# Chapter 7
# Implications of Existence and Equivalence Theorems

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# Main Points

- The existence of a discount factor means that $p = E(mx)$ is innocuous, and all content flows from the discount factor model.
- The theorems apply to sample moments too: the dangers of fishing up ex post or sample mean-variance efficient portfolios.
- Sources of discipline in factor fishing expeditions
- The joint hypothesis problem. How efficiency tests are the same as tests of economic discount factor models
- Factors vs. their mimicking portfolios
- Testing the number of factors
- Plotting contingent claims on the axis vs. mean and variance.

# $p = E(mx)$ is Innocuous

- Roll showed that mean-variance efficiency implies a single-beta representation, so some single-beta representation always exists since there is some mean-variance efficient return. The asset pricing model only serves to predict that a particular return(e.g., market return) will be mean-variance efficient. Thus, if one wants to "test the CAPM" it becomes much more important to be careful in choosing the reference portfolio.
- This insight led to the use of broader wealth indices in the reference portfolio. However, this approach has not caught on. Stocks are priced with stock factor, bonds with bond factors, and so on. More recently, stocks sorted on size, book/market, and past performance characteristics are priced by portfolios sorted on those characteristics.

# $p = E(mx)$ is Innocuous

- Since asset classes are not highly correlated, so risk premia from one source of betas have small impacts on another set of average returns. Also, more comprehensive  wealth measures that include human capital and real estate do not come with high-frequency price data, so adding them to a wealth portfolio has little effect on betas.
- From Roll's existence theorem, we know that there always exists a expected-return beta model, but the problem is how we choose the factor used in this model.
- Similarly, the law of one price implies that there exists some discount factor m such that $p = E(mx)$, and you impose almost no structure in doing so. But the problem is still in $m = f(data)$ .

# Ex Ante and Ex Post

- The theorems hold for any set of probabilities, hence they work equally well ex ante as ex post: $E(mx)$, $\beta$ , $E(\mathrm{R})$, and so forth can refer to agents' subjective probability distributions, objective population probabilities, or to the moments realized in a given sample.
- For example, if the law of one price holds in a sample, one may form an $x^*$ from sample moments that satisfies $p(x) = E(x^*x)$ where $p(x)$ refers to observed prices and $E(x^*x)$ refers to the sample average.
- This observation points to a great danger in the widespread exercise of searching for and statistically evaluating ad hoc asset pricing models. Most empirical asset pricing research posits an ad hoc pond of factors, fishes around a bit in that pond, and report statistical measures that show "success".

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# Discipline

* If we find an ex post efficient portfolio or $x^*$ that prices assets by construction, we will make mistakes. Because a portfolio is ex post efficient in one sample is unlikely to be mean-variance efficient ex ante or ex post in the next sample.
* Similarly, the portfolio $x^* = p'E(xx')^{-1}x$ that is a discount factor in one sample is unlikely to be a discount factor in the next sample; the required portfolio weights $p'E(xx')^{-1}$ change often drastically from sample to sample.
* The only solution is to impose some kind of discipline:1)use economic theory to carefully choose the variable; 2)use a battery of cross-sample and out-of-sample stability checks.
* Especially, we should understand the fundamental macroeconomic sources of risk, i.e., tie asset prices to macroeconomic events, to find a discount factor that are robust out of sample and across different markets.

# Mimicking Portfolios

- The pricing implications of any model can be equivalently represented by its factor-mimicking portfolio. If there is any measurement error in a set of economic variables driving m, the factor-mimicking portfolios for the true m will price assets better than an estimate of m that uses measured macroeconomic variables.
- That is to say, there is an important place for models that use returns as factors.
- But this does not tell us to circumvent the process of understanding the true macroeconomic factors by simply fishing for factor-mimicking portfolios.

# Irrationality and Joint Hypothesis

* Any test of "efficiency" is a joint test of efficiency and a "model of market equilibrium" (i.e., an asset pricing model).

* If(and only if) the discount factors that generate asset prices disconnected from marginal rates of substitution or transformation in the real economy, then the markets can be "irrational" or "inefficient" without requiring arbitrage opportunities.

* The existence theorems mean that there are no quick proofs of "rationality" or "efficiency", the only way to explain asset prices is thinking about economic models of the discount factor.

# The Number of Factors

- The equivalence theorems show that it is silly to focus on test the number of factors required to price a cross section of assets. Because with $m = b'f$, we can easily find a single-beta representation to any multiple-factor or multiple-beta representation, and they have the same pricing ability.
- For example, write

$$m = a + b_1 f_1 + b_2 f_2 + b_3 f_3 = a + b_1 f_1 + b_2 (f_2 + \frac{b_3}{b_2} f_3)$$
$$= a + b_1 f_1 + b_2 \hat{f}_2$$

to reduce a "three-factor" model to a "two -factor" model. In the ICAPM language, consumption itself could serve as a single state variable in place of the S state variables presumed to derive it.

# Discount Factor vs. Mean, Variance, and Beta

- Markowitz first derived mean-variance analysis to securities, and stated that investors' "utility functions" are defined over this mean and variance.
- Each security's mean return measures its contribution to the portfolio mean, and that regression betas on the overall portfolio give each security's contribution to the portfolio variance. So the mean-return versus beta description for each security followed naturally(Sharpe 1964).
- The transition from mean-variance frontiers and beta models to discount factors represents the realization that putting consumption in state in 1 and consumption in state 2 on the axes is a much more natural mapping of standard microeconomics into finance than putting mean, variance, etc. on the axes.
- The contingent-claim budget constraints are linear, while the mean-variance frontier is not.

# Discount Factor vs. Mean, Variance, and Beta

- Why prefer one language over another?
- The discount factor language has an advantage for its simplicity, generality, mathematical convenience, and elegance.
- The equation $p = E(mx)$ covers all assets, including bonds, options, and real investment opportunities.
- In expected return-beta formulation, there are several different asset pricing theories: expected return-beta for stocks, yield curve models for bonds, arbitrage models for options. In fact, they are just cases of $p = E(mx)$.

The End.