Chapter 5.
Mean-variance frontier
and beta representations

Main contents

- Expected return-Beta representation
- Mean-variance frontier: Intuition and Lagrangian characterization
- An orthogonal characterization of mean-variance frontier
- Spanning the mean-variance frontier
- A compilation of properties of R^*, R^{e*}, x^*
- Mean-variance frontiers for m: H-J bounds

5.1 Expected Return-Beta Representation

Expected return-beta representation

- Model:

$$E(R_i) = \alpha + \beta_{ia}\lambda_a + \beta_{ib}\lambda_b + ... \qquad (1)$$

- Restriction:

  α, λ are the same for all assets.

- β is estimated by time series regression on factors:

$$R_t^i = \alpha_i + \beta_{ia}\lambda_t^a + \beta_{ib}\lambda_t^b + \ldots\varepsilon_t^i, t = 1, 2, \ldots T \qquad (2)$$

Remark(1)

- In (1), the intercept is the same for all assets.
- In (2), the intercept is different for different asset.
- In fact, (2) is the first step to estimate (1).
- One way to estimate the free parameters α , λ is to run a cross sectional regression based on estimation of beta

$$E(R^i) = \alpha + \beta_{ia}\lambda_a + \beta_{ib}\lambda_b + \dots + \varepsilon_i$$

ε_i is the pricing errors

Remark(2)

- The point of beta model is to explain the variation in average returns across assets.
- The betas are explanatory variables,which vary asset by asset.
- The alpha and lamda are the intercept and slope in the cross sectional estimation.
- Beta is called as risk exposure amount, lamda is the risk price.
- Betas cannot be asset specific or firm specific.

Some common special cases

- If there is risk free rate, $R^f = \alpha$
- If there is no risk-free rate, then alpha is called (expected)zero-beta rate.
- If using excess returns as factors,

$$E(R^{ei}) = \beta_{ia}\lambda_a + \beta_{ib}\lambda_b + \ldots, i = 1, 2, \ldots N \quad (3)$$

- Remark: the beta in (3) is different from (1) and (2).
- If the factors are excess returns, since each factor has beta of one on itself and zero on all the other factors. Then,

$$E(R^{ei}) = \beta_{ia}E(f^a) + \beta_{ib}E(f^b) + \ldots, i = 1, 2, \ldots N$$

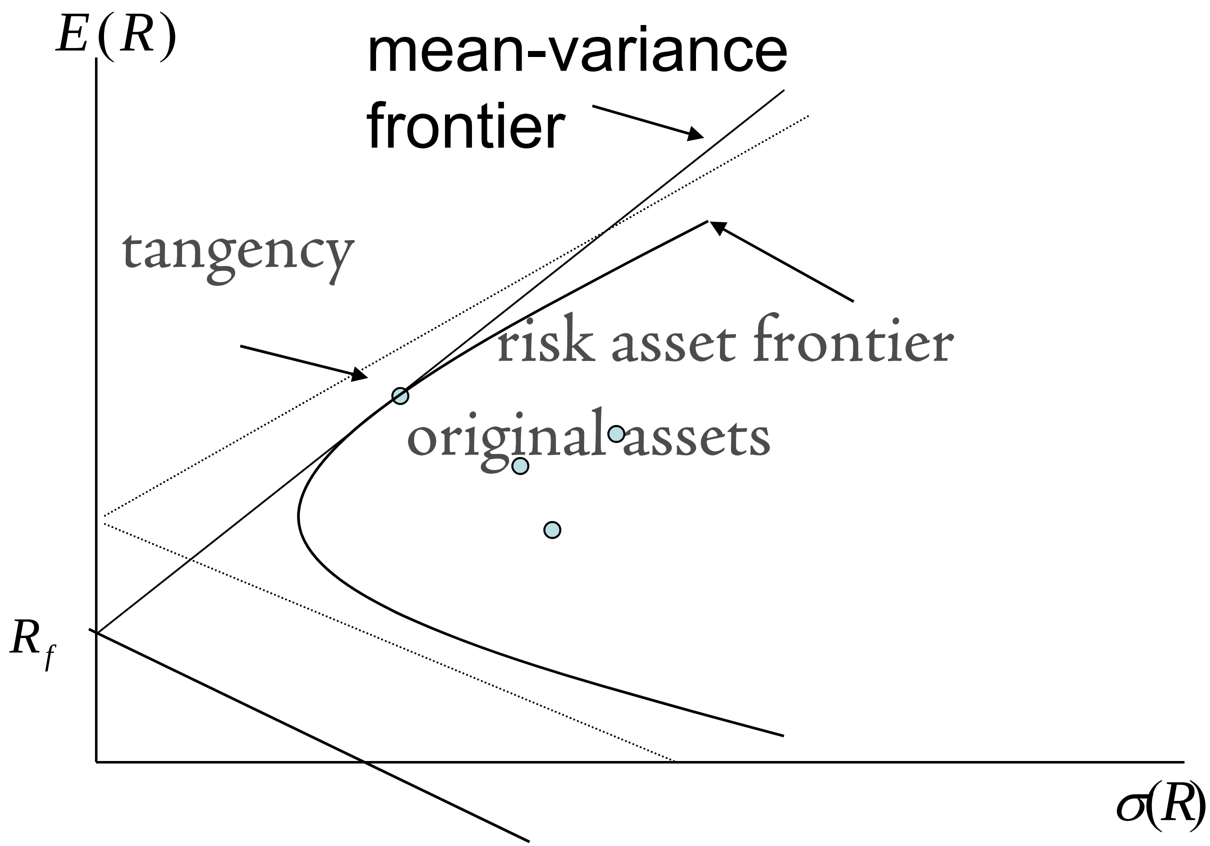
- 这个模型只研究风险溢酬，与无风险利率无关

5.2 Mean-Variance Frontier: Intuition and Lagrangian Characterization

Mean-variance frontier

- Definition: mean-variance frontier of a given set of assets is the boundary of the set of means and variances of returns on all portfolios of the given assets.
- Characterization: for a given mean return, the variance is minimum.

With or without risk free rate



When does the mean-variance exist?

- Theorem: So long as the variance-covariance matrix of returns is non singular, there is mean-variance frontier.
- Intuition Proof:
- If there are two assets which are totally correlated and have different mean return, this is the violation of law of one price. The law of one price implies the existence of mean variance frontier as well as a discount factor.

奇异矩阵的经济含义

- 如果$\begin{pmatrix} {\sigma_1}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & {\sigma_2}^2 \end{pmatrix}$奇异，则可得：${\sigma_1}^2 = a\rho\sigma_1\sigma_2$，$\rho\sigma_1\sigma_2$=$a{\sigma_2}^2$，其中$a$ 为常数,继而可得$\rho = \pm 1$.
- 所以，方差协方差阵奇异意味着：收益之间完全相关。
- 完全相关又分两种情形，一种是均值相同，一种是均值不同：
  - 均值不同时，违背了一价定律。
  - 均值相同时，不违背一价定律，但是这两种收益其实可以看作同一种收益来看，不作单独考虑。
- 因此，书中专门强调了完全相关但均值不同的情形。
- 一价定律成立，也就排除了上面的情形（收益完全相关但均值不同），也就满足了方差协方差矩阵非奇异，因而，根据定理，存在期望方差边界，同时存在随机贴现因子。

Mathematical method: Lagrangian approach

- Problem:

$$\min_{\{w\}} w'\sum w, s.t, w'E = u, w'1 = 1$$
$$E = E(R),$$
$$\sum = E[(R - E)(R - E)']$$

- Lagrangian function:

$$L = w'\sum w - \lambda(w'E - u) - \delta(w'1 - 1)$$

Mathematical method: Lagrangian approach(2)

- First order condition:

$$\frac{\partial L}{\partial w} = \sum w - \lambda E - \delta 1 = 0$$

- If the covariance matrix is non singular, the inverse matrix exists, and

$$w = \sum{}^{-1}(\lambda E + \delta 1),$$

$$E' w = E' \sum{}^{-1}(\lambda E + \delta 1) = u,$$

$$1' w = 1' \sum{}^{-1}(\lambda E + \delta 1) = 1,$$

$$\begin{bmatrix} E' \sum^{-1} E & E' \sum^{-1} 1 \\ 1' \sum^{-1} E & 1' \sum^{-1} 1 \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} u \\ 1 \end{bmatrix}$$

Mathematical method: Lagrangian approach(3)

- In the end, we can get

$$\lambda = \frac{Cu - B}{AC - B^2}, \delta = \frac{A - Bu}{AC - B^2},$$

$$w = \sum^{-1} \frac{E(Cu - B) + 1(A - Bu)}{AC - B^2},$$

$$\mathrm{var}(R^p) = \frac{Cu^2 - 2Bu + A}{AC - B^2},$$

$$A = E' \sum^{-1} E, B = E' \sum^{-1} 1, C = 1' \sum^{-1} 1$$

Remark

- By minimizing var(R^p) over u,giving

$$u^{\min var} = B/C, w = \sum^{-1} 1/(1' \sum^{-1} 1)$$

5.3 An orthogonal characterization of mean variance frontier

Introduction

- Method: geometric methods.
- Characterization: rather than write portfolios as combination of basis assets, and pose and solve the minimization problem, we describe the return by a three-way orthogonal decomposition, the mean variance frontier then pops out easily without any algebra.

Some definitions

- Definition of R^*: the return corresponding to the payoff x^* that can act as the discount factor.

$$R^* = \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})}$$

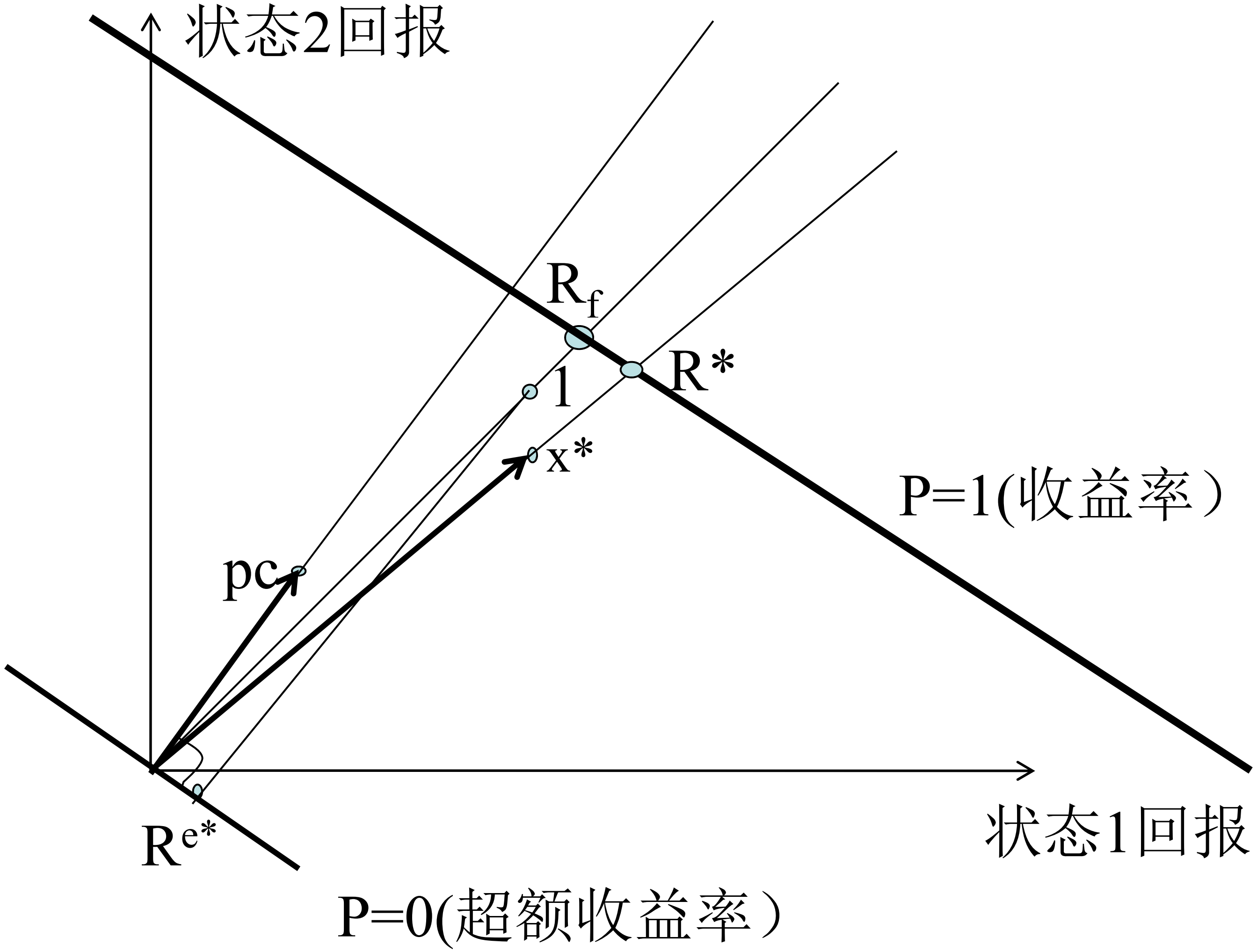
- Definition of R^{e*}:

$$R^{e*} \equiv proj(1 \mid \underline{R^e}),$$

假定$\pi(1) = \pi(2)$

$$\underline{R^e} = \{x \in \underline{X}, s.t. \, p(x) = 0\}$$

- R^{e*}反映了对不同状态的偏好程度。在状态偏好中性世界中，它等于0.


状态2回报
R_f
R^*
1
x^*
P=1(收益率）
pc
状态1回报
R^{e*}
P=0(超额收益率）

Theorem:

- Every return R^i can be expressed as:

$$R^i = R^* + \omega^i R^{e*} + n^i$$

- Where ω^i is a number, and n^i is an excess return with the property $E(n^i)=0$.
- The three components are orthogonal,

$$E(R^* R^{e*}) = E(R^* n^i) = E(R^{e*} n^i) = 0$$

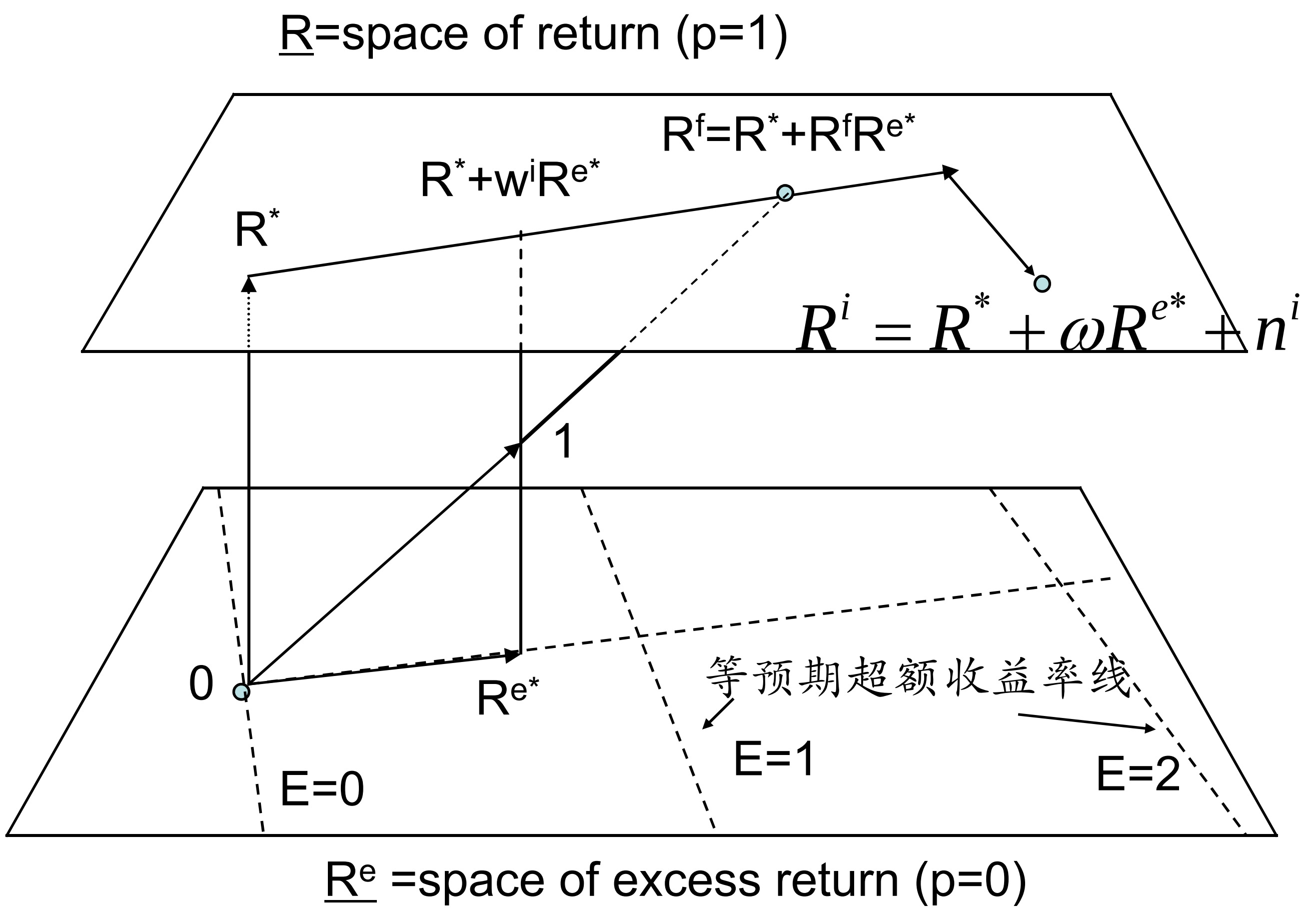
Theorem: two-fund theorem for MVF

- R^{mv} is on the mean-variance frontier iff

$$R^{mv} = R^{*} + \omega R^{e*}$$

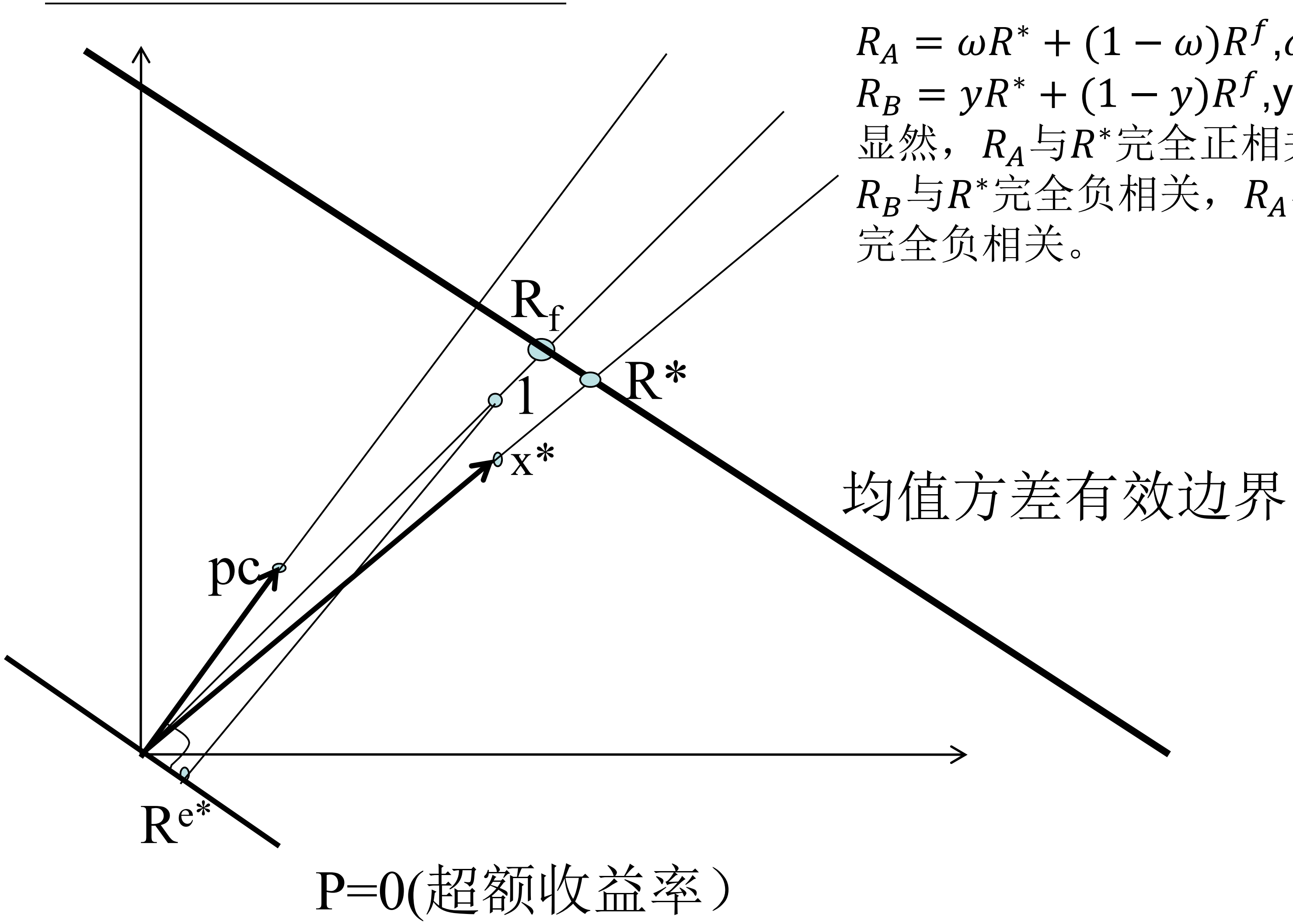
for some real number ω.

Proof: Geometric method



NOTE:1、回报空间为三维的。2、横的平面必须与竖的平面垂直。3、如果有无风险证券，则竖的平面过1点，否则不过，此时图上的1就是1在回报空间的投影。

若存在无风险资产，则R*和无风险资产可以构成一个平面（如下图），此时坐标轴不一定是原坐标轴。显然，R_f右下边的证券由于与x^*完全正相关，其贝塔系数等于σ_A/σ_{R^*},其预期收益率都应小于R_f，越右越小；而R_f左上边的证券由于与x^*完全负相关，其贝塔系数等于$-\sigma_A/\sigma_{R^*}$，其预期收益率都应大于R_f，越左越大。



$R_A = \omega R^* + (1-\omega)R^f$,$\omega > 0$
$R_B = yR^* + (1-y)R^f$,y< 0
显然，R_A与R^*完全正相关，R_B与R^*完全负相关，R_A与R_B完全负相关。

- 若不存在无风险资产，则0与均值方差有效边界构成的平面则只能由两个不完全相关的证券构成，否则的话就可以用两个完全相关的证券复制出无风险证券。此时，均值方差有效边界上的点不完全相关。

Proof: Algebraic approach

- Directly from definition, we can get

$$E(n^i) = E(R^{e*}n^i) = 0$$

$$E(R^i) = E(R^*) + w^i E(R^{e*})$$

$$\sigma^2(R^i) = \sigma^2(R^* + w^i R^{e*}) + \sigma^2(n^i)$$

只有$n^i = 0$时，方差在收益率给定情况下才最小。

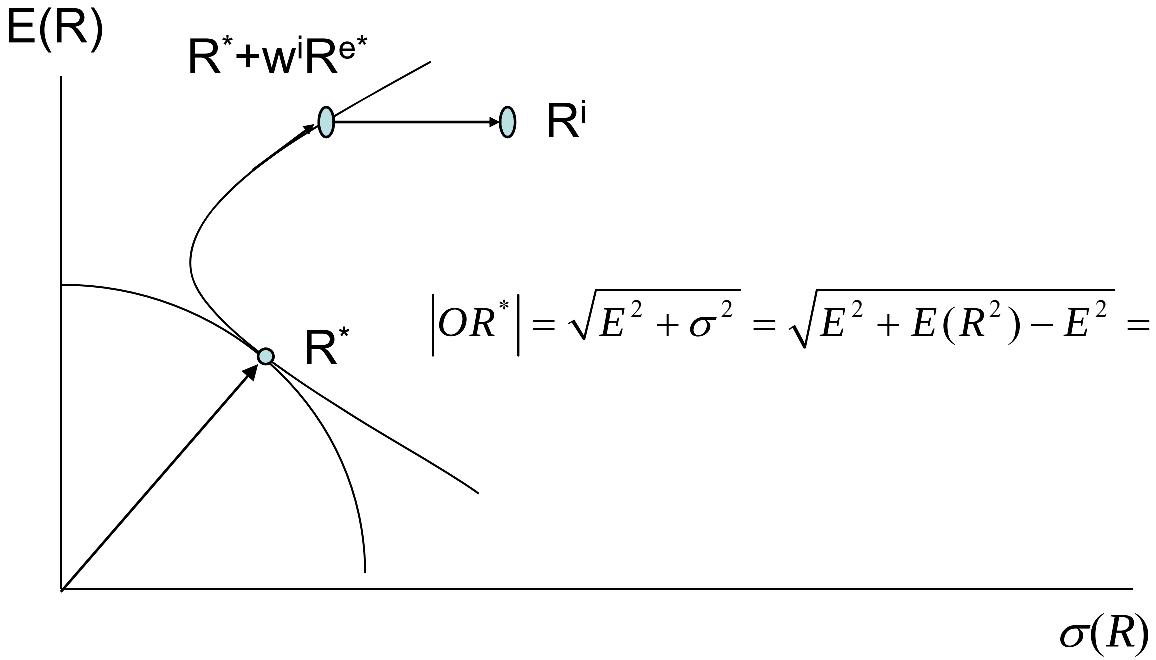
注意：正交不等于不相关。但$\text{cov}(R^*, n^i) = E(R^* n^i) - E(R^*)E(n^i) = 0$，而$\text{cov}(R^*, R^{e*}) = E(R^* R^{e*}) - E(R^*)E(R^{e*}) = -E(R^*)E(R^{e*})$

Decomposition in mean-variance space

- R^* is the minimum second moment return.
- Since $E(R^2) = E(R^{*2}) + w^2 E(R^{e*2}) + E(n^{i2})$
- When $\omega=0$ and n=0, $E(R^2)$ is smallest.
- In mean-standard deviation space, the line is circles, thus the minimum second moment return is the smallest circle the intersect the set of all assets.
- It is generally on the lower, or inefficient segment of mean-variance frontier.

Remark

- The minimum second moment return is not the minimum variance return.(why?)



$$\left|OR^*\right| = \sqrt{E^2+\sigma^2} = \sqrt{E^2+E(R^2)-E^2} = \sqrt{E(R^2)}$$

5.4 Spanning the mean variance frontier

Spanning the mean variance frontier

- With any two portfolios on the frontier. we can span the mean-variance frontier.
- Consider

$$R^{\alpha} = R^{*} + \gamma R^{e*}, \gamma \neq 0,$$

$$R^{e*} = \frac{R^{\alpha} - R^{*}}{\gamma},$$

$$R^{*} + wR^{e*} = R^{*} + \frac{w}{\gamma}(R^{\alpha} - R^{*}) = (1 - y)R^{*} + yR^{\alpha}$$

$$y = w / \gamma$$

5.5 A compilation of properties of R^*, R^{e*}, and x^*

Properties(1)

$$E(R^{*2}) = \frac{1}{E(x^{*2})},$$

- Proof:

$$R^{*} = \frac{x^{*}}{E(x^{*2})},$$

$$R^{*2} = \frac{x^{*}R^{*}}{E(x^{*2})},$$

$$E(R^{*2}) = \frac{E(x^{*}R^{*})}{E(x^{*2})} = 1/E(x^{*2})$$

Properties(2)

$$x^* = \frac{R^*}{E(R^{*2})}$$

- Proof:

$$R^* = \frac{x^*}{E(x^{*2})},$$

$$x^* = R^* E(x^{*2}) = \frac{R^*}{E(R^{*2})}$$

Properties(3)

- R^* can be used in pricing.
- Proof:

$$p(x) = E(x^* x) = \frac{E(R^* x)}{E(R^{*2})}$$

- For returns,

$$1 = p(R) = E(x^* R) = \frac{E(R^* R)}{E(R^{*2})} \Rightarrow$$

$$E(R^* R) = E(R^{*2})$$

Properties(4)

- If a risk-free rate is traded,

$$R^f = \frac{1}{E(x^*)} = \frac{E(R^{*2})}{E(R^*)}$$

- If not, this gives a "zero-beta rate" interpretation.

Properties(5)

- R^{e*} has the same first and second moment.
- Proof:

$$E(R^{e*}) = E(R^{e*}R^{e*}) = E(R^{e*2})$$

- Then

$$\mathrm{var}(R^{e*}) = E(R^{e*2}) - E(R^{e*})^2 = E(R^{e*})(1 - E(R^{e*}))$$

Properties(6)

- If there is risk free rate,

$$R^f = R^* + R^f R^{e*}$$

- Proof:

$$1 = proj(1 \mid \underline{R^e}) + proj(1 \mid R^*)$$

$$R^{e*} = 1 - proj(1 \mid R^*) = 1 - \frac{1}{R_f} R^*$$（从上图的两个相似三角形可以看出）

注意R^*前面的1和R_f都是标量，其余都是向量

$$R_f = R^* + R_f R^{e*}$$（这里的第二个R_f是标量，其余都是向量）

If there is no risk free rate

- Then the 1 vector can not exist in payoff space since it is risk free. Then we can only use

$$proj(1|\underline{X}) = proj(proj(1|\underline{X})|\underline{R^e}) + proj(proj(1|\underline{X})|R^*)$$

$$= proj(1|\underline{R^e}) + proj(1|R^*)$$

$$R^{e*} = proj(1|\underline{X}) - proj(1|R^*)$$

$$= proj(1|\underline{X}) - E(x^*)R^*$$

$$= proj(1|\underline{X}) - \frac{E(R^*)}{E(R^{*2})}R^*$$

Properties(7)

- Since

$$x^* = p'E(xx')^{-1}x$$

$$p(x^*) = E(x^*x^*)$$

- We can get

$$R^* = \frac{x^*}{p(x^*)} = \frac{p'E(xx')^{-1}x}{p'E(xx')^{-1}p}$$

Properties(8)

- Following the definition of projection, we can get

$$R^{e*} = proj(1 \mid R^e) = E(R^e)' E(R^e R^{e'})^{-1} R^e$$

- If there is risk free rate,we can also get it by:

$$R^{e*} = 1 - \frac{1}{R_f} R^* = 1 - \frac{1}{R_f} \frac{p' E(xx')^{-1} x}{p' E(xx')^{-1} p}$$

5.6 Mean-Variance Frontiers for Discount Factors: The Hansen-Jagannathan Bounds

Mean-variance frontier for m: H-J bounds

- The relationship between the Sharpe ratio of an excess return and volatility of discount factor.

$$E(mR^e) = E(m)E(R^e) + \rho_{m,R^e}\sigma(m)\sigma(R^e) = 0,$$

$$|\rho_{m,R^e}| = \left|\frac{E(m)E(R^e)}{\sigma(m)\sigma(R^e)}\right| \leq 1,$$

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}$$

- 从经济意义上讲，m的波动率不会太大，所有夏普比率也不应太大。
- If there is risk free rate, $E(m) = 1/R_f$

Remark

- We need very volatile discount factors with a mean near one to price the stock returns.

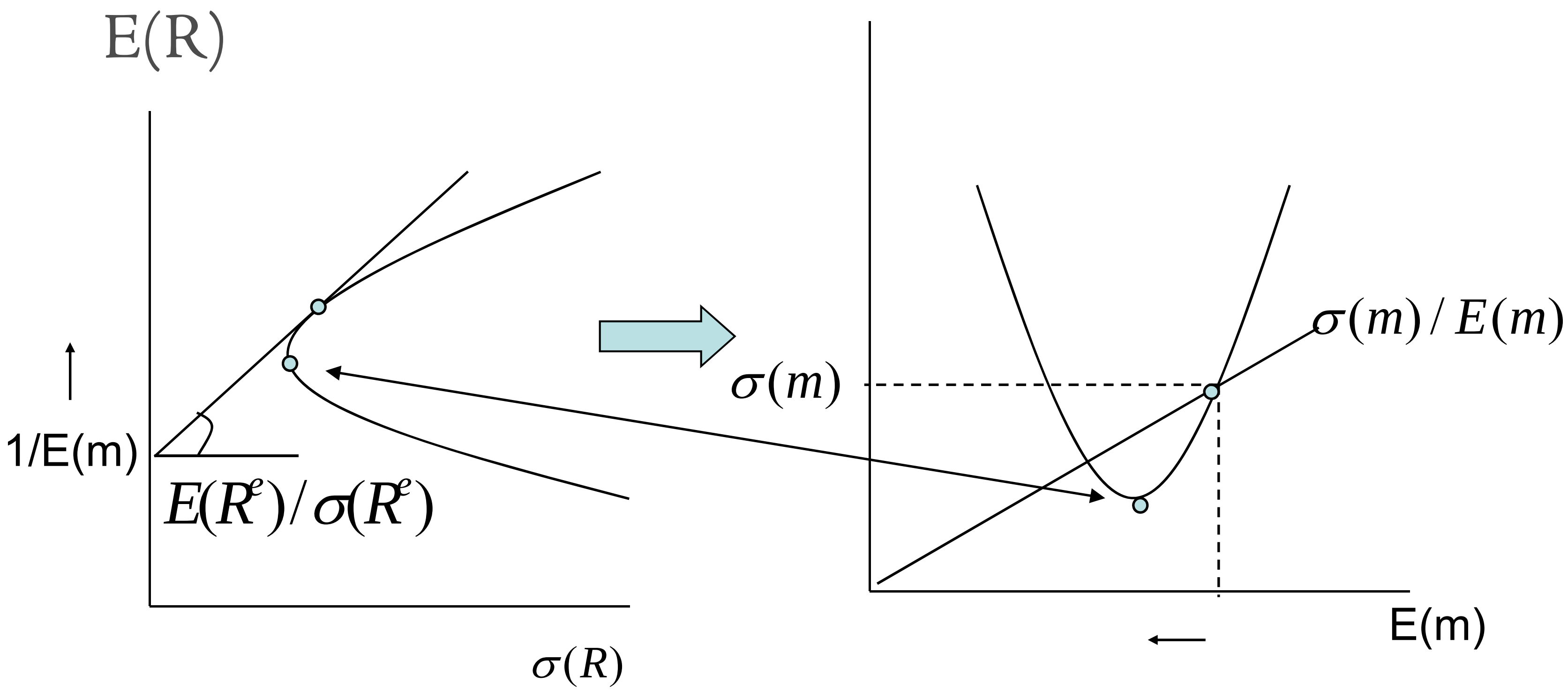
The behavior of Hansen and Jagannathan bounds

- For any hypothetical risk free rate, the highest Sharpe ratio is the tangency portfolio.
- Note: there are two tangency portfolios, the higher absolute Sharpe ratio portfolio is selected.
- If risk free rate is less than the minimum variance mean return, the upper tangency line is selected, and the slope increases with the declination of risk free rate, which is equivalent to the increase of E(m).

The behavior of Hansen and Jagannathan bounds

- On the other hand, if the risk free rate is larger than the minimum variance mean return, the lower tangency line is selected,and the slope decreases with the declination of risk free rate, which is equivalent to the increase of E(m).
- In all, when 1/E(m) is less than the minimum variance mean return, the H-J bound is the decreasing function of E(m). When 1/E(m) is larger than the minimum variance mean return, the H-J bound is an increasing function.

Graphic construction



Duality

- A duality between discount factor volatility and Sharpe ratios.

$$\min_{\{all\, m\ that\ price\ x \in X\}} \frac{\sigma(m)}{E(m)} = \max_{\{all\, excess\, returns\, R^e\ in\ \underline{X}\}} \frac{E(R^e)}{\sigma(R^e)}$$

Explicit calculation

- A representation of the set of discount factors is

$$m = E(m) + [p - E(m)E(x)]\Sigma^{-1}[x - E(x)] + \varepsilon,$$

$$\Sigma = \text{cov}(x, x'), E(\varepsilon) = 0, E(\varepsilon x) = 0$$

- Proof:

$$E(mx) = E((E(m) + [p - E(m)E(x)]\Sigma^{-1}[x - E(x)] + \varepsilon)x)$$

$$= E(m)E(x) + [p - E(m)E(x)]\Sigma^{-1}E[(x - E(x))x]$$

$$= E(m)E(x) + p - E(m)E(x) = p$$

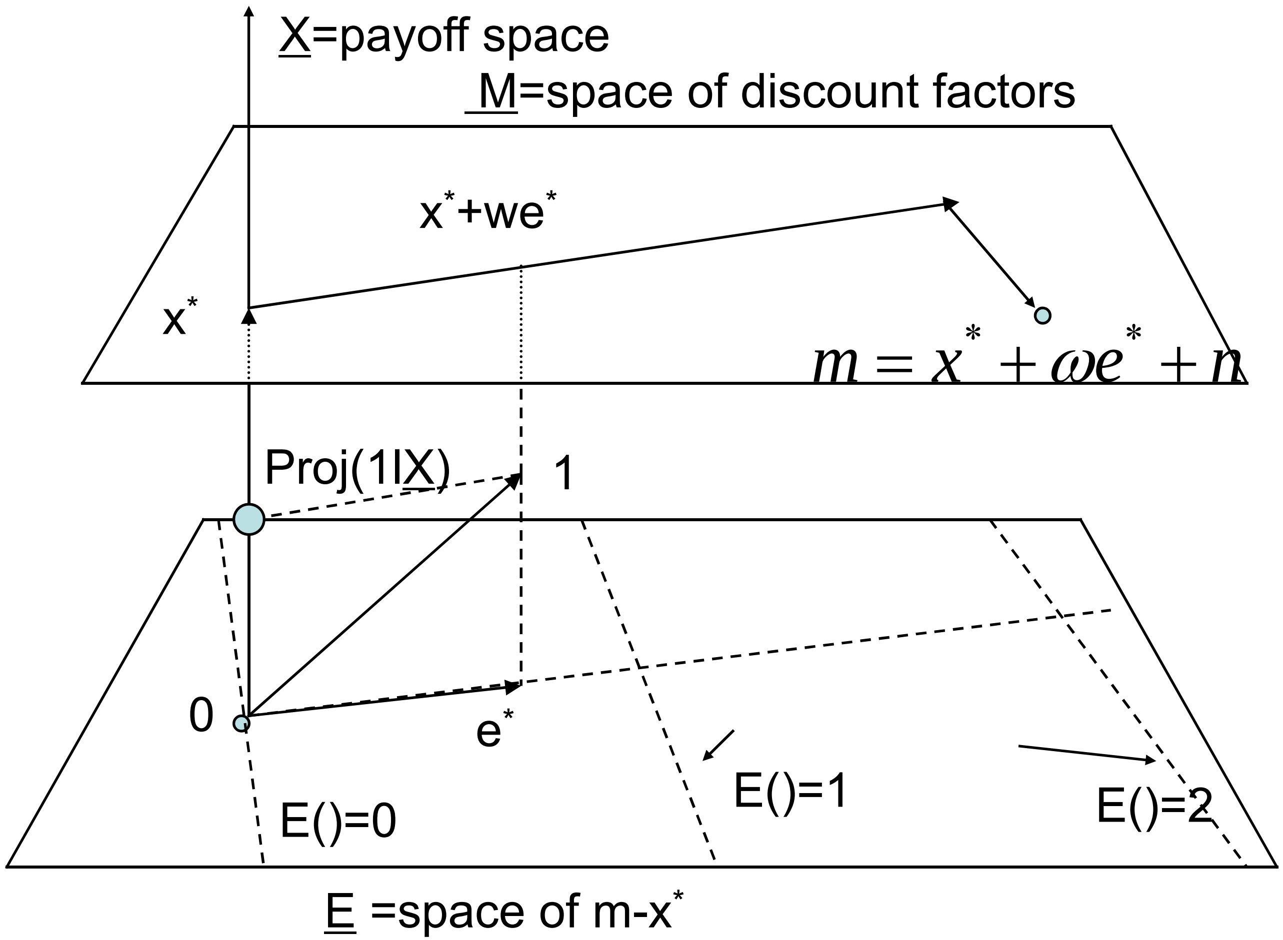
An explicit expression for H-J bounds

$$\sigma^2(m) \geq (p - E(m)E(x))' \sum^{-1} (p - E(m)E(x))$$

- Proof:

$$\begin{aligned} \sigma^2(m) &= \left(\left[p - E(m)E(x)\right]' \sum^{-1}\right)^2 \sum \; + \sigma^2(\varepsilon) \\ &= \left[p - E(m)E(x)\right]' \sum^{-1} \left[p - E(m)E(x)\right] + \sigma^2(\varepsilon) \\ &\geq \left[p - E(m)E(x)\right]' \sum^{-1} \left[p - E(m)E(x)\right] \end{aligned}$$

Graphic Decomposition of discount factor



NOTE:横的平面必须与竖的平面垂直。

Decomposition of discount factor

- Any discount factor must line in the plane perpendicular to payoff space through x*.

$$m = x^* + we^* + n$$

- Where

$$e^* \equiv 1 - proj(1 \mid \underline{X}) = proj(1 \mid \underline{E})$$

e^*反映了无风险资产不存在而产生的定价偏差。

- The mean-variance frontier of m is given by

$$m = x^* + we^*$$

Special case

- If unit payoff is in payoff space,

$$e^{*} = 1 - proj(1 \mid \underline{X}) = 0$$

- The frontier and bound are just $m = x^{*}$
- And

$$\sigma^{2}(m) \geq \sigma^{2}(x^{*})$$

- This is exactly like the case of state preference neutrality for return mean-variance frontiers, in which the frontier reduces to the single point R^{*}.

Mathematical construction

- We have got

$$
\begin{aligned}
m^* &= x^* + we^* \\
&= p' E(xx')^{-1} x + w\left(1 - proj\left(1 \mid \underline{X}\right)\right) \\
&= p' E(xx')^{-1} x + w\left(1 - E(x)' E(xx')^{-1} x\right) \\
&= w + \left[p - wE(x)\right]' E(xx')^{-1} x
\end{aligned}
$$

$$
E(m^*) = w + \left[p - wE(x)\right]' E(xx')^{-1} E(x)
$$

$$
\sigma^2(m^*) = \left[p - wE(x)\right]' \operatorname{cov}(xx')^{-1} \left[p - wE(x)\right]
$$

Some development

- H-J bounds with positivity. It solves

$$\min \sigma^2(m), s.t. p = E(mx), m > 0, E(m)\text{固定}$$

- This imposes the no arbitrage condition.
- Short sales constraint and bid-ask spread is developed by Luttmer(1996).
- A variety of bounds is studied by Cochrane and Hansen(1992).

The End