Hansen and Jagannathan bounds – Cochrane Chapter 5

Consumption based models don't work empirically – equity premium puzzle. Instead of just trying a bunch of different utility functions, it is helpful to characterize some properties that m must satisfy.

HJ bounds – bound on $\{\sigma(m), E(m),\ \text{other moments of m}\}$

<u>Purpose:</u>

(1) Give us a clearer understanding of why certain asset pricing models are rejected by the data.
(2) Allow us to compare asset pricing models against one another.
(3) Help to identify features of the data that present the most stringent restrictions on asset pricing models.

What is an asset pricing model?

<u>HJ bound using a single return</u>

$$0 = E(mR^e) \qquad\qquad R^e = R_i - R_f$$

$$0 = E(m)E(R^e) + \rho\sigma(m)\sigma(R^e)$$

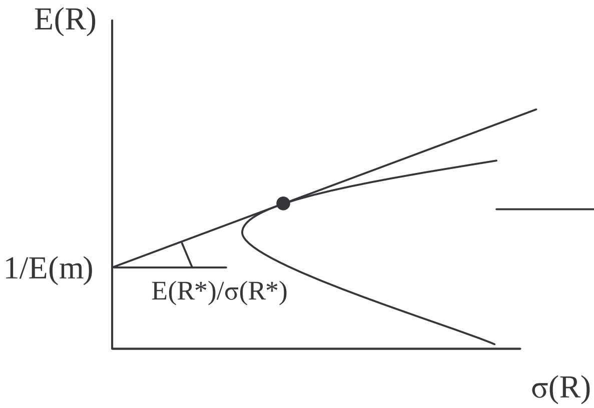
$$\sigma(m) = E(m)\frac{1}{(-\rho)}\frac{E(R^e)}{\sigma(R^e)}$$

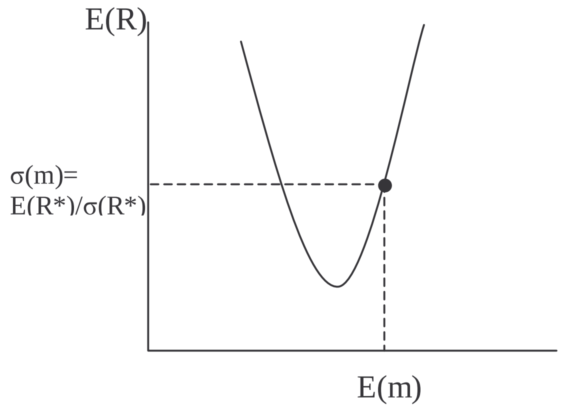
$$\boxed{\sigma(m) \geq E(m)\frac{E(R^e)}{\sigma(R^e)}}\ = \text{ when } \rho\text{=1} \qquad\qquad \text{What happens when } \rho\text{= -1}$$

$$\frac{\sigma(\text{m})}{\text{E(m)}} \geq \frac{E(R^e)}{\sigma(R^e)} \quad \leftarrow \text{ Sharpe ratio}$$

So $\frac{\sigma(\text{m})}{\text{E(m)}}$ must be at least as large as the maximum Sharpe ratio

$$\min\{\text{all m that price x}\}\frac{\sigma(\text{m})}{\text{E(m)}} = \max\{\text{all } \text{R}^e \text{ in } \underline{\text{x}}\}\frac{E(R^e)}{\sigma(R^e)}$$





What is σ(m)?

For CAPM, m = a-bRm
Suppose you have many stocks and CAPM holds. You observe σ(R) for each stock and find the maximum Sharpe ratio. You are not allowed to have an asset pricing model where σ(m) = 0.

Equity premium puzzle:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma} \qquad \text{power utility, RRA} = \gamma$$

$$E(R_m^e) = 6\%,\ \sigma(R_m^e) = 16\%,\ E(m) = \frac{1}{R_f} \approx 1 \Rightarrow \sigma(m) = 37.5\% \text{ or higher}$$

With log utility (γ=1), σ(m) $\approx$ 1% empirically
To get σ(m) = 50%, need a really high γ.

What are the bounds on E(m)?

Do these bounds apply to incomplete markets?

HJ bound using a vector of returns (no restrictions m $\geq$ 0)

Only use p = E(mx)

Think of regressing m_t on x_t:

(*) $m_t = E(m) + (x_t - E(x))'\beta + \varepsilon_t$

x_t, β are N*1 vectors assume iid

$p_{t-1} = E(m_t x_t)$

$p = E(m)E(x) + \text{cov}(m_t, x_t)$

$p = E(m)E(x) + \text{cov}[E(m) + (x_t - E(x))'\beta + \varepsilon_t, x_t]$

$p = E(m)E(x) + \text{cov}(x, x')\,\beta \qquad \leftarrow why?$

$p = E(m)E(x) + \Sigma\beta$, where Σ = var(x)

$\Rightarrow \beta = \Sigma^{-1}\left[p - E(m)E(x)\right] \quad or \quad \beta = \Delta^{-1}\left[1 - E(m)E(R)\right]$

what is this? - looks like a regression coefficient

Take variance of (*)

$\sigma^2(m) = \text{var}\left[(x_t - E(x))'\beta\right] + \sigma^2(\varepsilon_t)$

$\sigma^2(\varepsilon_i) \geq 0,$ so $\sigma^2(m) \geq \text{var}\left[(x_t - E(x))'\beta\right]$

$$\boxed{\sigma^2(m) \geq (p - E(m)E(x))'\Sigma^{-1}(p - E(m)E(x))}$$

hyperbola in $\{$E(m), σ(m)$\}$ space.

As we sweep through values of E(m), from higher to lower, the slope to the tangency portfolio on the efficient frontier falls until $\frac{1}{E(m)}$ equals the expected return on the minimum variance portfolio. As E(m) falls further, the Sharpe ratio increases.

You can find bounds on other moments of m. You can compute bounds with a further restriction that m > 0. In practice, bounds are not altered very much.

The m on the HJ bound is perfectly negatively correlated with the excess return of the tangency portfolio.

If we have an incomplete market and m is a valid discount factor, then $m+\varepsilon$ where $E(\varepsilon x)=0$ is also a valid discount factor, but it has more variance than m. This ε is just the ε_t in equation (*).

How can we use the HJ bound to rule out asset pricing models?

Suppose that we are testing the basic one-period consumption model with power utility.

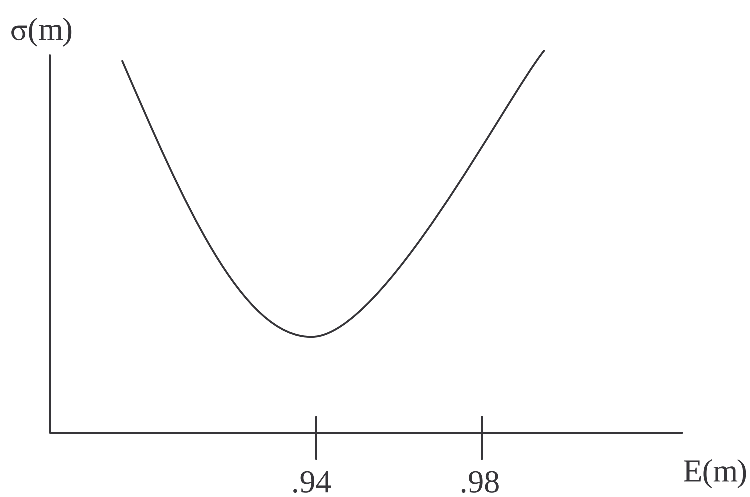
$$m=\beta\frac{u'(c_{t+1})}{u'(c_t)} \qquad u(c_t)=\frac{c_t^{1-\gamma}}{1-\gamma},\ u'(c_t)=c_t^{-\gamma}$$

$$so\ m=\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

Would γ=0 be a good model? $\mathrm{u(c_t)}=c_t$, no risk aversion

$m=\beta$

$\sigma(m)=0$



The HJ bound comes from E(mR) = 1 and the maximum Sharpe ratio that we observe.

If R_f = 1.02, then E(m) = 0.98, σ(m) = 0. Does this satisfy HJ bound?

Suppose γ=1 (log utility)

$$m = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-1} \quad \Rightarrow \quad \sigma(m) \approx 1\%$$

What models would be valid? need γ=25 or 30

Does adding an asset or asset class expand the efficient frontier? This is the same as asking if adding these assets cause the HJ bounds to go down.