Multi-Period Market Equilibrium. Asset Pricing

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1 Lucas Tree Model.

A special case of the infinite horizon, pure exchange economy with stochastic endowments is the so-called Lucas tree model with identical agents. The model shares features with most of the models used in macroeconomics and asset pricing.

1.1 The Economy

The model economy is given as follows:

- There is a large number of identical, infinitely lived agents each of whom maximizes lifetime expected utility.
- There is an equal number of trees, $i = 1, ..., N$. Each agent starts life at time zero with one tree. These trees are the only assets in the economy.
- At the beginning of period t, each tree i yields a stochastic dividend (or fruit) in the amount $d_{i,t}$ to its owner. The distribution of $d_{i,t}$ is identical for all trees. The process is known by the agents. The fruit is not storable.

- The market in the ownership of trees is perfectly competitive. In the equilibrium, asset prices clear the market. That is, the total stock positions of all agents are equal to the aggregate number of shares.

1.2 The representative consumer's problem

Due to the assumption that all agents are identical with respect to both preferences and endowments, we can work with a representative agent. The agents' economic activity in each period consists of trading shares, where a share represents a claim to the future stream of dividends.

We let $p_{i,t}$ be the price of tree i in period t (i.e., of a claim to the entire income stream, $\{d_{i,t+\tau}\}_{\tau=0}^{\infty}$), measured in terms of the consumption commodity and $z_{i,t}$ the quantity of tree i that the agent holds between t and $t+1$. Our aim is to find $p_{i,t}$ as a function of the stochastic payoffs, $\{d_{i,t+\tau}\}_{\tau=0}^{\infty}$. Stated formally, the representative consumer chooses $\{z_{1,t}, z_{2,t}, ..., z_{N,t}; c_t\}_{t=0}^{\infty}$ to maximize

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

where $u(c_t)$ is a concave function and $0 < \beta < 1$, subject to the period-by-period budget constraint

$$\sum_{i=0}^{N} z_{i,t+1} p_{i,t} + c_t \leq \sum_{i=0}^{N} z_{i,t} d_{i,t} + \sum_{i=0}^{N} z_{i,t} p_{i,t}$$

The period-by-period budget constraint has the following interpretation: The ownership of share i at the beginning of period t, $z_{i,t}$, entitles the owner to receive dividend $d_{i,t}$i in period t and to have the right to sell the tree at price $p_{i,t}$ in period t. Thus, the agent's source of funds in period t are given by the sum of shares that the agent holds times their respective dividends, $\sum_{i=0}^{N} z_{i,t} d_{i,t}$, plus the revenue of the sale of the shares at the prevailing prices, $\sum_{i=0}^{N} z_{i,t} p_{i,t}$. On the left hand side of the inequality we have got the agent's use of funds, saying that the proceedings are either consumed or spent for the number of shares the agent wants to hold in period $t+1$. This number of share, of course, has to be bought at the prevailing prices.

1.3 Identical agents and market clearing conditions

We will be looking for an equilibrium for this model. As usual, an equilibrium is a sequence of prices $p^*_{i,t}$ such that:

1. Given prices $p^*_{i,t}$, all agents optimize.

2. Markets clear.

In a model populated by identical agents (or by one representative agent), the market clearing condition for the asset market takes especial importance. Market clearing in the asset market means that for every agent who wishes to buy one unit of asset i at price $p_{i,t}$ there must be another agent who is willing to sell him that unit of the asset at $p_{i,t}$. This means that any positive demand $z^+_{i,t} > 0$ from one agent must be equalled by a negative demand (wish to sell) from another agent $z^+_{i,t} = z^-_{i,t}$. But now remember that in this model, all agents are identical. That means that if one agent wishes to buy some amount $z^+_{i,t} > 0$ of asset i at date t, then all of his identical colleagues will also want to buy the very same amount of the asset $z^+_{i,t} > 0$. The problem with this scenario is, of course, that there is no one on the other side of the market, no one who is willing to sell all of these agents the desired amounts. The reverse might also happen: if one agent wishes to sell $z^-_{i,t} > 0$, then all agents will want to sell $z^-_{i,t} > 0$ at the same time, creating an excess supply of the asset. Which only leaves one possibility to satisfy market clearing: that prices are exactly such that the representative agent wishes to neither buy nor sell the asset. As we shall see below, the relevant first-order conditions do nothing other than determine the price of the security i that is just sufficient to cause the agent to want neither to buy nor to sell.

1.4 The representative consumer's problem, restated

As we have shown, for a market equilibrium, the quantities of each tree demanded must equal to the (given) supply, that is, the number of shares for each tree must sum up to one. Since there is only one agent, equilibrium implies that $z_{i,t} = 1$ for all i, t. In other words, there is no other agent in the economy with whom to trade insurance, i.e., there is no trade. From the budget constraint, this implies that consumption of the representative agent must equal to output, which is the sum of dividends

$$c_t \leq \sum_{i=0}^{N} d_{i,t}.$$

Without loss of generality, we can assume that there is only one tree. Thus, the representative agent chooses $\{z_t, c_t\}_{t=0}^{\infty}$ to maximize

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

subject to the period-by-period budget constraint

$$z_{t+1}p_t + c_t \leq z_t d_t + z_t p_t$$

and market clearing conditions $c_t = d_t$.

1.5 Dynamic Optimization.

The representative agent wishes to choose a stream of consumptions $\{c_t\}_{t=0}^{\infty}$ to maximize the expected discounted sum of utilities:

$$\max_{\{z_{t+1}, c_t\}} E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \qquad (1)$$
$$z_{t+1}p_t + c_t \leq z_t d_t + z_t p_t$$

Note that maximization implies that the constraint will be binding, otherwise consumption can be increased.

The condition we are supposed to derive is nothing other than Euler equation.

The representative agent's problem is a problem of dynamic optimization, where c_t is a control variable, z_t is a state variable, and d_t and p_t are exogenous variables (recall that the consumer acts as a price-taker). Under strong Markovian assumptions [1] about the evolution of uncertainty, we usually write the following dynamic evolution equation, which is obtained from the budget constraint:

[1] This is just to say that all the relevant history is summarized in the current value of the state variable.

$$z_{t+1} = \frac{(d_t + p_t)\, z_t - c_t}{p_t} \tag{2}$$

We will use Bellman's optimality principle to solve the problem. The main idea behind the Bellman's optimality principle is to transform the multiperiod problem in (1) into a two period problem. This is achieved by noting that the representative agent commits to optimality in a dynamically consistent way - today's optimal choice is made with the knowledge that it will be optimal next period onwards as well. This reasoning together with the Markov structure and the constraint in (2), allows us to start looking for an optimal policy rule, rather then an optimal path. In other words, we now look for a functional representation of the optimal path - given the value of the state variable, what function (optimal policy rule) will give us the value of the optimal control variable. In our particular case, this will mean that we are looking for a function of share holding z, which will tell us what is the agent's optimal consumption for any particular share holding z.

We will maximize the value function

$$V_t(z_t) = \max_{c_s} E_t \left(\sum_{s=t}^{\infty} \beta^{(s-t)} u(c_s) \right)$$

The Bellman's optimality principle leads to the claim that the optimization problem in (1) is equivalent to the following two period one:

$$V_t(z_t) = \max_{c_t} \left[u(c_t) + \beta E_t \left(V_{t+1}(z_{t+1}) \right) \right] \tag{3}$$

s.t.

$$z_{t+1} = \frac{(d_t + p_t)\, z_t - c_t}{p_t} \tag{4}$$

Note that $V(z_t)$ is not the optimal policy rule we are looking for - this is just the value function of the problem - it gives us the maximum utility level achieved, given a particular path for the state variable z_t.

Substitute the budget constraint (4) in (3) and find

$$\frac{dV_{t+1}(z_{t+1})}{dc_t} = -\frac{1}{p_t} V'_{t+1}$$

Derive the F.O.C. with respect to the decision variables:

$$u^{'}(c_t) - \beta E_t \left(V_{t+1}^{'}(x_{t+1}) \right) \frac{1}{p_t} = 0$$

We now can use the envelope theorem[2] to derive $V_{t+1}^{'}(z_{t+1})$:

$$V_{t+1}^{'}(z_{t+1}) = u^{'}(c_{t+1}) \frac{\partial c_{t+1}}{\partial z_{t+1}}$$

By rewriting the binding budget constraint one period forward we get:

$$c_{t+1} = (d_{t+1} + p_{t+1}) z_{t+1} - p_{t+1} z_{t+2}$$

and

$$\frac{\partial c_{t+1}}{\partial z_{t+1}} = d_{t+1} + p_{t+1}$$

and thus

$$V_{t+1}^{'}(x_{t+1}) = u^{'}(c_{t+1})(d_{t+1} + p_{t+1})$$

If we substitute the last equation into Bellman equation we get

$$u^{'}(c_t) - \beta E_t \left(u^{'}(c_{t+1})(d_{t+1} + p_{t+1}) \right) \frac{1}{p_t} = 0$$

or

$$p_t u^{'}(c_t) = \beta E_t \left((p_{t+1} + d_{t+1}) u^{'}(c_{t+1}) \right)$$

which is the Euler equation we were supposed to derive.

1.6 Intertemporal Euler equation. Interpretation

The Euler equation has an intuitive interpretation. The left-hand side gives the marginal utility (loss) to giving up a small amount of consumption, and using it to buy some of the asset at price p_t. The right-hand side gives the discounted expected marginal utility (gain) at date $t + 1$ from having an increased amount of the asset: part of the utility gain comes from the expected resale value of the additional amount of the asset and part of it

[2]See the Appendix for the intuitive derivation and interpretation of the envelope theorem.

comes from the dividend which this additional amount of the asset brings. Thus, the Euler equation is simply saying that given prices p_t and dividends d_t, agents will find it optimal to increase their demand of the asset if the expected future gains to doing so (i.e. the RHS) are greater than the costs (i.e. the LHS). The Euler equation is also saying that the agents will find it optimal to decrease their demand of the assets whenever the costs in utility terms to buying an additional amount of the asset (i.e. the LHS) are greater than the expected future gains (the RHS). Taken together, that means that if agents are just indifferent between increasing and decreasing the amount of the asset which they demand, then they are already demanding the optimal amount of the asset.

1.7 Equilibrium

For the economy to be in equilibrium, the following should be true:

1. $z_t = z_{t+1} = ... = 1$: the agent owns the entire security.
2. $c_t = d_t$: the ownership of the entire security entitles the agent to all the economy's output.
3. Euler equation holds. The agent's holdings of the security are optimal given the prevailing prices. The equilibrium price must satisfy:

$$p_t u^{'}(d_t) = \beta E_t \Big((p_{t+1} + d_{t+1}) \, u^{'}(d_{t+1}) \Big) .$$

Iterating this equation forward and substituting the results for p_{t+s}, we get

$$\begin{aligned} p_t &= E_t[\beta d_{t+1} \frac{u^{'}(d_{t+1})}{u^{'}(d_t)} + \beta^2 d_{t+2} \frac{u^{'}(d_{t+2})}{u^{'}(d_t)} ... \\ &\quad + \beta^{(T-t)} y_{t+T} \frac{u^{'}(d_{t+T})}{u^{'}(d_t)} + \beta^{(T-t)} p_{t+T} \frac{u^{'}(d_{t+T})}{u^{'}(d_t)}] \end{aligned}$$

Taking the limit as $T \to \infty$:

$$p_t^* = E_t \left[\sum_{s=t+1}^{\infty} \beta^{(s-t)} \frac{u'(d_s)}{u'(d_t)} d_s + E_t \left(\lim_{T \to \infty} \frac{\beta^{(T-t)} u'(d_T) p_{t+T}^*}{u'(d_t)} \right) \right]$$

If we impose no-bubble condition, known also as *transversality* condition:

$$E_t\left(\lim_{T\to\infty}\beta^{(T-t)}p^*_{t+T}\right)=0$$

we have the "*fundamentals*" price that only depends on dividends:

$$p^*_t=E_t\left[\sum_{s=t+1}^{\infty}\beta^{(s-t)}\frac{u'(d_s)}{u'(d_t)}d_s\right]$$

2 Consumption CAPM

We want to see, in this model's context, what determines the amount by which the risky asset's expected return exceeds that of risk-free asset.

Define for security j:

$$1+r_{j,t+1}=\frac{p_{j,t+1}+d_{j,t+1}}{p_{j,t}}$$

Euler equation can be rewritten as:

$$1=\beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{j,t+1})\right] \tag{5}$$

Let p^B_t be the price in period t of one-period riskless discount bond in zero supply. The Euler equation for the bond is:

$$p^B_t u'(c_t)=\beta E_t\left(u'(c_{t+1})1\right)$$

Since the risk-free rate over the period from t to $t+1$ $r_{f,t+1}$ is defined as $p^B_t(r_{f,t+1})=1$, we have

$$\frac{1}{r_{f,t+1}}=p^B_t=\beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right] \tag{6}$$

Rewrite (5) in the form:

$$1=\beta E_t\left[\frac{u'(c_{t+1})}{u'(c_t)}\right]E_t\left[(1+r_{j,t+1})\right]+\beta cov_t\left[\frac{u'(c_{t+1})}{u'(c_t)},r_{j,t+1}\right] \tag{7}$$

Denote $E_t\left[(1+r_{j,t+1})\right]=1+\overline{r}_{j,t+1}$

Substitute (6) into (7):

$$1 = \frac{\overline{r}_{j,t+1}}{r_{f,t+1}} + \beta cov_t \left[\frac{u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right]$$

$$\frac{\overline{r}_{j,t+1}}{r_{f,t+1}} = 1 - \beta cov_t \left[\frac{u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right]$$

$$\overline{r}_{j,t+1} - r_{f,t+1} = -\beta (r_{f,t+1}) cov_t \left[\frac{u'(c_{t+1})}{u'(c_t)}, r_{j,t+1} \right] \quad (8)$$

Assets which have high returns when consumption is already high and marginal utility is low (and which have comparatively low returns when consumption is low and marginal utility is high) also have a high overall expected return: investors need to be compensated extra for holding an asset which does not pay that much when they really need it.

3 Appendix

3.1 Envelope Theorem

The envelope theorem can be understood as follows. Suppose that we want to maximize $f(x, q)$ over x. We can think of q as being a state variable and x as being a choice variable. For every value of q in this problem there will be a maximizing value of x. Define a function, $x(q)$ that gives the optimal x value for each value of q. We can also think of the value function in these terms as $V(q) = f(x(q), q)$. If we take the derivative of the value function with respect to the state variable, we obtain

$$\frac{\partial V(q)}{\partial q} = \frac{\partial f(x(q), q)}{\partial x} \frac{\partial x(q)}{\partial q} + \frac{\partial f(x(q), q)}{\partial q}$$

But we know that $x(q)$ is the value of x that maximizes f, so

$$\frac{\partial f(x(q), q)}{\partial x} \frac{\partial x(q)}{\partial q} = 0$$

and

$$\frac{\partial V(q)}{\partial q} = \left. \frac{\partial f(x(q), q)}{\partial q} \right|_{x=x(q)} .$$

This is a very simple statement of the envelope theorem. In the dynamic programming context, if we take the derivative of the value function with respect to the state variables and if we hold the choice variables (the actions) at their optimal levels, then we can consider the derivatives of the value function with respect to the choice variables to be equal to zero.