



Chp.4 The Discount Factor



Main Contents

- The Relationship between Law of One Price and Existence of Discount Factor;
 - The Relationship Between No Arbitrage and Existence of Positive Discount Factor;
 - An Alternative Formula to Compute the Discount Factor in Discrete and Continuous Time.
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4.1 law of one price and Existence of a Discount factor



Assumptions

- **A1:(Portfolio formation):** $x_1, x_2 \in \underline{X} \Rightarrow ax_1 + bx_2 \in \underline{X}$
for any real a and b.
- Remark: It's an important and restrictive simplifying assumption. short sales constraints, leverage limitations, and so on.
- **A2:(Law of one price, Linearity):**
$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2)$$
- Remark: if the payoff of asset A is the same as that of asset B in any case, then price of A=price of B. happy meal theorem. It rules out bid/ask spreads. 不考虑流动性。



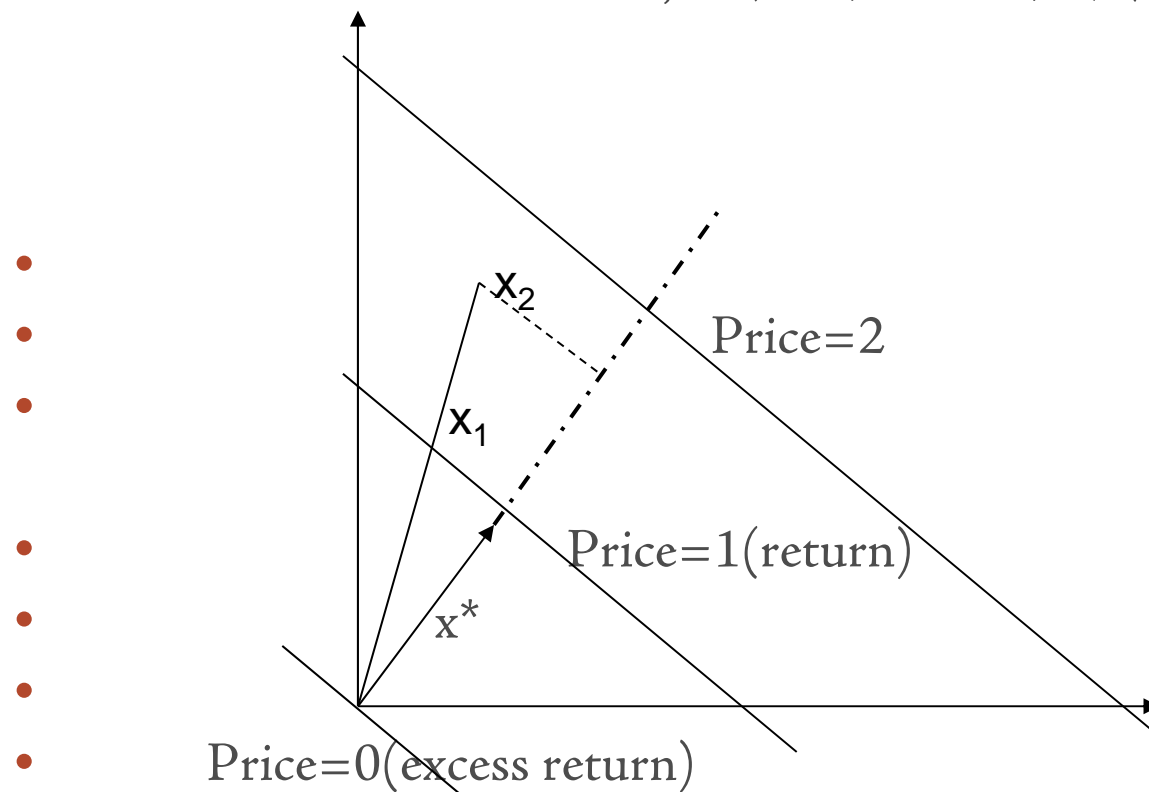
Theorem 1

- Given free portfolio formation A1, and the law of one price A2, there exists a unique payoff $x^* \in \underline{X}$ such that $p(x) = E(x^*x)$ for all $x \in \underline{X}$.
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Geometric Proof 1

- 一价定律=线性价格函数。
- 线性价格函数 \implies 等价线如下图所示。假设支付空间是二维的。
- 根据 $p=0$ 等价线可知 x^* 与之正交。（存在）（注意我们定义 $E(XY) = X \circ Y$ ，因此求内积时要乘以概率）





Geometric Proof 2

- 用 x^* 为 $p=1$ 等价线上的任一证券 x_1 定价可确定 x^* 的长度。
即：

$$|proj(x_1 | x^*)| \times |x^*| = 1 \Rightarrow |x^*| = 1 / |proj(x_1 | x^*)|$$

- 给定任意证券 x_2 ，它与0连线（或延长线）与 $p=1$ 等价线相交于 x_1 ，即 $x_2 = a x_1$ 。从图上可以看出，用 x^* 定价可得 $p(x_2) = a p(x_1)$ ，符合一价定律。
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Algebraic Proof

- Suppose the basis payoffs (after pruning redundant rows of x)

$$x = [x_1, x_2, \dots, x_N]', \text{ 注意: } x \text{ 在这里是矩阵}$$

- Then we want to find a discount factor x^* in payoff space, so it must be of the form

$$x^* = c'x$$

- 对于任意的证券组合 $a'x$, 我们用 x^* 来定价得:

$$p(a'x) = E((c'x)(a'x)) = c'aE(x'x)$$

$$\text{由一价定律可知 } p(a'x) = a'p$$

$$\therefore E(x'x)c = p \Rightarrow c = E(xx')^{-1} p$$

$$x^* = p' E(xx')^{-1} x$$

- 由于 x^* 对于任意证券都一样, 因此是唯一的。
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Other discount factors

- The discount factor in payoff space \underline{X} is unique.
- There are many other discount factors m not in \underline{X} . (unless the market is complete).
- If $p = E(mx)$, then $p = E[(m + \varepsilon)x]$ for any ε orthogonal to x , $E(\varepsilon x) = 0$.
- Any discount factor m can be represented as $m = x^* + \varepsilon$, with $E(\varepsilon x) = 0$.
- The pricing implication of any discount factor m for a set of payoff X are the same as those of projection m on X .

$$p = E(mx) = E[(proj(m | X) + \varepsilon)x] = E[proj(m | X)x]$$

- $proj(m | X)$ is called the mimicking portfolio for m .
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Theorem 2

- The existence of a discount factor implies the law of one price
 - Proof: if $x+y=z$, and there is a discount factor, then
$$p(x+y) = E(m(x+y)) = E(mz) = p(z)$$
-



4.2 No Arbitrage and Positive Discount Factors





Definition: No arbitrage

- D1: Every payoff x that is always nonnegative (almost surely), and positive with some positive probability, has positive price.
 - D2: If $x \geq y$ almost surely and $x > y$ with positive probability, then $p(x) > p(y)$.
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Theorem 3: $m > 0$ imply No arbitrage

- Proof:
 - For $x \geq 0$ and in some states $x > 0$.
 - Because $m > 0$ (positive in every state).
 - $P = E(mx) > 0$
-



No arbitrage implies a $m > 0$

□ 直觉：

- 无套利意味着正象限payoff的价格严格为正。 $p=0$ 线将正负价格区域分割开来。为了使正负价格区域不交叉，等价线必须经过0和第2、4象限，因此 m 必须经过0指向第一象限。
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Theorem 4: No arbitrage implies a $m > 0$, 可以对回报空间的任何 x 定价

- 证明：由于无套利蕴含着一价定律，也就意味着存在随机折现因子，故仅需证明 m 为正的。
- 联合 $(-p(x), x)$ 形成 $s+1$ 维空间 R^{s+1} 中的向量。令 M 表示所有的数对 $(-p(x), x)$ 构成的集合。

$$M = \{(-p(x), x); x \in \underline{X}\}$$

- 由一价定律， M 仍是一个线性空间。
 - 无套利意味着 M 的元素（ $s+1$ 维向量）不能够全部由正的分量组成。如果 x 是正的，那么 $-p(x)$ 一定为负（无套利保证的）。这样，超平面 M 就与正的向量空间 \mathbf{R}_+^{s+1} 只相交于原点。
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- 这样就存在一个函数 $F: \mathbf{R}^{s+1} \Rightarrow \mathbf{R}$ 使得对于 $(-p(x), x) \in M$ 的点 $F(-p, x) = 0$ ，并且除原点外的 $(-p(x), x)$ 的点 $F(-p, x) > 0$ （由超平面分离定理保证的）。
- 由于可以采用向量的内积来表示任何的线性函数，并且存在向量 $(1, m)$ 使得

$$F(-p, x) = (1, m) \cdot (-p, x) = -p + m \cdot x$$

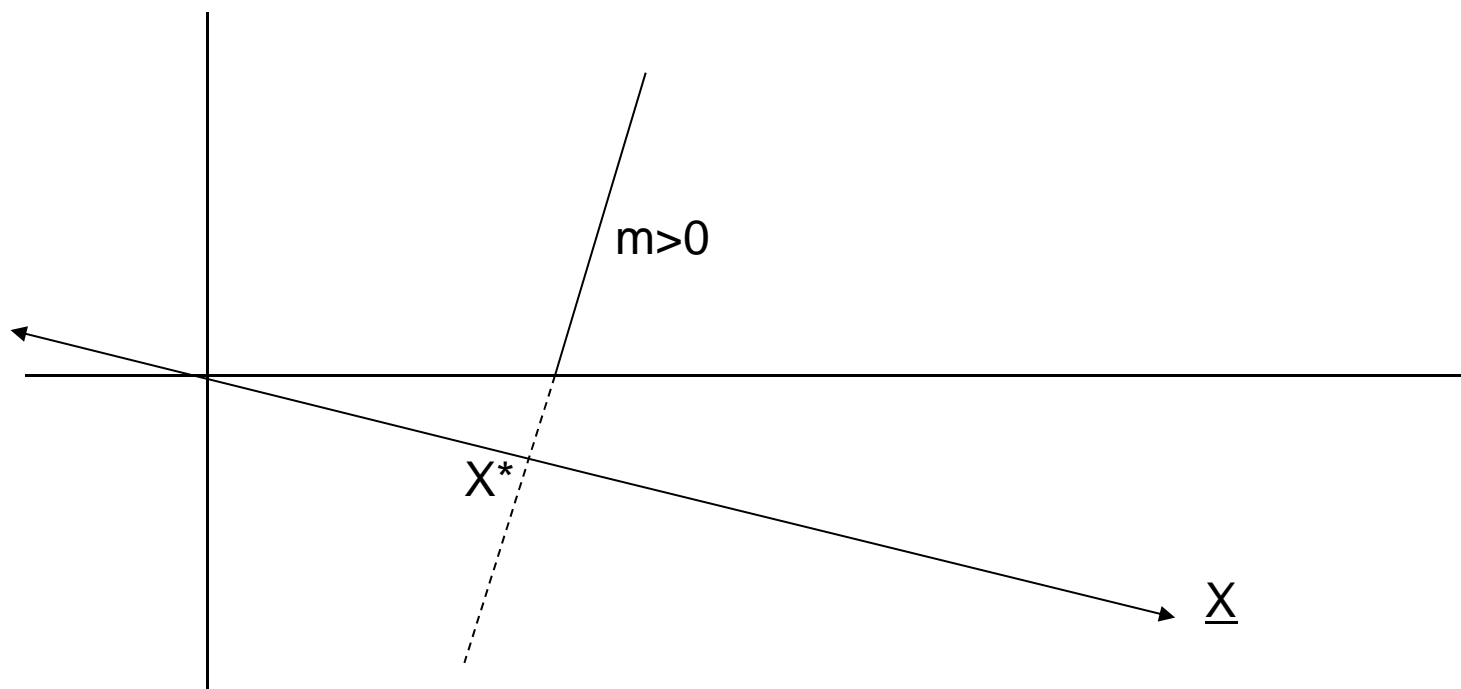
$$\text{or } -p + E(mx)$$

- 由于对所有 $(-p(x), x) > 0$ 的点 $F(-p, x)$ 都是正的，所以 m 必须是正的。
 - 在连续的情况下，可以由凸集分离定理和Riesz表示定理
同样得到结论
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Other discount factors

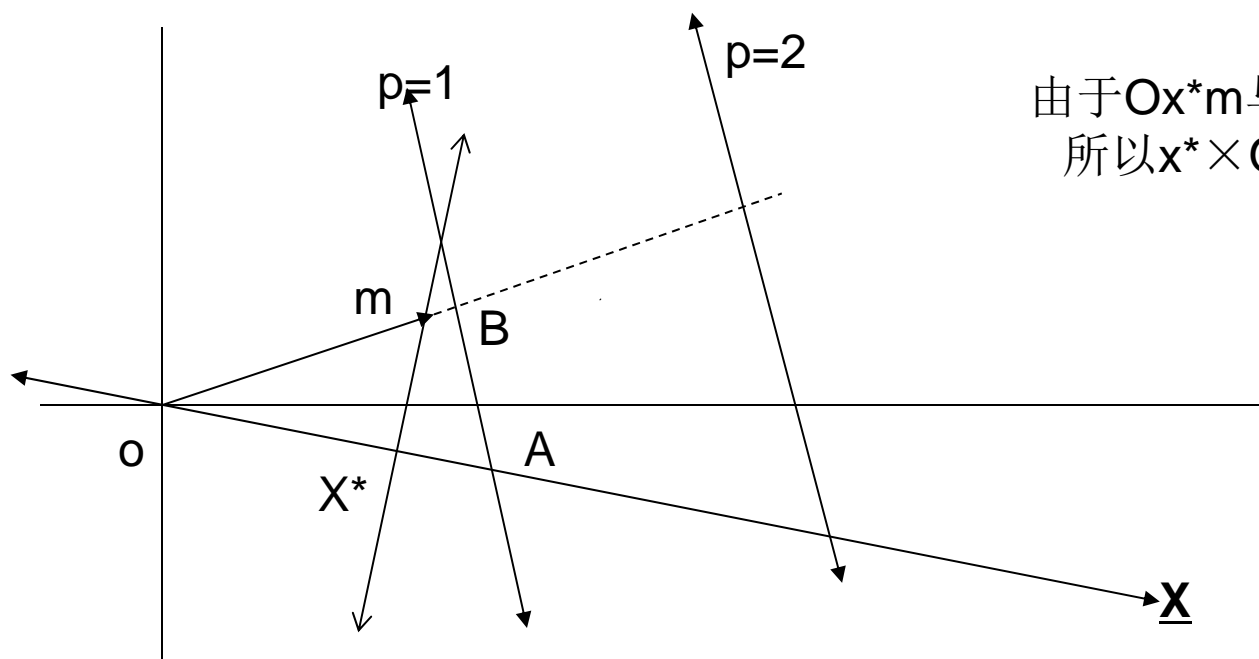
- 从经济意义上讲， m 应该为正。但 m 在支付空间中的投影不一定为正。
- In incomplete market, even x^* need not be positive.





Arbitrage-free extension of prices

- Each particular choice of $m > 0$ induces an arbitrage-free extension of prices on \underline{X} to all contingent claims



由于 Ox^*m 与 OBA 相似,
所以 $x^* \times OA = OB \times m$



No arbitrage and the law of one price

- No arbitrage is more strict than the law of one price.
 - No arbitrage implies the law of one price, but not vice versa.
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Why no arbitrage is more strict than law of one price?

- Law of one price implies the same payoff has the same price, but does not consider the situation of different payoffs. For example, if payoff $A > \text{payoff } B$ in any case, under the law of one price, $p(A) < p(B)$ may hold. This implies arbitrage opportunity.
 - No arbitrage implies positive payoff has positive price, which includes the law of one price.
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4.3 an alternative formula, and x^* in continuous time





Alternative formula

- $$x^* = E(x^*) + [p - E(x^*)E(x)]' \Sigma^{-1} (x - E(x))$$
- Proof:

$$\begin{aligned} E(x^* x) &= E\{[E(x^*) + (p - E(x^*)E(x))' \Sigma^{-1} (x - E(x))]x\} \\ &= E(x^*)E(x) + (p - E(x^*)E(x))' \Sigma^{-1} E\{[x - E(x)]x\} \\ &= E(x^*)E(x) + (p - E(x^*)E(x))' \Sigma^{-1} \Sigma = p \end{aligned}$$



Alternative formula(2)

- If a risk-free rate is traded, and the payoff space consists solely of excess returns($p=0$), then we have:

$$x^* = \frac{1}{R^f} - \frac{1}{R^f} E(R^e)' \Sigma^{-1} (R^e - E(R^e)); \Sigma \equiv \text{cov}(R^e)$$



X^* in continuous time

- Similarly, we can get

$$\frac{d\Lambda^*}{\Lambda^*} = -r_f dt - \left(\mu + \frac{D}{p} - r_f\right) \Sigma^{-1} \sigma dz$$

- Proof:

假设: $\frac{dp}{p} = \mu dt + \sigma dz, \frac{d\Lambda^*}{\Lambda^*} = -r^f dt - \sigma^\Lambda dz$

$$E_t \frac{dp}{p} + \frac{D}{p} dt - r_f dt = \left(\mu + \frac{D}{p} - r_f\right) dt = -E_t \left(\frac{d\Lambda^*}{\Lambda^*} \frac{dp}{p}\right) = \sigma \sigma_\Lambda dt,$$

$$\sigma \sigma_\Lambda = \left(\mu + \frac{D}{p} - r_f\right),$$

$$\sum \sigma_\Lambda = \left(\mu + \frac{D}{p} - r_f\right) \sigma',$$

$$\sigma_\Lambda = \sum^{-1} \left(\mu + \frac{D}{p} - r_f\right) \sigma' = \left(\mu + \frac{D}{p} - r_f\right)' \sum^{-1} \sigma$$



Other discount factors in continuous time

- Λ^* plus orthogonal noise will also act as a discount factor:

$$\frac{d\Lambda}{\Lambda} = \frac{d\Lambda^*}{\Lambda^*} + dw; E(dw) = 0; E(dzdw) = 0.$$



重要结论 (1)

- 在完全市场中, m 只有一个, 且严格为正。
 - 在不完全市场中, 即使处于无套利均衡状态, m 在回报空间中的投影也可能为负。
 - 在不完全市场中, 新产品 (只要不是原有产品的线性复制品) 可以使市场趋于完全。但若没有其他信息, 该产品就无法准确定价, 但可以确定价格区间。
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重要结论 (2)

- 在完全市场中，等价鞅测度是唯一的。概率测度可以自由转换。而在不完全世界中，等价鞅测度不是唯一的。
 - 在不完全世界，给定风险中性概率和任一资产（包括无风险资产）的价格，就可以求出在这种测度下任意新产品的价格。因为知道风险中性概率就知道所有状态价格之比，再利用已知价格就可算出所有状态价格，从而为所有资产定价。风险中性概率决定相对价格，该证券价格决定绝对价格。 $\pi^*(s) = pc(s) / E(m)$
 - 不完全世界中，即使我们知道现实的概率，和N (N 小于S) 种证券价格，我们仍无法对任意新产品定价。
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The End

