.5%), and a lower variance of the rate of return (0.048) than X (0.068). However, if the investors cannot observe the probability, or they strongly doubt their own beliefs on the probability distribution, by comparing the state payoff sets and following the maxmin principle, pessimistic investors will adopt contingent claim X, initially expecting that the worst state (S1) of each strategy will occur, and then select the strategy with the better result. Conversely, optimistic investors will expect that the best state will always occur, and then select the contingent claim with the better result. In contrast to risk preference, we term this pessimism and optimism[2] the ‘view bias’ of an investor facing imperfect information.

It is not necessarily the case that a risk-averse investor is pessimistic or a risk-taking investor is optimistic. Risk-averse investors can be either pessimistic or optimistic, reflecting their opinions in circumstances when probabilities are unobservable. We find the following stylized fact in Section 3: if the market return and consumption growth are positively

[2] Abel (2002) shows that pessimism and doubt in the subjective distribution of the growth rate of consumption reduce the equity premium puzzle. See also Guidolin (2006).

correlated, then either a high risk aversion coefficient with a neutral view bias coefficient, or a pessimistic view bias coefficient with a normal risk aversion coefficient, are substitutable explanations for the equity premium puzzle. If, on the other hand, the market return and consumption growth move in opposite directions, then either a high risk aversion coefficient with a neutral view bias coefficient, or an optimistic view bias with a normal risk aversion coefficient, are substitutable explanations for the equity premium puzzle; this would also contradict Bossaerts et al. (2010)'s finding that the attitudes towards risk and ambiguity are positively correlated.

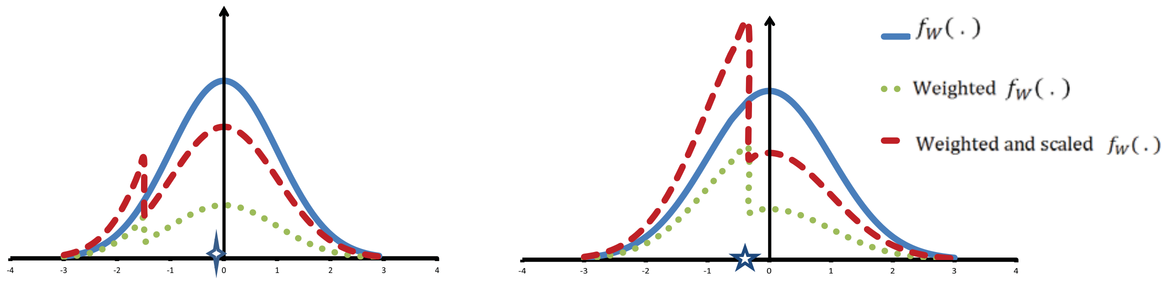
## 1.3 Necessity of employing expectile

What will happen if the extreme states of each strategy are identical? For example, the worst stock price is always zero; the worst rate of return is always negative infinity, etc. This is why many studies have been conducted using quantile to measure the reward of the rate of return instead of using extreme worst outcomes. The logic is that although the worst outcomes for each strategy are identical, their 5% quantiles might be sufficiently different for us to make a choice. However, there are two reasons that we do not consider quantile to be a valid measure. Firstly, quantile is not a sufficient statistic, and making decisions based on quantiles is a waste of information. Secondly, using quantile as a measure of reward does not produce a concise SDF pricing framework, which integrates all asset pricing problems at the contingent claim level. Without that framework, the expectation-based martingale theory is no longer available, and the elegant risk neutral valuation option pricing theory collapses. The above concerns motivate us to establish an expectile risk-reward measurement framework, by taking the merits of both expectation and quantile, without producing the drawbacks of either.

## 1.4 Six advantages of expectile approach

Firstly, the implicit function form makes the expectile a ‘self-trimming’ measure, where ‘self-trimming’ is defined as the superposition of weighted average and the weighting division point. That makes the expectile superior to the weighted average. See Figure 1. for a graphical illustration of how expectile is obtained and the differences between the expectile and the weighted average. For an arbitrarily chosen weighting division point, the weighted average is not necessarily being equal to that division point. When solving the complicated problems, we might have to apply the weighted average operator to another weighted average of a random variable. As the weighting division point differs from the weighted average, the results become increasingly unclear. However, the expectile ensures that irrespective of the complexity of the research problem, and how frequently the expectile operator needs to be layered onto another expectile operator, it is always possible to obtain a concise expression capable of completely addressing the essence of the problem without involving excessively technical aspects that are weakly supported by financial intuition.

Secondly, as a tool for resolving asset pricing puzzles, expectile can describe information ambiguity. We use relative entropy (Kukkback-leibler




(a) Weightd average

(b) Expectile

**Fig. 1.**
Comparison of non-self-trimming and self-trimming reward measure. The horizontal axis represents the value of $W$. Curve [ **—** $f_W(\cdot)$] represents the probability density of a standard normal distributed random variable. We assign a weight of $(1-\theta)$ to the left-hand side of the weighting division point, and $\theta$ to the right-hand side. We obtain the weighted density $\left[(1-\theta)\mathbf{1}_{\{W<q^*\}} + \theta\mathbf{1}_{W\{\geqslant q^*\}}\right] f_W(w)$, see curve [ **...** weighted $f_W(\cdot)$]. To transform it into a probability density, we scaled it by dividing it by the area under curve [ **...** weighted $f_W(\cdot)$], then we obtain a probability density, $\pi_W(\theta)$, see curve [ **- -** weighted and scaled $f_W(\cdot)$]. This graph illustrates the conceptual differences between weighted average [see Figure 1(a)] and expectile [see Figure 1(b)]. Four-pointed star in Figure 1(a) represents the weighted average of the standard normal distribution under [ **- -** weighted and scaled $f_W(\cdot)$], where the weighting division point is selected arbitrarily. The pentacle in Figure 1(b) represents the weighted average of the standard normal distribution under [ **- -** weighted and scaled $f_W(\cdot)$], where the weighting division point is selected to be to equal to the weighted average (pentacle).

Distance), a measure of the extent to which the distorted distribution differs from the original one, to quantify the information lost. We find that the relative entropy of expectile is less than that of quantile, $0 \leqslant D(f\|h_{expextile}) < D(f\|h_{quantile}) = +\infty$, which means that expectile can model the process where the information is being lost continuously w.r.t the view bias adjustment. However, when we use quantile to measure the reward, the probability information cannot be adjusted to control for the view bias. Moreover, the expectile is a sufficient statistic; thus all of the information contained in the sample is retained when a sample-based expectile estimator is employed. However, the quantile is not a sufficient statistic, and the information contained in the sample will be lost when employing a sample-based quantile estimator.

Thirdly, the risk measure ('variancile' - $VAR_\theta$) and reward measure (‘expectile’ - $\mathbb{E}_\theta$) can be defined simultaneously,

$$VAR_\theta(W) \triangleq \int (W - \mathbb{E}_\theta(W))^2 \pi_W(\theta) f_W(w) dw, \tag{3}$$

and the monotonic increasing of reward and the invariability of risk hold. Girsanov theory describes how the stochastic dynamics change when the original measure is changed to an equivalent measure. The Doléans exponential shifts the original distribution density rightward (leftward) when the risk preference of the world switches from the risk averse (risk taking) to risk neutral. The risk preference will not change an investor’s assessment of the amount of risk, i.e., the variance remains constant; but risk preference will affect the price of the risk, namely the average excess rate of return an investor will demand as compensation for taking a unit of market risk. The change of view bias under expectile risk-reward framework also describes a change in measures. Instead of shifting the original distribution density, the view bias reshapes the distribution by asymmetrically weighting the good state and bad state. Similarly, the invariability of variancile guarantees that the view bias will not change an investor’s assessment of the amount of risk. The monotonicity of expectile is compatible with how investors adjust their anticipation of the Brownian motion by accounting for the risk of information ambiguity. We term the adjustment as the ‘first category information premium’ in subsection 1.1.

However, based on the assumption that stock prices are characterized by geometric Brownian motion, the rate of return of a stock is a linear function of Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{4}$$

Moreover, the portfolio rate of return is the weighted rate of return of each security. Thus to conduct portfolio management, we need to investigate the coherency, i.e., the additivity, homogeneity, and risk-free condition of the reward measure. We also need to check the coherency, namely the sub-additivity, homogeneity, and risk-free condition of the risk measure. The result is that all these conditions are satisfied, except the additivity of the expectile: if there is no risk-source dimensional reduction, i.e., if an investor knows the joint distribution of investor stock returns within the portfolio, his expectile of the rate of return of the portfolio is equal to the sum of the expectile of each security. If an investor only knows the distribution density of the rate of return of the portfolio and does not know the joint distribution of the stock returns, his expectile of the rate of return of the portfolio is not equal to the sum of the expectile of each security.

Hence, fourthly, the expectile risk-reward measure framework completes the market.

**Proposition 1** (*Extended law of one price*) Under imperfect information, when portfolios are repackaged, the return remains the same if and only if investors are view neutral. If pessimism exists, the investors can combine the securities into a portfolio to engage in riskless arbitrage; if optimism exists, the investors can split the portfolio to earn non-zero excess returns.

**Proof.** See Appendix. ■

There is no conflict between this result and the traditional no arbitrage theory. Under this general risk-reward measurement framework, the amount of the information contained in the asset is another factor that influences asset pricing in addition to time, market risk, and information ambiguity risk. Individuals who earn a non-zero excess rate of return through repackaging must have additional information about the joint distribution associated with the securities. The premium does not compensate for bearing the market or information ambiguity risk. It is a reward for providing additional information. We term this the 'second category information premium'. In a general risk-reward measurement framework, we reassess the market completeness of a contingent claim market, Arrow-Debreu securities market and ordinary securities market. The market completeness expands itself from the security level to a portfolio level. The policy recommendation is that there should not be any portfolio repackaging constraints to ensure each elementary adopted consumption process obtainable. Obtaining the second category information premium by repackaging portfolios can improve the welfare of both parties and is a Pareto equilibrium allocation process.

Fifthly, the expectile reflects 'ex party effect'. We have checked the traditional coherency conditions for a reward measure, however, under imperfect information, the above conditions are not sufficient. As a reward measure, it should reflect the 'ex parte effect'. An illustrative hypothetical example is a father with two sons who worries about his elder son, a fisherman, because tomorrow might be a rainy day; he worries about his younger son, an umbrella seller, because tomorrow might be a sunny day.

Under an expectation-based framework, where only risk aversion affects outcomes, if the expected rate of return of asset X is negative, and there is a perfect negative correlation between the assets X and Y, then the expected rate of return of asset Y is positive. However, in the above example, the father's reward measures for X and Y are both negative. Therefore, it is not appropriate to describe the father's behaviour as risk averse, and that behaviour is more indicative of his pessimism. To be an appropriate reward measure, the expectile should be able to capture that meaningful difference. We formalized the 'ex parte effect' as another coherency condition under imperfect information in Lemma 1.

**Lemma 1** (*Ex parte effect*[3]) Assume $X$ and $Y$ are the rates of return of two assets. Assume $Y = bX$, then $\mathbb{E}_\theta(Y) = b\mathbb{E}_\theta(X)$, iff $b \geqslant 0$, and $\mathbb{E}_\theta(Y) = b\mathbb{E}_{1-\theta}(X)$, iff $b \leqslant 0$.

**Proof.** See Abdous and Remillard (1995). ∎

Lemma 1 considers the case of perfect correlation between two assets. Naturally, we are interested in the case where the correlation is imperfect. To address this issue, we need to develop an approach to construct two standard normal distributed random variables $X$ and $Y$ with a correlation being $\rho$. Under perfect information, $Y$ can always be equivalently constructed as $Y_1 = \rho X + \sqrt{1-\rho^2}Z$ or $Y_2 = \rho X - \sqrt{1-\rho^2}Z$, where $X$, $Z$ are two independent standard normal distributions $N(0,1)$. However, under imperfect information, $\mathbb{E}_\theta(Y_1) \neq \mathbb{E}_\theta(Y_2)$. According to the non-additivity of expectile with reduced information, $\mathbb{E}_\theta(Y_1)$ and $\mathbb{E}_\theta(Y_2)$, both differ from

[3] Due to the 'ex parte effect', it is not always correct to modify the existing expectation-based theory to generate an expectile-based theory by simply replacing the expectation with an expectile.

$\mathbb{E}_\theta(Y)$. We should choose between constructions of $Y_1$ and $Y_2$ to ensure that the expectile based on the joint distribution of $(X, Z)$ and the expectile based on the one-dimensional distribution of Y distort the anticipation in the same direction.

**Lemma 2** (*Construction of correlated standard normal distributions*[4]) Assume $X$, $Z$ are independent $N(0, 1)$. If $Y = \rho X + sign(\rho)\sqrt{1-\rho^2}Z$, then $Y \sim N(0, 1)$, $\rho = \rho_{XY}$, where the correlation between $X$ and $Y$. $\mathbb{E}_\theta(Y) = \mathbb{E}_\theta(\overline{\rho X + sign(\rho)\sqrt{1-\rho^2}Z})$ and $\mathbb{E}_\theta(\rho X + sign(\rho)\sqrt{1-\rho^2}Z)$ will take the same sign (i.e., $> 0$, $= 0$ or $< 0$) for a given view bias $\theta$.

**Proof.** Trivial and omitted.

Thus far, we have checked the coherency of the reward measure. To check the risk measure and understand how risk can be diversified through portfolio management under the new measurement framework, we still need to define the following concepts.

**Definition 2** The *linear conditional expectile*[5] of $Y$ w.r.t. $X$ is defined as

$$\mathbb{E}_\theta(Y|X) \triangleq \underset{g\in A}{argmin}\mathbb{E}\left[\pi_X(\theta)\pi_{Y|X}(\theta)(Y - g(X))^2\right] \quad (5)$$

where $X$ and $Y$ are random variables with normal distribution, set $A = \{g : \mathcal{R} \mapsto \mathcal{R}|g(X) = \beta_0 + \beta_1 X\}$, and $\pi_{Y|X}(\theta)$ is conditional view bias

[4] $\mathbb{E}_\theta(\overline{\rho X + sign(\rho)\sqrt{1-\rho^2}Z})$ means that the expectile is calculated based on the one-dimensional distribution of the random variable $Y$, where $Y$ is constructed as $\rho X + sign(\rho)\sqrt{1-\rho^2}Z$.

[5] Why is a linear relationship sufficient for asset pricing? When we use $n$-dimensional geometric Brownian motion to model the stock price dynamics, the rate of return of the stock is a linear function of the independent increments. Why do we study expectiles in a normal distribution or a function of the normal distribution? This ensures that the extended law of one price holds. Why is the conditional expectile $\mathbb{E}_\theta(Y|X) = b_0 + b_1 X$ considered as adequate for pricing purposes? When we solve the portfolio optimization problem, the expectile on a $n$-dimensional rate of return can be converted into the summation of the expectiles of the product of two standard normal random variables.

adjustment[6].

**Definition 3** The *unconditional expectile* of measurable function $h(X,Y)$ is well defined, if $\mathbb{E}[\pi_X(\theta)\pi_{Y|X}(\theta)h(X,Y)] = \mathbb{E}\left[\pi_Y(\theta)\pi_{X|Y}(\theta)h(X,Y)\right]$, then it is defined as $\mathbb{E}_\theta\left[h(X,Y)\right] \triangleq \mathbb{E}\left[\pi_X(\theta)\pi_{Y|X}(\theta)h(X,Y)\right]$.

Finally, within the expectile framework, the view bias adjustment $\pi_X(\theta)$ and the risk aversion adjustment, i.e., SDF, are product-separable, which is essential for simplifying the problem. When we apply the above coherent risk-reward measure to the portfolio optimization problem, we obtain the first order condition (FOC), which is an extended SDF-based asset pricing formula,

$$p(g(X)) = \mathbb{E}_\theta(m \cdot g(X)) = \int_\Omega \pi_X(\theta) \cdot m \cdot g(x) f_X(x) dx \tag{6}$$

where $X$ is the rate of return, a normally distributed random variable, and $g(X)$ is the payoff of the contingent claim, a measurable function of $X$. Assuming the probability setting holds, $m$ is the SDF, and $f_X(x)$ is the probability distribution function of $X$. To conclude Section 1, we compare the view bias approach to the traditional risk preference approach and summarize the results in Table 2.

## 2 Expectile CAPM and the Anomalies

In this section, we first revise the expected utility maximization axiom to produce an expectile utility maximization axiom. We then reformulate

[6] $\pi_{Y|X}(\theta) \triangleq \frac{(1-\theta)\mathbf{1}_{\{sign(\beta_1)(Y-g(X)<0)\}}+\theta\mathbf{1}_{\{sign(\beta_1)(Y-g(X)\geqslant 0)\}}}{\int\left[(1-\theta)\mathbf{1}_{\{sign(\beta_1)(Y-g(X)<0)\}}+\theta\mathbf{1}_{\{sign(\beta_1)(Y-g(X)\geqslant 0)\}}\right]f_{Y|X}(x)dy}$.

**Table 2**
**Comparison of risk preference approach and view bias approach**

| Risk preference approach | View bias approach |
|---|---|
| Panel A: Similarity | |
| Change the price of market risk, keep the quantity of market risk unchanged | |
| Self-trimming measure | |
| Sufficient statistics | |
| Define reward measure and risk measure simultaneously | |
| Risk diversification | |
| Panel B: Dissimilarity | |
| Expectation/Variance | Expectatile/Variancile |
| Perfect and imperfect information | Imperfect information |
| Shift | Reshape |
| Character | Attitude |
| Stable | Variable |
| No ex parte effect | Ex parte effect |
| Market risk premium | First $\&$ Second category information premium |
| No instant payoff from repackaging | Instant payoff from repackaging |
| Panel C: Relationship | |
| Product separable, $p(g(X)) = \mathbb{E}_\theta(mg(X)) = \int_\Omega \pi_X(\theta)mg(x)f_X(x)dx$ | |

Table 2 compares the traditional risk preference approach and the newly developed view bias approach. The similarities include: (1) Both risk preference and view bias adjustments change the probability measure from real to subjective ones. (2) As risk preference (view bias) deviates from risk neutrality (view neutrality), only the reward measure, expectation (expectile), will be adjusted, and the value of the risk measure, variance (variancile), remains unchanged. In other words, both approaches change the price of the market risk, and neither approach changes the quantity of the market risk. (3) Both the expectation and expectile measures are self-trimming, which allows us to obtain concise results. (4) As both the expectation and expectile measures are sufficient statistics, the empirical analysis will never waste sample information. (5) Both approaches simultaneously define the risk and reward measure. The logic of the framework is internally consistent. (6) Risk can be diversified through portfolio management under both measurement frameworks. The differences include: (1) Risk preference approach employs expectation and variance as reward and risk measure respectively; view bias approach employs expectile and variancile as reward and risk measure respectively. (2) Risk preference is relevant under both perfect and imperfect information; view bias is only relevant under imperfect information. (3) Risk preference shifts the probability distribution curve; view bias reshapes the probability distribution curve. (4) Risk preference describes an investor's character; view bias describes an investor's attitude. (5) Risk preference is stable and can be determined experimentally; view bias is variable and is affected by the status quo. (6) Expectation does not capture the 'ex parte effect'; expectile captures the 'ex parte effect'. (7) Risk preference will compensate the investor for holding market risk, and the premium is market risk premium; view bias will compensate the investor for holding information ambiguity risk, and the premiums include the first and second category information premiums. (8) The law of one price holds, and repackaging portfolios does not induce non-zero excess returns; the extended law of one price holds, and repackaging portfolios induces non-zero excess returns. (9) The market is complete at the security level; the market is complete at the portfolio level. The relationship between two approaches is that the expectile-based asset pricing formula implies that the view bias adjustment and SDF are product-separable.

the Merton problem under the expectile framework and extend the CAPM theory. Finally, we explain the equity premium puzzle and the momentum using the expectile-based framework.

## 2.1 Expectile CAPM with view bias adjustment

The technical difficulty we face in replacing the expected utility maximization axiom with an expectile utility maximization axiom is that, thus far, we have only defined the expectile of a measurable function with a maximum of two random variables. However, recall the basic consumption-based optimum, the utility of intermediate consumption and the utility of terminal wealth are functions of high dimensional random variables. Fortunately, the essential component of the problem can eventually be converted into an expectile for pairs of the product of two standard normal distributions.

**Theorem 1** (*Expectile CAPM*) Suppose Assumption A.1 through A. 6 in the Appendix hold. Then the following representation holds for $\mathbb{E}_\theta(\cdot)$:

$$\mathbb{E}_\theta(r_i - r_f) = \beta^\theta \mathbb{E}_\theta(r_M - r_f), i = 1, 2, \cdots, n \tag{7}$$

where $\beta^\theta \triangleq \dfrac{\Phi^2\widetilde{\sigma}_{iM} + \Psi\sigma_{iM}}{\Phi^2\widetilde{\sigma}_M^2 + \Psi\sigma_M^2}$, and $\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^n \varpi_j\sigma_i\sigma_j sign(\rho_{ij})\sqrt{1-\rho_{ij}^2}$, $\widetilde{\sigma}_M^2 \triangleq \sum_{j=1}^n \varpi_j\sigma_M\sigma_j sign(\rho_{jM})\sqrt{1-\rho_{jM}^2}$, $\sigma_{iM} \triangleq \sum_{j=1}^n \varpi_j\sigma_i\sigma_j\rho_{ij}$, $\sigma_M^2 \triangleq \sum_{j=1}^n \varpi_j\sigma_M\sigma_j\rho_{jM}$, $\Phi \triangleq \mathbb{E}_\theta(w)$, $\Psi \triangleq \mathbb{E}_\theta(w^2)$, $\varpi_i$ is the weight of security $i$ within the portfolio, and $w \sim N(0,1)$.

**Proof.** See Appendix. ■

We denote $\mathbb{E}(r_i)$ as $\mu_i$, $\mathbb{E}(r_M)$ as $\mu_M$, and convert Equation (7) into an equivalent expectation-based formula as below,

$$\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Phi - r_f = \beta^\theta(\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Phi - r_f), i = 1, 2, \cdots, n \tag{8}$$

The left-hand side of the equation is the expected excess rate of return with view bias adjustment $\frac{\sigma_i}{\sqrt{dt}}\Phi$ to compensate for taking information ambiguity risk. $\sigma_{iM}$ is the amount of the systematic risk of security $i$, $\sigma_M^2$ is a benchmark, the amount of systematic risk inherent in the market portfolio, $\widetilde{\sigma}_{iM}$ is the amount of latent risk of security $i$, $\widetilde{\sigma}_M^2$ is the amount of latent risk inherent in the market portfolio. $\beta^\theta$ is the relative amount of risk of security $i$. In contrast to the traditional $\beta$, it does not represent the relative amount of systematic risk but the weighted average of systematic risk and latent risk, with the weights being $\frac{\Psi}{(\Phi^2+\Psi)}$ and $\frac{\Phi^2}{((\Phi^2+\Psi)}$, respectively. Under perfect information, the investor will have a neutral view bias, and hence $\Phi$ becomes zero and $\Psi$ becomes one. Then the extended expectile-based CAPM will degenerate into a tradition CAPM. We are able to explain the equity premium puzzle using this new theoretical framework.

## 2.2 Theoretical explanation for equity premium puzzle

Mehra and Prescott (1985) first documented the equity premium puzzle; also see Cochrane (2001). Over the last 50 years, real U.S. stock returns have averaged 9% with a standard deviation of about 16% while the actual return on treasury bills has been about 1%. Thus, the historical annual market Sharpe ratio has been about 0.5. Aggregate consumption growth has been about 1%. We can only reconcile these facts with the theory if investors have a risk aversion coefficient of 50! If we consider the correlation between the consumption growth rate and market returns, the risk aversion coefficient has to be 250. However, the experimental data demonstrate that the risk aversion coefficient for the average person is between 1 and 4.

**Theorem 2** Assume that the correlation between the consumption growth rate and the market return is $\rho_{CM}$, then

$$\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Phi - r_f = \alpha\sigma_C\sigma_M(\Psi\rho_{CM} + \Phi^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2}), \tag{9}$$

where $\sigma_C$ is the volatility of the aggregate consumption growth rate, and $\alpha$ is the risk aversion coefficient.

**Proof.** See Appendix. ■

For the same setting in Cochrane (2001), under the view bias adjusted expectile framework, with a reasonable constant risk aversion coefficient of 3, the required daily and annual view bias, for eliminating the equity premium puzzle, are 0.47 and 0.22 respectively. We find that view bias' deviation from neutral (0.5) under imperfect information will significantly reduce the required risk aversion in the equity premium puzzle.

## 2.3 Theoretical explanation for momentum

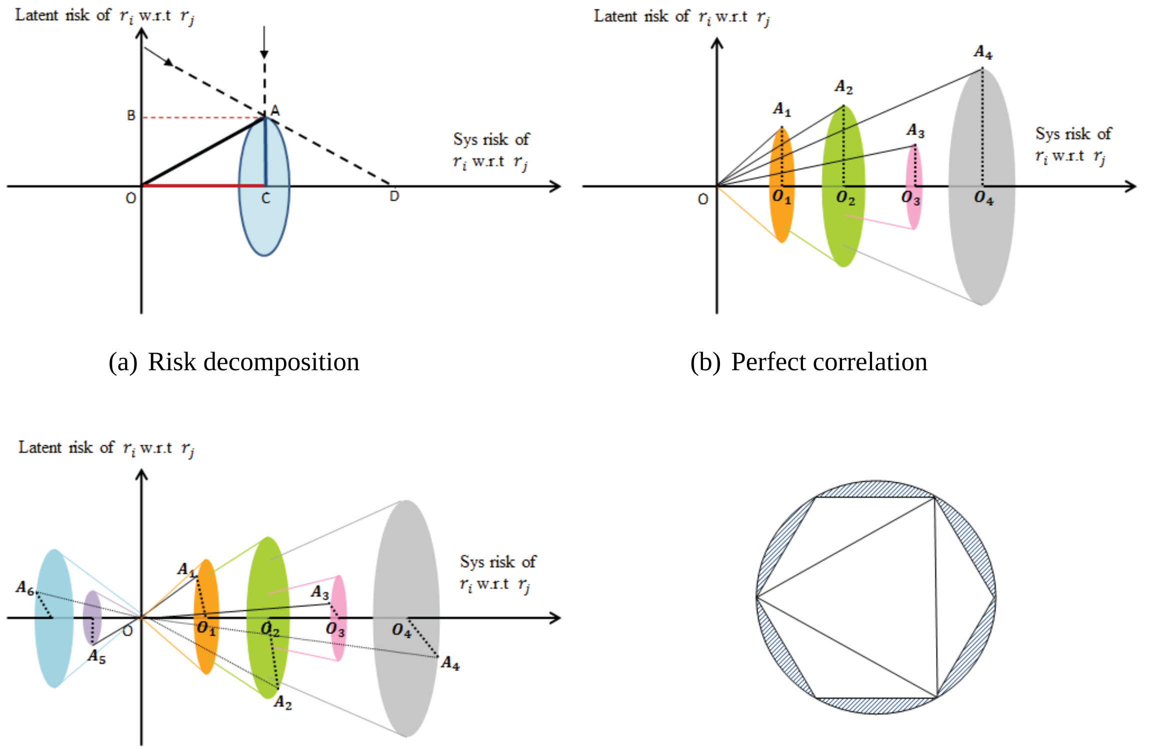
In this subsection, we will explain how the momentum effect is generated. Firstly, we discuss the risk decomposition between each pair of the securities ($i$ and $j$) and the risk decomposition of security $i$ w.r.t. the market portfolio, and elaborate the difference between idiosyncratic risk and latent risk. Secondly, we exploit how the view bias $\theta$ towards ambiguity affects $\beta^\theta$. We then hypothesize that view bias moves periodically and claim that if this is the case, the period of $\beta^\theta$ is half of or the same as the period of $\theta$. Finally, we explain the momentum based on the above analysis.

### 2.3.1 Systematic, latent and idiosyncratic risks

The difficulty of estimating $\beta^\theta$ of a given security lies in estimating its latent risk $\widetilde{\sigma}_{iM}$ and the corresponding benchmark $\widetilde{\sigma}^2_M$, which requires us to know the correlation between each pair of securities across the whole market portfolio.

Firstly, we discuss the risk decomposition of each pair of securities. The risk of security $i$ can be decomposed into a risk being perfectly correlated with the risk of security $j$ (see Figure 2(a): OC), i.e., $\sigma_{ij} \triangleq \sigma_i\sigma_j\rho_{ij}$ and a risk being independent to the risk of security $j$ (see AC), i.e., $\widetilde{\sigma}_{ij} \triangleq \sigma_i\sigma_j sign(\rho_{ij})\sqrt{1-\rho^2_{ij}}$. Surprisingly, under the view bias based expectile framework, the compensated risk is the weighted sum of both (see OD), i.e., $\Phi^2\sigma_i\sigma_j\rho_{ij} + \Psi\sigma_i\sigma_j sign(\rho_{ij})\sqrt{1-\rho^2_{ij}}$. This differs from the traditional risk aversion based expectation framework, under which compensation will only be provided for $\sigma_{ij}$.

Secondly, we consider the risk decomposition of security $i$ w.r.t. the market portfolio. As discussed in the last paragraph, the risk of security $i$ can be decomposed into $\sigma_{ij}$ and $\widetilde{\sigma}_{ij}$, where $j = 1, 2, \cdots, n$. In Figure 2(b), $\sigma_{ij}$ are the vectors from the origin to the centres of the circles, and $\widetilde{\sigma}_{ij}$ are




(a) Risk decomposition

(b) Perfect correlation

(c) Imperfect correlation

(d) Latent risk by industry

**Fig. 2.**

**Risk decomposition**

This graph intuitively explains why we need each pair of the security returns to estimate the latent risk of security $i$ w.r.t the market portfolio, and illustrate why an approximate estimation by industry works. Figure 2(a) shows that not only the projection of return $j$ on return $i$, but also the independent component perpendicular to return $i$ should be taken into account as the latent risk. The total risk of $r_i$ w.r.t $r_j$ can be decomposed into OC and AC, where OC is the systematic risk, AC is the latent risk. Figure 2(b) assumes that, the securities ($j = 1, 2, ..., n$, where $n$ represents the number of securities included in the market index apart from security $i$) in the market index are all perfectly correlated. Then the latent risk equals the idiosyncratic risk ($\widetilde{\sigma}_{iM} = \sigma_i\sigma_M\sqrt{1-\rho_{iM}^2}$). Systematic risk $\sigma_{iM}$ is the sum of distances from origin $O$ to $O_j$. Latent risk $\widetilde{\sigma}_{iM}$ is the sum of radius $O_jA_j$ of the circles, where $j = 1, 2, \cdots, n$. Figure 2(c) describes the real world situation that the securities ($j = 1, 2, ..., n$) in the market index are not perfectly correlated. Then the latent risk does not equal the idiosyncratic risk. Again systematic risk $\sigma_{iM}$ is the sum of distances from origin $O$ to $O_j$. Latent risk $\widetilde{\sigma}_{iM}$ is the sum of radius $O_jA_j$. Since the rate of return of each investor security might be negatively correlated within the market index, if we consider the market portfolio as a one-dimensional security $j^*$, and calculate the latent risk of security $i$ w.r.t $j^*$, by doing so, some latent risks will be cancelled out with each other, hence it will wrongly estimate the latent risk of security $i$ w.r.t the market portfolio. See Figure 2(d), the securities in the same industry tend to have strong positive correlations; hence it makes sense to approximate the latent risk of security $i$ w.r.t the market portfolio by dividing the market portfolio into several sub-portfolios by industry. The accuracy improves as the sections become finer. The marginal improvement of latent risk estimation accuracy decrease rapidly.

the radiuses of the circles. Suppose that the rate of return for each pair of securities are all perfectly correlated, then all vectors $(OA_1, OA_2, \cdots, OA_n)$ point at $12$ o'clock if $\rho_{ij} \geqslant 0$ (as shown in Figure 2(b)) or $6$ o'clock if $\rho_{ij} < 0$, otherwise the vectors points in disordered directions as in Figure 2(c).

$$\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j sign(\rho_{ij}) \sqrt{1 - \rho_{ij}^2} = \sigma_i \sigma_M sign(\rho_{iM}) \sqrt{1 - \rho_{iM}^2} \quad (10)$$

The left-hand side of Equation (10) is defined as the latent risk of security $i$ w.r.t. the market portfolio, and right-hand side is defined as idiosyncratic risk of security $i$ w.r.t. the market portfolio. The latent risk equals the idiosyncratic risk if and only if the investor securities in the market portfolio are all perfectly correlated, otherwise the negatively correlated securities tend to offset each other, and the latent risk will be distinguished from idiosyncratic risk. Therefore, we cannot simply estimate $\widetilde{\sigma}_{iM}$ and $\widetilde{\sigma}_M^2$ using the aggregate rate of return of the market portfolio as we did when estimating $\sigma_{iM}$ and $\sigma_M^2$. We need the joint distribution of each pair of securities in the market portfolio. Therefore, an accurate estimate of $\beta^\theta$ is practically infeasible.

Thirdly, we develop an approximate estimation of the latent risk $\widetilde{\sigma}_{iM}$ and the benchmark $\widetilde{\sigma}_M^2$. Specifically, we divide the market portfolio by industry, and then estimate the latent risk by $\widetilde{\sigma}_{iM} = \sum_{l=1}^{N} \varpi_l \sigma_i \sigma_l sign(\rho_{il}) \sqrt{1 - \rho_{il}^2}$, where $N$ represents the number of industry, $\sigma_l$ is the volatility of the rate of return of sub-portfolio of industry $l$. The securities in the same industry tend to have strong positive correlations; they are located in the same section of the graph. We determine the latent risk of security $i$ w.r.t. each industry portfolio and then calculate the weighted average. In this manner, we approximate the latent risk of security $i$ w.r.t. the market portfolio. The accuracy improves as the categories become narrower (see Figure 2(d)). If the number of industries increases from $3$ to $6$, the marginal improvement in the accuracy of the estimation of latent risk is significant. However, if the number of industries increases from $6$ to $12$, this improvement is no greater than the shaded area. We summarize the fact in Lemma 3.

**Lemma 3** Latent risk can be approximated using industry rates of return by

$$\lim_{N \to n} \sum_{m=1}^{N} \varpi_l \sigma_i \sigma_l sign(\rho_{il}) \sqrt{1 - \rho_{il}^2} = \widetilde{\sigma}_{iM} \tag{11}$$

**Proof.** Obvious and omitted.

### 2.3.2 Periodicity analysis of view bias, beta and return

We exploit how view bias towards ambiguity affects $\beta^\theta$. If the view bias is neutral, i.e., $\theta = 50\%$, then $\Phi^2 = 0$, $\Psi = 1$, and $\beta^\theta = \frac{\sigma_{iM}}{\sigma_M^2}$, which is the projection of total risk of security $i$ on the market portfolio risk, see Figure 2(a). Intuitively $\beta^\theta$ is akin to the shadow OC that OA would cast on the horizontal axis perpendicularly, under the sun at noon. If the view bias is pessimism or optimism, i.e., $\theta \neq 50\%$, then $\Phi^2 > 0$, $0 < \Psi < 1$. $\beta^\theta$ is the relative weighted average of systematic risk and latent risk of security $i$ w.r.t. the benchmark, market portfolio. Again, see Figure 2(a), if an investor holds a biased view, $\beta^\theta$ is akin to the shadow OD that OA would cast on the vertical axis under early morning or dawn sun. Under imperfect information, it takes time for the investor to process the current and past information, adjust his view bias, and then update his view bias to accommodate the new information. Therefore, the investor will overreact to ambiguity. For example, he begins with overly pessimistic view bias and adopts a very conservative trading strategy, but he subsequently realizes that the real situation is much better than what he had thought, and he begins to take an optimistic view of new information. This overreaction would be eliminated if he were reacting to the same information because he will learn from the experience. However, as the information he receives is progressing rather than stationary, such overreactions can never be eliminated. Therefore, it is reasonable to hypothesize that view bias moves periodically, which allows us to explain the momentum effect. See Corollary 1.

**Corollary 1** $\forall\theta, \beta^{\theta} = \beta^{1-\theta}$. If the dynamics of $\theta$ exhibits symmetric periodic movements around neutral (as with a sine wave or a square wave), the period of $\beta^{\theta}$ is half the length of the period of $\theta$; if the dynamics of $\theta$ exhibits asymmetric periodic movements around neutral, the period of $\beta^{\theta}$ is equal to the period of $\theta$. The rate of return of the portfolio will have the same periodicity as view bias.

**Proof.** Obvious from the definition of $\beta^{\theta}$, $\Phi$, and $\Psi$, and Equation (8).

### 2.3.3 Two alternative explanations for momentum

Momentum initially documented by Jegadeesh and Titman (1993) shows that past winners continue to outperform past losers; while Wang (2005) claim the beta estimated for the winner portfolio is even lower than that of the loser. Fama and French (1996) find that, among several CAPM anomalies, momentum is the only one that is unexplained by the three-factor model. We claim that the actual factor causing the momentum effect is the fluctuation of investors' view bias, which adjusts the price of the risk (the market excess rate of return) by $\frac{\sigma_M}{\sqrt{dt}}\Phi$ and causes the amount of relative risk $\beta^\theta$ to fluctuate accordingly, either with a period of half of or the same length as the period of the view bias. $\beta^\theta$'s periodicity depends on whether the dynamics of $\theta$, when it represents pessimism, is a phase lagged mirror reflection of the dynamics of $\theta$ when the parameter indicates optimism. There are two alternative explanations.

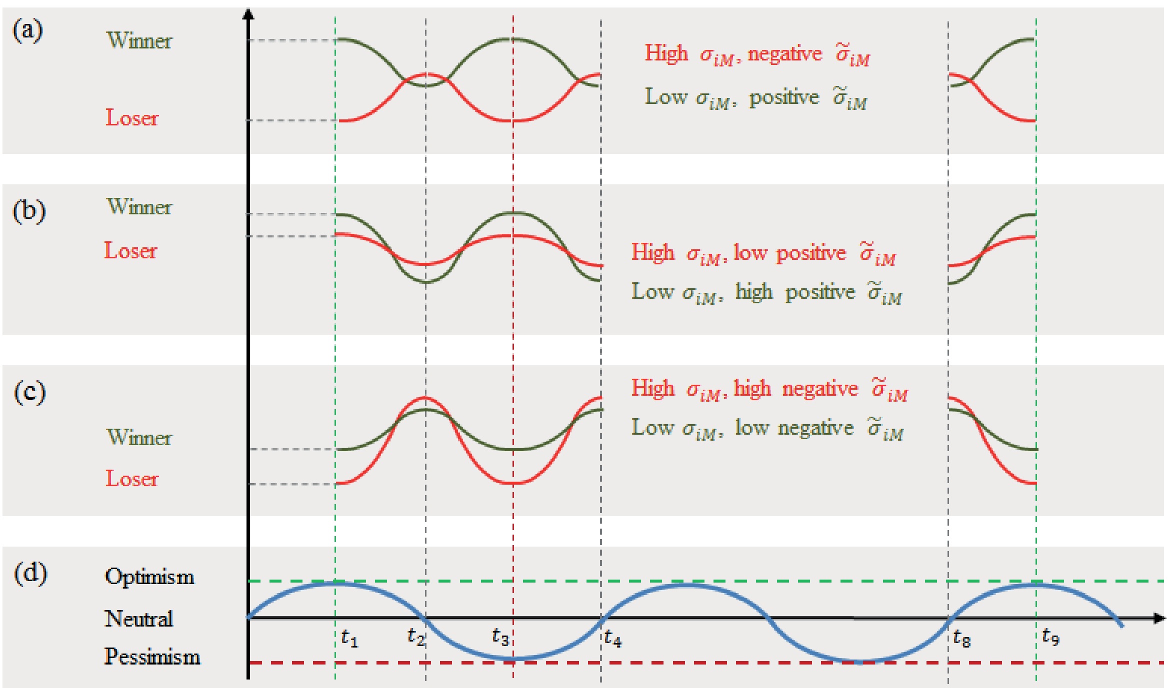



**Fig. 3.**
Explain momentum - A view bias adjusted beta approach. This graph illustrates how the view bias reciprocation explains the momentum effect. Figure 3.(d) describes a symmetric view bias fluctuation w.r.t view neutral. Figure 3.(a) to (c) depict the behaviour of three types of winners and losers under the corresponding view bias at different times. If the winner portfolio is more composed of the securities with low $\sigma_{iM}$ and high $\tilde{\sigma}_{iM}$, and the loser portfolio is more composed of the securities with high $\sigma_{iM}$, low $\tilde{\sigma}_{iM}$, then momentum effect appears, namely past winners continue to outperform past losers while the beta $\frac{\sigma_{iM}}{\sigma_M^2}$ estimate for the winner portfolio is even lower, under the condition that the view bias adjusted price of risk is positive.

**Explanation 1** (View bias adjusted beta approach) After being adjusted by view bias, a relatively large amount of priced risk and a positive market price of risk cause the winner portfolio to outperform the loser portfolio with a lower beta. Assume that, after the view bias adjustment, the price of risk $\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Phi - r^f$ is positive. See Figure 3.(d), the view bias at time $t_1$ is extreme optimism, according to the expectile CAPM, $\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Phi - r^f = \beta^\theta(\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Phi - r^f)$, the optimism will adjust the expected rate of return for different stocks in the same direction, by $\frac{\sigma_i}{\sqrt{dt}}\Phi$. If we scale the portfolios to ensure that they exhibit the same level of volatility, e.g., we make $\sigma_i = \sigma_j = \sigma_M$, optimistic view bias will shift the expected rate of return by the same amount, and it will not change the rank of the security. However, as view bias changes, it will also change the amount of priced risk from $\beta$ to $\beta^\theta$, and that will change the ranking of the stock accordingly. At time $t_1$, see Figure 3.(a) to (c), the risk amount that determines the excess rate of return is $\frac{\Phi^2\widetilde{\sigma}_{iM}+\Psi\sigma_{iM}}{\Phi^2\widetilde{\sigma}_M^2+\Psi\sigma_M^2}$. If the winner portfolio has a relatively high share of securities with low systematic risk $\sigma_{iM}$ and high latent risk $\widetilde{\sigma}_{iM}$, and the loser portfolio contains a relatively high share of securities with high systematic risk $\sigma_{iM}$ and low latent risk $\widetilde{\sigma}_{iM}$, then this will explain the momentum effect where past winners continue to outperform past losers while the beta $\frac{\sigma_{iM}}{\sigma_M^2}$ estimate for the winner portfolio is even lower. Thus, accepting this explanation of the momentum effects requires that we should not find any empirical evidence against the following five statements. Firstly, the view bias adjusted market excess rate of return is positive. Secondly, the $\sigma_{iM}$ of the winner portfolio is low, and the $\sigma_{iM}$ of the loser portfolio is high. Thirdly, the $\widetilde{\sigma}_{iM}$ of the winner portfolio is greater than the $\widetilde{\sigma}_{iM}$ of the loser portfolio. Fourthly, the periods of both winner portfolio and loser portfolio are the same, either half of or the same length as the period of view bias. Fifthly, momentum occurs whenever view bias deviates from neutral to the greatest extent; momentum is reduced or eliminated when view bias is close to neutral.

**Explanation 2** (View bias adjusted market price of risk approach) After being adjusted by view bias, a relatively low amount of risk and a negative market price of risk will cause the winner portfolio to outperform the loser portfolio with a lower beta. Assume that, after the view bias adjustment, the price of the risk, $\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Phi - r^f$, is negative. Then the greater the amount of the risk and the lower the compensation for taking the risk are, the lower the ranking of the portfolio is. For example[7], if the market excess rate of return is 5% on average, the adjustment $\frac{\sigma_M}{\sqrt{dt}}\Phi$ is -8% under pessimistic view bias. Thus, the new price of risk is -3%. Assume that the view bias adjusted beta for the winner portfolio is 0.8 and is 2 for loser portfolio. Then the winner's compensation for taking risk is -2.4%, and compensation for the loser is -6%. Assuming that the view bias adjustments $\frac{\sigma_i}{\sqrt{dt}}\Phi$ are the same for both, 8%, then the excess rate of return for the winner portfolio is 5.6%, and that of the loser portfolio is 2%. That explains the momentum effect. Thus, if we are to accept this alternative explanation of the momentum effects, we should not find any empirical evidence against the following two statements. Firstly, the view bias adjusted market excess rate of return is negative. Secondly, the view bias adjusted beta for the winner portfolio is even lower.

[7] New price of risk $= 5\% - 8\% = -3\%$; Winner's compensation for taking risk $= -3\% \times 0.8 = -2.4\%$; Loser's compensation for taking risk $= -3\% \times 2 = -6\%$; Excess rate of return for the winner portfolio $= -2.4\% + 8\% = 5.6\%$; Excess rate of return for the loser portfolio $= -6\% + 8\% = 2\%$.

Banz (1981) initially documented the size effect in U.S. stock returns, where whether size premium is a compensation for systematic risk was discussd. Recent studies consider the possibility that liquidity is a priced state variable, and the returns on small stocks are sensitive to this state variable (van Dijk, 2011). However, Amihud (2002) finds that liquidity risk can only account for part of the size effect. Another anomaly is the value effect, first documented by Sanjoy (1983). The returns are predicted by the ratios of market value to accounting measures such as earnings or the book value of equity. This paper explains the size and value effects as follows: under imperfect information, the size and value of a firm are signals, which investors use to extrapolate future performance from past performance and adjust their view bias towards the ambiguity, and consequently view bias leads to price distortions.

## 3 Empirical Implications

In this section, we provide empirical evidence on the advantages of the expectile-based model in explaining asset pricing anomalies. Firstly, we determine data availability. Secondly, by assuming a reasonable risk aversion coefficient, we obtain an aggregate view bias time series implied by solving equity premium puzzle. Thirdly, we construct momentum portfolios. We use time series rankings of 10 momentum portfolios' returns as proxies for $\beta_t^\theta$. We conduct spectral density of the ranking and the excess rate of return for all momentum portfolios. The periods of the ranks and the excess rate of return are both compatible with the period of the view bias coefficient.

We didn't find any evidence against the fact that view bias towards ambiguity has been priced into those momentum portfolios. Therefore, fourthly, we test the expectile-based unconditional CAPM on 10 momentum portfolios using the average aggregate equity premium puzzle implied view bias coefficient over the observation period. We reach dramatically different conclusions when testing the moment conditions. GMM rejects the null hypothesis suggested by traditional CAPM, $\mathbb{E}(r_i^e - \beta_0 - \beta_1 r_M^e) = 0$. The bias adjusted GMM accepts the null hypothesis suggested by the expectile-based CAPM $\mathbb{E}_\theta(r_i^e - \beta_0 - \beta_1 r_M^e) = 0$. Finally, we analyse the systematic risk and latent risk of the winner portfolio and the loser portfolio, as well as the view bias adjusted excess rate of return, we find that Explanation 2 satisfactorily explains the momentum.

## 3.1 Data availability

To apply the above theory to the data, we need stock return, market index, risk-free rate, and consumption per capita. Daily stock return data including the common shares of all NYSE-, AMEX-, and NASDAQ-listed firms are obtained from CRSP. We select stocks with stock share codes of $10$ or $11$ to exclude closed-end funds, real estate investment trusts, American depository receipts and foreign stocks. We obtain the monthly average of daily common share returns and market returns (including dividends) and volatility. The annualized volatility is obtained by multiplying $\sqrt{252}$. The daily risk-free rate is proxied by the return on $30$ T-bill data from CRSP divided by $30.4$. U.S. quarterly aggregate nondurable goods and service consumption per capita are obtained from John Campbell's website. We interpolate it into a monthly time series. We use Gauss-Laguerre quadrature to numerically approximate $\Phi$ and $\Psi$ w.r.t any given $\theta$.

## 3.2 Equity premium puzzle Implications

We solve Equation (9) to obtain an equity premium puzzle implied aggregate U.S. view bias coefficient time series from May.$1926$ to Sep.$1999$, by assuming a reasonable constant risk aversion coefficient of $3$. The average view bias is $0.473$, a slight deviation from view neutral while the maximum and minimum are $0.871$ and $0.219$ respectively. The pessimistic part of view bias is not a phase lagged mirror reflection of the dynamics of optimistic part of view bias.

We compare the equity premium puzzle implied view bias coefficient under a constant risk averse coefficient of $3$ and the equity premium puzzle implied risk averse coefficient under view neutrality. The conclusion is that if we neglect the view bias factor, which is slightly deviates from neutral, we need a substantial risk preference to reconcile $\mathbb{E}(r_M) - r_f = \alpha\sigma_C\sigma_M\rho_{CM}$ with the actual data, which is the stylized fact termed equity premium puzzle. We conduct the view bias spectral analysis for the time span from May. $1926$ to Sep. $1999$. The first significant period is approximately 50 years, while the second significant period is approximately 90 months.

We construct subsamples for implied view bias and implied risk preference using the sign of the correlations $\rho_{CM}$. For the implied risk averse coefficient, we truncate it to $[-300, 300]$ to exclude the outliers, then the average is 30. We run regressions for each group for the data pairs that have risk aversion levels within the range of $[-300, 300]$. The result indicates that implied view bias and implied risk preference are strongly correlated in both groups. We find that the implied risk preference under view neutral and the implied view bias given a constant risk preference of $3$ are positively (negatively) correlated[8] when the aggregate consumption growth rate and market portfolio returns are positively (negatively) correlated. In other words, a high risk aversion coefficient with a neutral view bias coefficient, or a pessimistic view bias coefficient with a normal risk aversion coefficient, are substitutable explanations for the equity premium puzzle, only when the market return and consumption growth tend to move in the same direction. Otherwise, a high risk aversion coefficient with a neutral view bias coefficient, or an optimistic view bias with a normal risk aversion coefficient, are substitutable explanations.

## 3.3 Linkage between two anomalies

As there always will be a modelling risk when we employ the actual data to do empirical test, we should get rid of the modelling risk to the great practical extent before we employ expectile CAPM to solve momentum anomaly. Thanks to Corollary 1 and Theorem 1 and the fact that the pessimistic part of view bias is not a phase lagged mirror reflection of the dynamics of optimistic part of view bias, if expectile CAPM is correct, we

[8] The implied risk preference is positively correlated with the implied view bias when the aggregate consumption is positively correlated with market portfolio return. The intercept (-361) and the slope (712) are both statistically significant under 99% confidence, the adjusted $R^2$ is 0.58. The implied risk preference is negatively correlated with the implied view bias when the aggregate consumption is negatively correlated with market portfolio return. The intercept (341) and the slope (-689) are both statistically significant under $99\%$ confidence, the adjusted $R^2$ is 0.60.

should not find any evidence against that the period of the excess rate of return of the portfolio is equal to the period of view bias. This is a unique perspective, from which we hypothesize and demonstrate that view bias impacts the market.

We obtain the ranks of the selected stocks in deciles. The sample range[9] is from Jun. 1963 to Dec. 1970. Following many studies, the ranking periods have a length of six months. For any given month, the rank of a certain stock is determined based on the returns from the past 6 months. We construct momentum portfolios in July 1967, the middle of the sample range. We group the stocks by rank to obtain 10 portfolios and derive the value-weighted monthly return of each portfolio. We scale the portfolio returns to ensure that each has the same volatility to control for the $\frac{\sigma_i}{\sqrt{dt}}\Phi$ adjustment in the expectile CAPM. We then conduct the spectral analysis to determine the period from Jun. 1963 to Dec. 1970. The first and second significant periods for view bias are 62 and 19 months.

The empirical results in Table 3 confirm that both the $\beta^\theta$ and the excess rate of return have periods consistent with periods of view bias $\theta$. As we are interested in the period rather than the magnitude of the $\beta_t^\theta$ dynamic, it is sensible to use the time series of the ranking of returns as a proxy for its $\beta_t^\theta$. We compare the periods of view bias and the ranking period of each momentum portfolio $\beta_t^\theta$. As shown in Table 3, many of the periods are consistent with the predictions of the expectile CAPM. Therefore, we

[9]We select this sample range for the reason that we need the volatility of each momentum portfolio and market index to be stable in order to carry out the following analysis, hence the 50 years view bias cycle is too long. The sample range (Jun. 1963 to Dec. 1970) covers 93 months, roughly the second statistical significant period, and it is the golden age for the united states' economy after the second world war.

consider that view bias towards ambiguity is priced into securities. None of the ranking periods has a length of approximately 62, and the p-value for white noise test is large. This is because we use rank as a proxy for beta $\beta_t^\theta$, and the range of the rank is restricted to [1, 10]; hence it is unable to reflect the long term trend or cycle.

**Table 3**
**Value weighted momentum portfolio spectral analysis**

| | Rank spectral analysis | | Return spectral analysis | | |
|---|---|---|---|---|---|
| Portfolio | Period | p-value | 1st period | 2st period | p-value |
| P1 | 14 | 0.3187 | 57 | 21 | <0.0001 |
| P2 | 19 | 0.6497 | 57 | 18 | <0.0001 |
| P3 | 19 | 0.2384 | 56 | 18 | <0.0001 |
| P4 | 24 | 0.9419 | 57 | 19 | <0.0001 |
| P5 | 24 | 0.7953 | 60 | 23 | <0.0001 |
| P6 | 20 | 0.5192 | 58 | 22 | <0.0001 |
| P7 | 20 | 0.5519 | 63 | 24 | <0.0001 |
| P8 | 23 | 0.0019 | 68 | 23 | <0.0001 |
| P9 | 19 | 0.1282 | 70 | 24 | <0.0001 |
| P10 | 26 | 0.0618 | — | 25 | <0.0001 |
| Average | 20.6 | | 60.7 | 21.7 | |

Table 3 compares the period of the ranking time series and the periods of the return time series of 10 momentum portfolios. Both of them are compatible with view bias periods. The average first period of the excess rate of return of the momentum portfolios is 60.7 months, which is close to 62 months, the first period of view bias. The average second period of the excess rate of return and the average second period of the ranking of the momentum portfolios are 21.7 months and 20.6 months, respectively, which are both close to 19 months, the second period of the view bias. The sample range is from Jun. 1963 to Dec. 1970.

## 3.4 Momentum implications

We provide an example of empirical evidence on the advantages of the expectile-based model for testing CAPM. We first demonstrate the empirical methodology, and then claim the results.

### 3.4.1 Empirical methodology

In this subsection, we employ view bias restricted conditional GMM to test view bias based expectile CAPM theory empirically. We first test the unconditional expectile CAPM,

$$\mathbb{E}_\theta(r_i - r_f) = \beta^\theta \mathbb{E}_\theta(r_M - r_f),$$

We rewrite it into

$$\mathbb{E}_\theta\left(r_i^e - \beta^\theta r_M^e\right) = 0,$$

where $r_i^e$ is the excess rate of return of security $i$, and $r_M^e$ is the excess rate of return of the market portfolio. Hence testing CAPM is testing a moment condition. The difficulty is that there is only one equation, but with two parameters (zero intercept and statistically significant non-zero slope) to be tested. We know that only when the number of moment conditions is greater than the dimension of the parameter vector, the model is said to be overidentified. Over-identification allows us to check whether the model's moment conditions match the data well or not. Fortunately, the unconditional expectile CAPM is just a special case of conditional expectile CAPM. Therefore, we solve the problem by estimating an unconditional expectile CAPM model using conditional GMM, which is restricted by the view bias condition. The econometric model is as follows,

$$\begin{cases} \mathbb{E}\left(\pi_{r_{i,t}^e} r_{i,t}^e - \beta_0 - \beta_1 \pi_{r_{M,t}^e} r_{M,t}^e\right) = 0 \\ \mathbb{E}\left(\pi_{r_{i,t-1}^e} \pi_{r_{i,t}^e | r_{i,t-1}^e} r_{i,t}^e r_{i,t-1}^e - \beta_0 \pi_{r_{i,t-1}^e} r_{i,t-1}^e - \beta_1 \pi_{r_{i,t-1}}^e \pi_{r_{M,t}^e | r_{i,t-1}^e} r_{M,t}^e r_{i,t-1}^e\right) = 0 \\ \mathbb{E}\left(\pi_{r_{M,t-1}^e} \pi_{r_{i,t}^e | r_{M,t-1}^e} r_{i,t}^e r_{M,t-1}^e - \beta_0 \pi_{r_{M,t-1}^e} r_{M,t-1}^e - \beta_1 \pi_{r_{M,t-1}}^e \pi_{r_{M,t}^e | r_{M,t-1}^e} r_{M,t}^e r_{M,t-1}^e\right) = 0 \end{cases} \quad (12)$$

*St:* $\pi_{Y|X}$ *are implied by* $\mathbb{E}_\theta(Y|X) = \alpha_0 + \alpha_1 X$, *where* $Y \in \{r_{i,t}^e r_{M,t}^e\}$, $X \in \{r_{i,t}^e r_{M,t}^e\}$

where $r_{i,t}^e$ is the excess rate of return of security $i$ at time $t$, $r_{M,t}^e$ is the excess rate of return of the market portfolio at time $t$. The first equation of GMM is an unconditional moment condition, the second equation of GMM is a moment condition conditioning on the information of the previous period's excess rate of return of security $i$, and the last equation is a moment condition conditioning on the information of the previous period's excess rate of return of the market portfolio. We will test the null hypothesis $\mathbb{H}_0 : \beta_0 = 0, \beta_1 = 0$, and the moment condition by the case of conditional homoskedasticity and heteroskedasticity according to the following steps.

- Step 1: obtain the equity premium puzzle implied view bias $\theta$, and take this $\theta$ as given, search for the sample expectile to get $\pi_{r^e_{i,t}}$ and $\pi_{r^e_{M,t}}$; using the same sample set with one period lag to get $\pi_{r^e_{i,t-1}}$, and $\pi_{r^e_{M,t-1}}$.
- Step 2: run expectile regression[10] of $r^e_{i,t}$ on $r^e_{i,t-1}$ to obtain the estimated parameters, which we use to estimate $\pi_{r^e_{i,t}|r^e_{i,t-1}}$. Repeat the same procedure to estimate $\pi_{r^e_{i,t}|r^e_{M,t-1}}$, $\pi_{r^e_{M,t}|r^e_{i,t-1}}$, $\pi_{r^e_{M,t}|r^e_{M,t-1}}$.
- Step 3: test the statistical significance of the estimated slope in $\pi_{r^e_{i,t}|r^e_{i,t-1}}$.
- Step 4: if the slope is insignificant, then replace $\pi_{r^e_{i,t}|r^e_{i,t-1}}$ using $\pi_{r^e_{i,t}}$.
- Step 5: repeat step 3 and step 4 for $\pi_{r^e_{i,t}|r^e_{M,t-1}}$, $\pi_{r^e_{M,t}|r^e_{i,t-1}}$, $\pi_{r^e_{M,t}|r^e_{M,t-1}}$.
- Step 6: run GMM to estimate $\beta_0$ and $\beta_1$.
- Step 7: test null hypothesis $\mathbb{H}_0$: $\beta_0 = 0$, $\beta_1 = 0$.
- Step 8: test the moment condition $\mathbb{H}_0$: $\mathbb{E}_\theta(r^e_i - \beta_0 - \beta_1 r^e_M) = 0$.

[10] See Salvati et al. (2012) and Sobotka et al. (2013).

### 3.4.2 Empirical results

We assume the view bias is exogenuously given as the average of view bias implied from equity premium puzzle, then we estimate the view bias based beta by running GMM on expectile CAPM and test if the corresponding moment conditions are satisfied. The sample range is again from Jun. 1963 to Dec. 1970, at a total of 93 months. We run GMM and classical OLS regressions to estimate the CAPM model to compare to a view bias adjusted GMM estimation for each momentum portfolio. We first group the sample records of the return time series of each momentum portfolio and the market portfolio by the subdivision of view bias (optimism and pessimism). For each portfolio, we then perform the expectile regression to obtain the view bias adjustments for both groups. The slopes estimated by the expectile regression are not statistically significant; hence we replace the conditional $\pi_{r^e_{j,t}|r^e_{i,t-1}}$ with the unconditional $\pi_{r^e_{j,t}}$. We finally combine two data groups to obtain a complete return time series for each momentum portfolio and the market portfolio, and then we run the GMM (Equation (12)). See Table 4, the view bias adjusted betas, estimated by $\mathrm{GMM}_\theta$, GMM, and OLS exhibit the same pattern for the 10 momentum portfolios. The beta for the best performing winner portfolio estimated by $\mathrm{GMM}_\theta$ is 0.7687, which is the lowest among the 10 momentum portfolios. Therefore, the real world data do not support Explanation 1.

We claim that the actual data support Explanation 2. The amount of risk of the winner portfolio is lower than that of the loser portfolio, and the adjusted market price of risk is negative. Firstly, we first study the risk composition of each portfolio, see Table 5. We decompose the total risk of each portfolio into systematic risk $\sigma_{iM}$ and idiosyncratic risk $\sigma_i\sigma_M\sqrt{1-\rho_{iM}^2}$. According to Theorem 1 and Equation (10), the real priced risk is the weighted average of systematic risk and latent risk $\widetilde{\sigma}_{iM}$, where the weights are $\Psi$ and $\Phi^2$ respectively. Secondly, we arbitrarily set a view bias. For any view bias, there exists a pair of $\Psi$ and $\Phi^2$, which we are able to use to obtain the view bias adjusted beta $\beta^\theta$. Thirdly, we calculate the sample average of the market excess rate of return, which is positive. However, the market excess rate of return adjusted by the first category of the information premium $\frac{\sigma_M}{\sqrt{dt}}\Phi$ is negative (-2.991 annually), which we multiply by $\beta^\theta$ to find that the adjusted expected portfolio excess returns $\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Phi - r_f$ are all negative. By subtracting the item $\frac{\sigma_i}{\sqrt{dt}}\Phi$, we obtain the expected portfolio excess return $\mu_i - r_f$, which are all positive. Fourthly, we determine the view bias for each portfolio that makes the expected portfolio excess returns implied by the expectile CAPM equal to the sample average excess return; we then obtain the momentum effect implied view bias for each momentum portfolio. The corresponding view bias adjusted betas for each portfolio is reconciled with the $\beta^\theta$ estimated by GMM $_\theta$ in Table 4. Fifthly, we find that the differences between $\beta$ and $\beta^\theta$ are small. The $\beta^\theta$ value for the best performing winner portfolio is the lowest among all 10 portfolios, which is consistent with the GMM$_\theta$ results. Finally, we demonstrate the advantages of the expectile CAPM and of using a view bias distorted GMM to test the expectile CAPM. As shown in Table 4, the J-statistics for the GMM$_\theta$ are all greater than 5%, which implies that the moment conditions in Equation

(12) are all satisfied, and the model is correctly specified. Of the 10 J-statistics for the GMM, 6 are below 5%, which implies that their moment conditions in Equation (12) with all distortion multipliers $\pi \equiv 1$ are rejected, and traditional CAPM models are misspecified. The dramatically different results provide an empirical evidence on the advantages of the expectile-based theoretical and empirical models. All of the above analysis supports Explanation 2.

**Table 4**

**Summary of view bias adjusted GMM under expectile CAPM, GMM, and OLS under alternative traditional CAPM**

| | | P1 | P2 | P3 | P4 | P5 | P6 | P7 | P8 | P9 | P10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Alpha (e-3) | −0.110 | 0.553 | 2.243** | 2.302** | 2.878** | 2.281** | 2.731** | 5.753*** | 8.424*** | 7.063*** |
| | t-statistic | [-0.12] | [0.56] | [2.85] | [2.74] | [2.72] | [2.70] | [3.62] | [4.08] | [8.22] | [4.69] |
| | Beta (e-1) | 8.975*** | 9.529*** | 9.056*** | 9.095*** | 9.181*** | 9.325*** | 9.383*** | 9.058*** | 9.143*** | 7.687*** |
| $GMM_\theta$ | t-statistic | [42.67] | [38.15] | [47.58] | [30.85] | [29.62] | [36.25] | [43.52] | [24.84] | [41.65] | [30.12] |
| | J-statistic | [27.47] | [29.10] | [25.65] | [24.80] | [30.57] | [33.81] | [31.81] | [32.13] | [31.74] | [29.14] |
| | P-value (%) | 87.29 | 81.97 | 92.01 | 93.74 | 76.35 | 61.96 | 71.09 | 69.65 | 71.37 | 81.84 |
| | Alpha (e-3) | 0.281 | -0.460 | -1.071 | 2.449 | 3.437 ** | 2.221 | 2.376* | 5.790*** | 7.495*** | 5.588** |
| | t-statistic | [0.21] | [-0.38] | [0.86] | [1.90] | [2.95] | [1.81] | [2.56] | [3.65] | [5.56] | [2.81] |
| GMM | Beta (e-1) | 8.924*** | 9.361*** | 9.282*** | 9.286*** | 9.050*** | 9.313*** | 9.569*** | 9.302*** | 9.340*** | 8.172*** |
| | t-statistic | [29.17] | [29.05] | [26.18] | [27.20] | [26.86] | [26.74] | [35.86] | [20.28] | [32.30] | [18.19] |
| | J-statistic | [21.16] | [22.98] | [21.36] | [20.18] | [23.61] | [29.22] | [28.94] | [26.46] | [24.24] | [16.81] |
| | P-value (%) | 6.98 | 4.20 | 6.62 | 9.08 | 3.49 | 0.61 | 0.67 | 1.48 | 2.90 | 20.80 |
| | Alpha (e-3) | 0.526 | 1.377 | 2.334 | 2.297 | 2.706 | 2.731 * | 2.818* | 8.849*** | 7.339*** | 8.622** |
| | t-statistic | [0.32] | [0.97] | [1.52] | [1.48] | [1.66] | [2.00] | [2.39] | [3.55] | [4.72] | [3.34] |
| OLS | Beta (e-1) | 9.111*** | 9.371*** | 9.263*** | 9.220*** | 9.160*** | 9.412*** | 9.580*** | 8.718*** | 9.212*** | 7.689*** |
| | t-statistic | [20.67] | [24.99] | [22.81] | [22.56] | [21.30] | [26.17] | [30.83] | [17.13] | [22.44] | [11.29] |
| | R-square (%) | 82.93 | 87.65 | 85.54 | 85.25 | 83.75 | 88.61 | 91.52 | 76.93 | 85.13 | 59.15 |

Table 4 compares the view bias adjusted GMM regression results of expectile CAPM to GMM and OLS regression results of traditional CAPM using 10 momentum portfolios' monthly data of U.S. stock market from Jun. 1963 to Dec. 1970. P1 is the bottom loser portfolio; P2 is the bottom second loser; P3 is the bottom third loser... P10 is the top winner; P9 is top second winner; P8 is top third winner... All regression coefficients marked by $*(**, ***)$ are significant at the 5% (1%, <0.0001)level.

**Table 5**

**Value weighted momentum portfolio risk composition**

| | Market | P1 | P2 | P3 | P4 | P5 | P6 | P7 | P8 | P9 | P10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sys risk (e-3) | 1.417 | 1.291 | 1.327 | 1.309 | 1.310 | 1.297 | 1.335 | 1.356 | 1.242 | 1.308 | 1.091 |
| Idyosyn (e-4) | 0.000 | 5.847 | 4.978 | 5.435 | 5.399 | 5.709 | 4.773 | 4.136 | 6.827 | 5.458 | 9.041 |
| Total risk (e-3) | 1.417 | 1.876 | 1.825 | 1.852 | 1.850 | 1.868 | 1.812 | 1.769 | 1.925 | 1.854 | 1.996 |
| Latent risk (e-3) | 2.339 | 2.389 | 2.392 | 2.396 | 2.397 | 2.419 | 2.392 | 0.544 | 2.413 | 2.399 | 2.506 |
| Priced risk (e-3) | 1.441 | 1.315 | 1.351 | 1.333 | 1.334 | 1.321 | 1.358 | 1.368 | 1.265 | 1.332 | 1.115 |
| View bias (e-1) | 4.500 | 4.526 | 4.483 | 4.463 | 4.453 | 4.447 | 4.436 | 4.426 | 4.336 | 4.290 | 4.319 |
| $\Theta^2$ (e-3) | 6.400 | 5.762 | 6.868 | 7.413 | 7.666 | 7.841 | 8.170 | 8.463 | 11.356 | 12.997 | 11.941 |
| $\Phi$ | 1.0064 | 1.0058 | 1.0069 | 1.0074 | 1.0077 | 1.0078 | 1.0082 | 1.0085 | 1.0114 | 1.0130 | 1.0119 |
| Beta | 1.000 | 0.911 | 0.936 | 0.924 | 0.925 | 0.915 | 0.942 | 0.956 | 0.876 | 0.923 | 0.770 |
| Adjusted beta | 1.000 | 0.912 | 0.937 | 0.925 | 0.926 | 0.917 | 0.943 | 0.947 | 0.879 | 0.925 | 0.776 |
| Mkt $r^e$ | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 | 0.653 |
| Adjusted mkt $r^e$ | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 | -2.991 |
| Adjusted portf $r^e$ | -2.991 | -2.728 | -2.803 | -2.766 | -2.769 | -2.742 | -2.819 | -2.831 | -2.630 | 2.767 | -2.321 |
| Expected portf $r^e$ | 0.653 | 0.730 | 0.971 | 1.156 | 1.219 | 1.291 | 1.297 | 1.359 | 2.224 | 2.426 | 2.656 |
| Averaged portf $r^e$ | 0.653 | 0.730 | 0.971 | 1.156 | 1.219 | 1.291 | 1.297 | 1.359 | 2.224 | 2.426 | 2.656 |

Table 5 decomposes the risk for each momentum portfolio from Jun. 1963 to Dec 1970. Sys risk is systematic risk $(\sigma_{iM} = \rho_{iM}\sigma_i\sigma_M)$. Idyosyn is idyosyncratic risk $(\sigma_\epsilon = \sigma_i\sigma_M\sqrt{1-\rho_{iM}^2})$. Total risk is the sum of systematic risk $(\sigma_{iM})$ and idyosyncratic risk $(\sigma_\epsilon)$. According to expectile CAPM, Priced risk , namely the risk being priced $(\Theta^2\widetilde{\sigma}_{iM} + \Phi\sigma_{iM})$, is the weighted average of systematic risk $(\sigma_{iM})$ and Latent risk $(\widetilde{\sigma}_{iM})$. All numbers marked by (e-n) is expressed in scientific notation to a level of $10^{-n}$. Assume view bias $\theta$ for each momentum portfolio are different. For any set of view bias, there is a corresponding pair of $\Theta$ and $\Phi$ . Beta $(\beta = \frac{\sigma_{iM}}{\sigma_M^2})$ is the relative amount of risk in traditional CAPM. Adjusted beta $(\beta^\theta = \frac{\Theta^2\widetilde{\sigma}_{iM}+\Phi\sigma_{iM}}{\Theta^2\widetilde{\sigma}_M^2+\Phi\sigma_M^2})$ is the relative amount of risk in expectile CAPM. Mkt $r^e$ $(\mu_M^e = \mathbb{E}(r_M - r_f))$, representing the expected market excess rate of return, is the price of risk in traditional CAPM. Adjusted mkt $r^e$ $(\widetilde{\mu}_M^e = \mu_M^e + \frac{\sigma_M}{\sqrt{dt}}\Theta))$, is the price of risk in expectile CAPM. Adjusted portf $r^e$ $(\widetilde{\mu}_i^e = \beta_i^\theta \times \widetilde{\mu}_M^e)$, is portfolio $i$'s reward for taking market risk in exectile CAPM. Expected portf $r^e$ $(\mu_i^e = \widetilde{\mu}_i^e - \frac{\sigma_i}{\sqrt{dt}}\Theta)$, is the model implied expected excess rate of return of each portfolio. Averaged portf $r^e$ is the sample average of excess rate of return of each portfolio. View bias is chosen to make the model forcasted Expected portf $r^e$ equal to the sample Averaged portf $r^e$.

# 4 Conclusion

This research developed an expectile-based asset pricing framework to resolve two categories of asset pricing anomalies simultaneously, either from the perspective of SDF's mean and variance, e.g., the equity premium puzzle, or from the perspective of SDF's factor structure, e.g., momentum.

The equity premium puzzle can be explained by the fact that information ambiguity introduces view bias, where, under imperfect information, a slight deviation of view bias from neutral will significantly reduce the required risk aversion in the equity premium puzzle. We also use cyclical movements in view bias to explain momentum. Systematic risk is the full amount of priced risk only when view bias is neutral. Whenever view bias exhibits pessimism or optimism, the total amount of priced risk will become the weighted average of systematic risk and latent risk. The price of risk will be adjusted by two categories of information premium: compensation for taking information ambiguity risk, and reward for providing additional information. Real data analysis shows that the negative adjusted price of risk and the low total amount of priced risk provide explanations for the observed momentum. Empirical analysis indicates that the statistically insignificant moment conditions of GMM from traditional CAPM are statistically significant in a view bias adjusted GMM from expectile CAPM.

Future work might focus on 1) Estimating view bias, alpha and beta simultaneously. So far, the view bias in expectile CAPM estimation is exogenously given by the equity premium puzzle. It would be interesting to compare the endogenously estimated view bias with the view bias implied by the equity premium puzzle. We are expecting that the exogeneity can fully remove alpha. 2) How the view bias serves as a transducer: many risk factors indirectly affect asset pricing by affecting the view bias factor. Regression analysis of view bias on fundamental risk factors will provide insight to explore the real determinants of view bias.

# Appendix

In this appendix, we provide the assumptions and detailed mathematical proofs of the results in the paper.

**Assumption A. 1** Time interval between each decision is infinitesimal.

**Assumption A. 2** Prices follow diffusion processes.

**Assumption A. 3** Only consumption and portfolio process are controllable.

**Assumption A. 4** There is no exogenous endowment.

**Assumption A. 5** Investors are homogenous.

**Assumption A. 6** Information is imperfect, and pessimism or optimism view bias exists.

**Proof** of **Proposition 1**

To prove the extended law of one price, equivalently, we only need to prove the following statement. If $X_i \sim N(\mu_i, \sigma_i)$, the correlation between $X_i$ and $X_j$ is $\rho_i j$, and $\exists \rho_{ij} \neq 1$, then

$$\mathbb{E}_\theta(\sum_{i=1}^n X_i) = \mathbb{E}_\theta(\overline{\sum_{i=1}^n X_i}) = \begin{cases} > 0 \theta > 50\% \\ > 0 \theta > 50\% \\ < 0 \theta < 50\% \end{cases} .$$

Case 1: if $\theta = 50\%$, then $\mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^n X_i}) = \mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\sum_{i=1}^n X_i) = 0$.

Case 2: if $\theta > 50\%$ and $\exists \rho_{ij} \neq 1$,

- Step 1: if $n = 2$, $X_i \sim N(\mu_i, \sigma_i)$, we have $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12})$, then $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) = \mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\sum_{i=1}^2 X_i) = (\sigma_1 + \sigma_2)\mathbb{E}_\theta(W) - \mathbb{E}_\theta(W)\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho_{12}}$.

  If $\rho_{ij} \neq 1$, and due to the fact that $\mathbb{E}_\theta(W)$ is a monotonously increasing function w.r.t $\theta$, $\mathbb{E}_\theta(W) > 0$, hence $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) > 0$, otherwise if $\rho_{ij} = 1$, $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) = \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i})$.

- Step 2: we make assumption that, if $n = k$, then $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) \geqslant 0$.

- Step 3: if $n = k + 1$, then $\mathbb{E}_\theta(\sum_{i=1}^n X_i) = \mathbb{E}_\theta(\sum_{i=1}^{k+1} X_i) = \mathbb{E}_\theta(\sum_{i=1}^k X_i) + \mathbb{E}_\theta(X_k) \geqslant \mathbb{E}_\theta(\overline{\sum_{i=1}^k X_i}) + \mathbb{E}_\theta(X_k) \geqslant \mathbb{E}_\theta(\overline{\sum_{i=1}^{k+1} X_i})$.

Case 3: if $< 50\%$ and $\exists \rho_{ij} \neq 1$, since $\mathbb{E}_\theta(W) < 0$, ceteris paribus, we get $\mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) \leqslant 0$.

Case 4: if $\forall \rho_{ij} = 1$, then $\mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^n X_i}) = \mathbb{E}(\sum_{i=1}^n X_i) - \mathbb{E}(\sum_{i=1}^n X_i) = 0$.

■

**Proof of Theorem 1**

We first define the following variables. $W(t)$ $\triangleq$Total wealth at time $t$. $P_i(t)$ $\triangleq$Price of the ith asset at time $t(i = 1, \cdots, n)$. $S_j(t)$ $\triangleq$ Value of the jth state variable at time $t(j = 1, \cdots, m)$. $C(t)$ $\triangleq$ Consumption per unit time at time $t$. $w_i(t)$ $\triangleq$ Proportion of total wealth in the ith asset at time $t(i = 1, \cdots, n)$. Note $\sum_{i=1}^n w_i(t) \equiv 1$ We model the consumption and portfolio choosing process as follows,

$$J[W(t), S(t), t] \equiv \underset{C(s),\varpi(s)}{max} \mathbb{E}_{\theta,t}\{\int_t^T U_1\left[C(s), s\right] ds + U_2\left[W(T), T\right]\} \quad \textbf{(A.1)}$$

St: Boundary condition:

$$J\left[W(T), S(T), T\right] = U_2\left[W(T), T\right].$$

Budget equation:

$$W(t) = \sum_{i=1}^{n} w_i(t_0)\frac{P_i(t)}{P_i(t_0)}\left[W(t_0) - C(t_0)h\right]$$

Assumption 1: $t \equiv t + h$, $h \to 0$ Assumption 2:

$$\frac{dP_i(t)}{P_i(t)} = \mu_i(S,t)dt + \sigma_i(S,t)\sqrt{dt}w_i, i = 1, 2, \cdots, n$$

$$\frac{dP_i(t)}{P_i(t)} = \mu_i(S,t)dt + \sigma_i(S,t)\sqrt{dt}w_i, i = 1, 2, \cdots, n$$

$$V = \left[\sigma_i l\right], \sigma_{il} = \sigma_i\sigma_l\rho_{il}, i, l = 1, 2, \cdots, n$$

$$dS_j(t) = f_j(S,t)dt + g_i(S,t)\sqrt{dt}q_j, j = 1, 2, \cdots, m$$

By Taylor's theorem and the mean value theorem for integrals, Equation (A.1) can be rewritten as,

$$
\begin{aligned}
J\left[W(t_0), S(t_0), t_0\right] = \underset{C(s),\varpi(s)}{max} \mathbb{E}_{\theta,t_0}\{ & U_1\left[C(\bar{t}), \bar{t}\right] h + J\left[W(t_0), S(t_0), t_0\right] \\
& + \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial dt} h + \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial W}\left[W(t) - W(t_0)\right] \\
& + \sum_{j=1}^{m} \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial S_j}\left[S_j(t) - S_j(t_0)\right] \\
& + \frac{1}{2}\frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right])}{\partial W^2}\left[W(t) - W(t_0)\right]^2 \\
& + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right]}{\partial S_k \partial S_j}\left[S_k(t) - S_k(t_0)\right]\left[S_j(t) - S_j(t_0)\right] \\
& + \sum_{j=1}^{m} \frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right]}{\partial W \partial S_j}\left[W(t) - W(t_0)\right]\left[S_j(t) - S_j(t_0)\right] + O(h^2)\}
\end{aligned}
$$

where $\bar{t} \in [t_0, t]$, take limit as $h \to 0$, take the $\theta$-adjusted expectation operators onto each term, and subtracting, $J[W(t_0), S(t_0), t_0]$ of both sides, we have

$$
\begin{aligned}
0 = \underset{C(s),\varpi(s)}{max} \{ & U_1\left[C(t), t\right] dt + \frac{\partial J\left[W(t), S(t), t\right]}{\partial dt} dt \\
& + \frac{\partial J\left[W(t), S(t), t\right]}{\partial W}\mathbb{E}_{\theta,t}\left[dW(t)\right] \\
& + \sum_{j=1}^{m} \frac{\partial J\left[W(t), S(t), t\right]}{\partial S_j}\mathbb{E}_{\theta,t}\left[dS_j(t)(t\right] \\
& + \frac{1}{2}\frac{\partial^2 J\left[W(t), S(t), t\right])}{\partial W^2}\mathbb{E}_{\theta,t}\left[dW(t)\right]^2 \\
& + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 J\left[W(t), S(t), t\right]}{\partial S_k \partial S_j}\left[S_k(t) - S_k(t)\right]\mathbb{E}_{\theta,t}\left[dS_j(t)\right] \\
& + \sum_{j=1}^{m} \frac{\partial^2 J\left[W(t), S(t), t\right]}{\partial W \partial S_j}\mathbb{E}_{\theta,t}\left[dW(t) dS_j(t)\right] + O(dt^2)\}
\end{aligned}
\tag{A.2}
$$

By subtracting $W(t_0)$ on both sides, the budget equation is rewritten as,

$$W(t) - W(t_0) = \left[\sum_{i=1}^{n} \varpi_i(t_0) \frac{P_i(t) - P_0(t)}{P_i(t_0)}\right] [W(t_0) - C(t_0)h] - C(t_0)h$$

The expectile of the limit process as $h \to 0$ is

$$\begin{aligned}\mathbb{E}_{\theta,t}[dW(t)] &= \{\sum_{k=1}^{n} \varpi_i(t) W(t) \mathbb{E}_{\theta,t}\left(\frac{dP_i(t)}{P_i(t)}\right) - C(t)dt\} + O(dt^2) \\ &= \{\sum_{k=1}^{n} \left(\varpi_i(t) W(t) \left[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(w_i)\right] - C(t)\right)\} dt\end{aligned} \tag{A.3}$$

Applying the same limit process to other terms,

$$\mathbb{E}_{\theta,t}[dW(t)]^2 = \sum_{l=1}^{n} \sum_{i=1}^{n} \varpi_i(t) \varpi_l(t) W(t) 2\sigma_i(S,t) \sigma_l(S,t) \mathbb{E}_(\theta,t)[w_i w_l] dt \tag{A.4}$$

$$\mathbb{E}_{\theta,t}[dS_j(t)] = f_j(S,t)dt + g_j(S,t)\sqrt{dt}\mathbb{E}_{\theta,t}(q_i) \tag{A.5}$$

$$\mathbb{E}_{\theta,t}[dS_k(t)dS_j(t)] = g_k(S,t)g_j(S,t)\mathbb{E}_{\theta,t}[q_k(t)q_j(t)]dt + O(dt^2) \tag{A.6}$$

$$\mathbb{E}_(\theta,t)[dW(t)dS_j(t)] = \left[\sum_{i=1}^{n} \varpi_i(t) W(t) \sigma_i(S,t) g_j(S,t) \mathbb{E}_{\theta,t}[w_i q_j(t)]\right] dt \tag{A.7}$$

Take Equations (A.3) to (A.7) into Equation (A.2), and assume the *n*th asset is risk free asset, we get the following HJB function:

$$0 = \max_{C(s),\varpi(s)} \{U_1 [C(t), t] dt + \frac{\partial J [W(t), S(t), t]}{\partial dt} dt + \frac{\partial J [W(t), S(t), t]}{\partial W}$$
$$\times \{ \left[ \left( \sum_{i=1}^{n-1} \left( \varpi_i(t) \left( \mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(w_i) \right) - r_f \right) + r_f \right) W(t) \right] - C(t) \}$$
$$+ \sum_{j=1}^{m} \frac{\partial J[W(t), S(t), t]}{\partial S_j} \left[ f_j(S,t) + \frac{g_j(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(q_i) \right]$$
$$+ \frac{1}{2} \partial^2 J[W(t), S(t), t]/W^2 \sum_{l=1}^{n-1} \sum_{i=1}^{n-1} \varpi_i(t) \varpi_l(t) W(t)^2 \times \sigma_i(S,t) \sigma_l(S,t) \mathbb{E}_(\theta, t)(w_i w_l)$$
$$+ \frac{1}{2} \sum_{k=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 J[W(t), S(t), t]}{\partial S_k \partial S_j} g_k(S,t) g_j(S,t) \mathbb{E}_(\theta, t)(q_k q_j)$$
$$+ \sum_{j=1}^{m} \partial^2 J[W(t_0), S(t_0), t_0] \partial W \partial S_j [\sum_{i=1}^{n-1} \varpi_i(t) W(t) \sigma_i(S,t) \times g_j(S,t) \mathbb{E}_(\theta, t)(w_i q_j)]\}$$

We let the derivatives of HJB w.r.t consumption $C(t)$ and weight assigned to each risky asset, $\varpi_1(t)$ to $\varpi_{n-1}(t)$, equal to zero, the first order conditions are,

$$U_{1,C} [C^*(t), t] - \frac{\partial J [W(t), S(t), t]}{\partial W} = 0$$

$$\frac{\partial J[W(t), S(t), t]}{\partial W} \{\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(w_i) - r^f\}$$
$$+ \frac{\partial^2 J[W(t), S(t), t]}{\partial W^2} \sum_{l=1}^{n-1} \varpi_l^* W(t) \sigma_i(S,t) \sigma_l(S,t) \mathbb{E}_{\theta,t}(w_i w_l)$$
$$+ \sum_{j=1}^{m} \frac{\partial J[W(t), S(t), t]}{\partial W \partial S_j} \{\sigma_i(S,t) g_j(S,t) \mathbb{E}_{\theta,t}(w_i w_j)\}$$
$$= 0, i = 1, 2, \cdots, n-1$$

Define a more compact expression in the following way,

$$V = [\sigma_i \sigma_l \mathbb{E}_{\theta,t}(w_i w_l)], i, l = 1, 2, \cdots, n-1$$

$$\mathit{\Gamma} = [\sigma_i g_j \mathbb{E}_{\theta,t}(w_i q_j)], i, l = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m$$

Then, we can get the optimized portfolio process,

$$\varpi^* = -\frac{J_W[W(t), S(t), t]}{W(t)J_{WW}[W(t), S(t), t]}V^{-1}\{\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w_i) \\ - r^f\} - V^{-1}\Gamma\frac{J_{SW}[W(t), S(t), t]}{W(t)J_{WW}[W(t), S(t), t]}$$

We write the above formula in form of vectors, and sum K homogeneity investors portfolio weight, we get the aggregated market portfolio weight.

$$\varpi_M = \frac{\sum_{k=1}^{K}\varpi^K W^K}{\sum_{k=1}^{K} W^K} \\ = \frac{A}{M}V^{-1}\{\mu(S,t) + \frac{\sigma(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f\} + V^{-1}\Gamma\frac{B}{M}$$

where

$$A = \sum_{k=1}^{K}\left(-\frac{J_W^k[W(t), S(t), t]}{J_{WW}^k[W(t), S(t), t]}\right); B = \sum_{k=1}^{K}\left(-\frac{J_{SW}^k[W(t), S(t), t]}{J_{WW}^k[W(t), S(t), t]}\right); M = \sum_{k=1}^{K} W^k.$$

The expectile excess return vector satisfies the following equation,

$$\mu(S,t) + \frac{\sigma(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f = \varpi'_M V\frac{M}{A} - \Gamma\frac{B}{A}$$

We denote each scalar of the vector $\varpi'_M V$ as $\sigma_{iM}^{\theta}$, and assuming that state variables are constants,

$$\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - r^f = \sigma_{iM}^{\theta}\frac{M}{A} \quad \text{(A.8)}$$

We specify the term of $\sigma_{iM}^{\theta}$, and get

$$\begin{aligned}\sigma_{iM}^{\theta} &= \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \mathbb{E}\varpi\varpi_{\theta,t}(w_i w_j) \\ &= \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \{[\mathbb{E}_{\theta,t}(w)]^2 \sqrt{1-\rho_{ij}^2} \times sign(\rho_{ij} + \mathbb{E}_{\theta,t}(w^2)\rho_{ij}\} \\ &= \mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \sqrt{(1-\rho_{ij}^2} \times sign(\rho_{ij}) + \mathbb{E}_{\theta,t}(w^2) \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \rho_{ij}\end{aligned} \tag{A.9}$$

We consider a market portfolio as a whole, and then it is a one-dimension random variable. We use symbol $\bar{M}$ to distinguish it from $n$ dimensional market portfolio,

$$\mu_{\bar{M}}(S,t) + \frac{\sigma_{\bar{M}}(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(\varpi) - \gamma^f = \sigma_{\bar{M}M}^{\theta} \frac{M}{A} \tag{A.10}$$

where

$$\begin{aligned}\sigma_{\bar{M}M}^{\theta} &= \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \mathbb{E}_{\theta,t}(w_{\bar{M}} w_j) \\ &= \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \{[\mathbb{E}_{\theta,t}(w)]^2 \sqrt{1-\rho_{j\bar{M}}^2} \times sign(\rho_{j\bar{M}}) + \mathbb{E}_{\theta,t}(w^2)\rho_{j\bar{M}}\} \\ &= [\mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \sqrt{(1-\rho_{j\bar{M}}^2} \times sign(\rho_{j\bar{M}}) + \mathbb{E}_{\theta,t}(w^2) \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \rho_{j\bar{M}}\end{aligned}$$

The reason why we use symbol $\bar{M}$ to distinguish the market portfolio being taken as one security and as an $n$ dimensional portfolio is that the expectile does not satisfy the additivity when there is a risk dimensional-receding. After the coefficients being taken out of the expectile operator, there is no difference between $\bar{M}$ and $M$, therefore we can rewrite the above expression as follows,

$$\sigma^{\theta}_{\bar{M}M} = [\mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_M \sigma_j \sqrt{1-\rho^2_{jM}} \times sign(\rho_{jM}) + \mathbb{E}_{\theta,t}(w^2)\sigma^2_M \tag{A.11}$$

Take Equations (A.9) and (A.11) into Equation (A.8) and Equation (A.10) respectively, and denote $\mu_i \triangleq \mu_i(S,t)$, $\sigma_i \triangleq \sigma_i(S,t)$

$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f}{\mu_M + \frac{\sigma_M}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f} = \frac{\sigma^{\theta}_{iM}}{\sigma^{\theta}_{\bar{M}M}}$$

We denote

$$\widetilde{\sigma}_{iM} \triangleq \textstyle\sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j sign(\rho_{ij}) \sqrt{1-\rho^2_{ij}},$$
$$\widetilde{\sigma}^2_M \triangleq \textstyle\sum_{j=1}^{n} \varpi_j \sigma_M \sigma_j sign(rho_{jM}) \sqrt{1-\rho^2_{jM}},$$
$$\Phi \triangleq (w) \text{ and } \Psi \triangleq \mathbb{E}_{\theta}(w^2),$$

where $\varpi_i$ is the weight of security $i$ within the portfolio, and $w \sim (0,1)$. Hence we have two equivalent expressions of expectile CAPM,

$$\mathbb{E}_{\theta}(r_i - r_f) = \beta^{\theta}\mathbb{E}_{\theta}(r_M - r_f), i = 1, 2, \cdots, n.$$

or

$$\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Phi - r^f = \beta^{\theta}(\mu_M + \frac{\sigma_i}{\sqrt{dt}}\Phi - r^f), i = 1, 2, \cdots, n.$$

■

**Proof of Theorem 2.**

We assume the investor is solving the following portfolio optimization problem.

$$\underset{\varpi(t)}{max}\mathbb{E}_{\theta,t}\{\int_t^\infty e^{-\delta s}U\left(C(s)\right)ds\}$$
$$=\mathbb{E}_{\theta,t}\{\int_t^{t+h} e^{-\delta s}U\left(C(s)\right)ds\}+\mathbb{E}_{\theta,t}\{\int_{t+h}^\infty e^{-\delta s}U\left(C(s)\right)ds\}$$
$$=\mathbb{E}_{\theta,t}\{\int_t^{t+h} e^{-\delta s}U\left(C(s)\right)ds\}+\mathbb{E}_{\theta,t}\{\int_0^\infty e^{-\delta(t+h+s)}U\left(C(t+h+s)\right)ds\}$$

$$St: C(t)=e(t)-\varpi(t)P_M(t);$$
$$C(t+s+h)=e(t+s+h)+\varpi(t)D(t+s+h)h;\ h\to 0$$

where $\varpi(t)$ is number of shares of market portfolio, $\delta$ is time preference, $e(t)$ is endowment at time $t$, $D(t)$ is dividend at time $D$. We take the same approach shown in Cochrane (2001) and get the following similar result, which is expressed under expectile operator.

$$\mathbb{E}_{\theta,t}\left(\frac{dP_M(t)}{P_M(t)}\right)+\frac{D(t)}{P_M(t)}dt-r^f dt=\alpha\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right] \qquad \text{(A.12)}$$

Taking the diffusion processes into the right hand side of the above equation, we have

$$\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right]=\mathbb{E}_{\theta,t}\left[(\mu_C dt+\sigma_C\sqrt{dt}w_C)(\mu_M dt+\sigma_M\sqrt{dt}w_M)\right]$$

where $w_C$, $w_M$ are standard normal distributed, their correlation is $\rho_{CM}$.

Omitting the high order derivatives, we obtain

$$\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right] = \sigma_C\sigma_M dt\mathbb{E}_{\theta,t}[w_C w_M]$$
$$= \sigma_C\sigma_M dt\left([\mathbb{E}_{\theta,t}(w)]^2\sqrt{1-\rho_{CM}^2}\times sign(\rho_{CM}) + [\mathbb{E}_{\theta,t}(w)]^2\rho_{CM}\right) \tag{A.13}$$

Taking Equation (A.13) into Equation (A.12), and cancel dt on both sides, we have

$$\mu_M + \frac{D(t)}{P_M(t)} + \sigma_M\sqrt{dt}\mathbb{E}_{\theta,t}(w) - r^f$$
$$= \alpha\sigma_C\sigma_M\left([\mathbb{E}_(\theta,t)(w)]^2\sqrt{1-\rho_{CM}^2}\times sign(\rho_{CM}) + [\mathbb{E}_{\theta,t}(w)]^2\rho_{CM}\right)$$

We drop $t$, assume $D(t) = 0$, and denote $\Phi \triangleq (w)$, $\Psi \triangleq \mathbb{E}_\theta(w^2)$, we get

$$\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}}\Phi - r_f = \alpha\sigma_C\sigma_M\left(\Phi^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2} + \Psi\rho_{CM}\right)$$

If the volatility of aggregation consumption growth rate and the volatility of rate of return of market portfolio are completely correlated, that is, $\rho_{CM} = 1$.

$$\mathbb{E}(r_M) + \frac{\sigma_M}{dt}\Phi - r_f = \alpha\sigma_C\sigma_M(\Psi\rho_{CM}).$$

■

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